



PME 42

July 3-8, 2018 --- Umeå, Sweden

Proceedings

Of the 42nd Conference of the International Group
for the Psychology of Mathematics Education

Editors: Ewa Bergqvist, Magnus Österholm,
Carina Granberg, and Lovisa Sumpter

Volume 2

Research Reports A – Haa

**Proceedings of the 42nd Conference
of the International Group
for the Psychology of Mathematics Education**

Umeå, Sweden
July 3-8, 2018

Editors

Ewa Bergqvist
Magnus Österholm
Carina Granberg
Lovisa Sumpter

Volume 2

Research Reports A – Haa



Cite as:

Bergqvist, E., Österholm, M., Granberg, C., & Sumpter, L. (Eds.). (2018). *Proceedings of the 42nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2). Umeå, Sweden: PME.

Website: <http://www.pme42.se>

The proceedings are also available via <http://www.igpme.org>

Copyright © 2018 left to the authors

All rights reserved

ISSN: 0771-100X

ISBN (volume 2): 978-91-7601-903-0

Logo Design: Catarina Rudälv and Amanda Rudälv

Printed by CityPrint i Norr AB, Umeå

TABLE OF CONTENTS

VOLUME 2

RESEARCH REPORTS A-HAA

WHY IS CALCULATING THE AVERAGE SPEED DIFFICULT?.....	3
Linda Marie Ahl, Ola Helenius	
ACCOUNTABILITY AND ASSESSMENT: GAPS AND GRIDS	11
Julie Alderton, Nick Pratt	
PRESERVICE MATHEMATICS TEACHERS' CURRICULUM VISUALIZATION	19
Julie Amador, Darrell Earnest	
LINKS BETWEEN TEACHERS' PEDAGOGICAL TECHNOLOGICAL KNOWLEDGE AND THEIR PERSONAL CHARACTERISTICS	27
Ahlam Adnan Anabousy, Michal Tabach	
MAKING MATHS MATTER: ENGAGING STUDENTS FROM LSES SCHOOLS THROUGH SOCIAL JUSTICE CONTEXTS	35
Judy Anne Anderson, Alice Kriesler	
CLASSROOM DIALOGUE AS A FRENCH BRAID: A CASE STUDY FROM TRIGONOMETRY	43
Mette Andresen, Bettina Dahl	
USING SELF-VIDEO ANALYSIS TO PROMOTE TEACHER GROWTH.....	51
Nick Andrews, Jenni Ingram, Andrea Pitt	
DEVELOPING PROSPECTIVE TEACHERS' MATHEMATICS ORIENTATIONS IN THE CONTENT COURSES	59
Aina Appova	
BIG BLOCKS OF PROOF	67
Amir Hossein Asghari, Sharareh Taghi Dastjerdi, Fateme Ahmadvpour Mobarakeh	
MIDDLE GRADE STUDENTS' PERFORMANCE ON ARITHMETIC CALCULATIONS PRESENTED AS WORD PROBLEMS OR NUMERIC PROBLEMS	75
Mike Askew, Hamsa Venkat	
MOTIVATION AND CANING IN GHANAIIAN SECONDARY SCHOOL: EVIDENCE FROM A SURVEY AND INTERVIEWS	83
Florence Christianah Awoniyi	

LEARNING TO ASSESS: EXPLORING CHANGES TO PRE-SERVICE TEACHERS' CRITERIA FOR A QUADRATICS TASK	91
Michal Ayalon, Karina Wilkie	
ELEMENTARY SCHOOL TEACHERS' IMPLEMENTATION OF DYNAMIC GEOMETRY USING MODEL LESSON VIDEOS	99
Anna E. Baccaglini-Frank, Pietro Di Martino, Nathalie Sinclair	
AFFORDANCES AND TENSIONS IN TEACHING BOTH COMPUTATIONAL THINKING AND MATHEMATICS.....	107
Sandy Bakos, Mathieu Thibault	
THE TRANSITION FROM HIGH SCHOOL TO UNIVERSITY MATHEMATICS: ENTERING A NEW COMMUNITY OF PRACTICE.....	115
Amalia-Christina Bampili, Charalampos Sakonidis, Theodossios Zachariades	
SITUATIONAL AND DISTAL SOURCES OF MEANING IN A MULTILINGUAL MATHEMATICS CLASSROOM.....	123
Richard Barwell	
KNOWLEDGE OF STATISTICAL TESTS BY PROSPECTIVE HIGH SCHOOL TEACHERS	131
Carmen Batanero, María del Mar López-Martín, María Magdalena Gea, Pedro Arteaga	
THE EFFECTIVENESS OF INTEGRATING GEOGEBRA HOTS ACTIVITIES ON THE DEVELOPMENT OF CREATIVE MATHEMATICAL THINKING	139
Nimer Fayez Baya'a, Wajeeh Mahmood Daher, Dania Jamal Abo-Mokh, Ahlam Adnan Anabousy	
INDIVIDUAL DIFFERENCES IN FRACTIONS' CONCEPTUAL AND PROCEDURAL KNOWLEDGE: WHAT ABOUT OLDER STUDENTS?	147
Maria Bempeni, Stavroula Pouloupoulou, Ioanna Tsiplaki, Xenia Vamvakoussi	
PRIMARY SCHOOL CHILDREN'S (9 YEAR OLDS') UNDERSTANDING OF QUADRILATERALS	155
Melania Bernabeu, Mar Moreno, Salvador Llinares	
VALUE OF PICTURES IN MODELLING PROBLEMS FROM THE STUDENTS' PERSPECTIVE	163
Matthias Böckmann, Stanislaw Schukajlow	
CALCULUS STUDENTS' USE OF VISUALIZATIONS WHEN SOLVING VOLUME PROBLEMS	171
Krista Kay Bresock, Vicki Lynn Sealey	
A COMPARISON OF APPROACHES TO STIMULATED RECALL INTERVIEWS WITH MATHEMATICS TEACHERS IN ORDER TO IDENTIFY SHIFTS IN ATTENTION	179
Julian Tom Brown	

DECISION-MAKING IN NOTICING STUDENTS' PROPORTIONAL REASONING	187
Àngela Buform, Ceneida Fernández, Salvador Llinares	
CLUMPS OR CHUNKS? - CONTEXTUAL RELEVANCE OF STUDENTS' FEATURES OF THE DATA	195
Christian Büscher	
A LARGE SCALE CASCADE MODEL IN THE CONTEXT OF MATHEMATICS CURRICULUM REFORM: INTERACTIVE FACTORS OF INFLUENCE ON MULTIPLIERS' WORK	203
Ana Paula Canavarro, Leonor Santos	
RESPONSE PROCESS VALIDITY EVIDENCE: A PROPORTIONAL REASONING EXAMPLE	211
Michele Carney, Katie Paulding	
MATHEMATICAL INDUCTION AT THE TERTIARY LEVEL: LOOKING BEHIND APPEARANCES	219
Gemma Carotenuto, Cristina Coppola, Pietro Di Martino	
AN INNOVATIVE QUALITATIVE VIDEO ANALYSIS INSTRUMENT TO ASSESS THE QUALITY OF POST-SECONDARY ALGEBRA INSTRUCTION	227
Anne Cawley, Irene Duranczyk, Angeliki Mali, Vilma Mesa, April Strom, Laura Watkins, Patrick Kimai, Dexter Lim	
THE PREDICTIVE ROLE OF DIFFERENT REASONING FORMS TO STUDENTS' EARLY ALGEBRAIC THINKING ABILITIES	235
Maria Chimoni, Demetra Pitta-Pantazi	
PEER OBSERVATION AS A TOOL TO FACILITATE MATHEMATICS TEACHERS' SELF-REFLECTION IN A PROFESSIONAL DEVELOPMENT INTERVENTION	243
Brantina Chirinda, Patrick Barmby	
A SOCIAL EXPRESSION OF NUMBER USING TOUCHCOUNTS	251
Sean Chorney	
THE DESIRED TEACHER REFLECTED IN RESEARCH ARTICLES ON PRACTICUM	259
Iben Maj Christiansen, Lisa Österling	
THE NATURAL NUMBER BIAS IN ARITHMETIC OPERATIONS: THE CASE OF THE REPRESENTATIONAL FORM OF THE NUMBERS	267
Konstantinos Christou	
RE-THINKING 'CONCRETE TO ABSTRACT': TOWARDS THE USE OF SYMBOLICALLY STRUCTURED ENVIRONMENTS	275
Alf Coles, Nathalie Sinclair	

TEACHER BELIEFS AND SUPPORT FOR ARGUMENTATION	283
AnnaMarie Conner, Laura Marie Singletary	
FINANCIAL LITERACY: PRACTICING MONEY MANAGEMENT AS YOUR FUTURE-SELF	291
Mary Connolly, Cynthia Nicol	
BOUNDARY CROSSING IN DESIGN BASED RESEARCH - LESSONS LEARNED FROM TAGGING DIDACTIC METADATA	299
Jason Cooper, Shai Olsher	
JOURDAIN AND DIENES EFFECTS REVISITED - PLAYING TIC TAC TOE OR LEARNING NON-EUCLIDEAN GEOMETRY?.....	307
Jason Cooper, Alon Pinto	
FROM ‘FROWNS AND GROANS’ TO ‘ASTONISHMENT AND DELIGHT’: SEEKING INDICATORS OF A MATHEMATICS TEACHERS IDENTITY	315
Cosette Crisan, Melissa Rodd	
LINKING THEORY AND PRACTICE: PROSPECTIVE TEACHERS CREATING FICTIONAL CLASSROOM DISCUSSIONS	323
Annalisa Cusi, Francesca Morselli	
ELEMENTARY STUDENTS' CONDITIONAL REASONING SKILLS IN MATHEMATICAL AND EVERYDAY CONTEXTS	331
Anastasia Datsogianni, Stefan Ufer, Beate Sodian	
THE FIRST-TIME PHENOMENON: SUCCESSFUL STUDENTS' MATHEMATICAL CRISIS IN SECONDARY-TERTIARY TRANSITION	339
Pietro Di Martino, Francesca Gregorio	
ROLE OF USING AN ALTERNATIVE CONCEPT DEFINITION IN CONDUCTING MATHEMATICAL TASKS OF TEACHING: THE CASE OF EXPLAINING WHY AN ALGORITHM WORKS	347
Merve Dilberoglu, Cigdem Haser	
CONCEPTUALIZING AN EXPERT TEACHER’S EXPERTISE IN A LESSON DESIGN STUDY IN SHANGHAI	355
Liping Ding, Keith Jones	
MATHEMATICIANS' CRITERIA FOR ACCEPTING THEOREMS AND PROOFS – AN INTERNATIONAL STUDY	363
Anika Dreher, Aiso Heinze	
FROM EVERYDAY PROBLEM TO A MATHEMATICAL SOLUTION — UNDERSTANDING STUDENT REASONING BY IDENTIFYING THEIR CHAIN OF REFERENCE	371
Jonas Dreyøe, Dorte Moeskær Larsen, Morten Misfeldt	

CONCEPTUALISING TRANSLATIONS BETWEEN REPRESENTATIONS	379
Anneli Dyrvold	
STUDENTS' CONCEPTIONS FOR CURVE LENGTH: A HYPOTHETICAL LEARNING TRAJECTORY APPROACH	387
Cheryl Lynn Eames	
PREDICTORS OF DEMAND FOR MATHEMATICS SUPPORT	395
Antony Edwards, David Carroll	
ONE STUDENT'S DISCURSIVE DEVELOPMENT ON ROTATION IN RELATION TO INSTRUCTION FROM A COMMUNICATIVE PERSPECTIVE	403
Elcin Emre-Akdogan, Beste Gucler, Ziya Argun	
FROM PRINCIPLES OF VISION AND DIVISION TO A SYSTEM PREMISED ON AND SUBJECT TO INTERANIMATED DIMENSIONS: SOME REFLECTIONS ON IDENTITY IN MATHEMATICS EDUCATION	411
Osnat Fellus	
THINKING IN MOVEMENT AND MATHEMATICS: A CASE STUDY	419
Francesca Ferrara, Giulia Ferrari	
USE OF STUDENT-PRODUCED VIDEOS IN THE TEACHING OF COMBINATORICS	427
Anders Støle Fidje	
WHAT VOCABULARY DO TEACHERS USE WHEN ANALYSING THE USE OF REPRESENTATIONS IN CLASSROOM SITUATIONS?	435
Marita Friesen, Carmel Mesiti, Sebastian Kuntze	
SOLVING ARITHMETIC-ALGEBRAIC WORD PROBLEMS BY 10- TO 12-YEAR-OLD STUDENTS	443
Torsten Fritzlar, Nadja Karpinski-Siebold	
ACADEMIC PROCRASTINATING IN THE TRANSITION FROM SCHOOL TO UNIVERSITY MATHEMATICS	451
Sebastian Geisler, Katrin Rolka	
GAP AND CONGRUENCY EFFECT IN FRACTION COMPARISON	459
Juan Manuel González-Forte, Ceneida Fernández, Wim Van Dooren	
GENDER SPECIFICITIES IN A SUPPORT PROJECT FOR ENGINEERING STUDENTS	467
Birgit Griesse	

Table of Contents – Volume 2

TEACHER’S GAZE BEHAVIOR WHEN SCAFFOLDING PEER INTERACTION AND MATHEMATICAL THINKING DURING COLLABORATIVE PROBLEM-SOLVING ACTIVITY	475
---	-----

Eeva Haataja, Enrique Garcia Moreno-Esteva,
Miika Toivanen, Markku S. Hannula

INDEX

INDEX OF AUTHORS AND CO-AUTHORS (VOLUME 2)	485
--	-----



RESEARCH REPORTS A – HAA

WHY IS CALCULATING THE AVERAGE SPEED DIFFICULT?

Linda Marie Ahl and Ola Helenius

Kriminalvården and Stockholm University

National Center for Mathematics Education, University of Gothenburg

Speed may seem like a simple concept, but similar to other rate-based concepts, tasks involving speed can be significantly challenging for students. We examine adult and 16-year-old students' work with a task involving the average speed over two equal distances. Solutions are analyzed by applying scheme theory and the distinction between the predicative and operative forms of knowledge. We show that a major obstacle in achieving successful solutions of the problem is related to the linguistic properties in the problem formulation, inducing students to calculate the arithmetic average instead of the average speed. Students who solve the task successfully use predicative forms of knowledge related to speed, or they reason directly with the rate concept.

INTRODUCTION

In a mathematics pretest for my adult students, one particular task caused more confusion than any other. It was a multiple-choice question in which students were asked to decide on whether a statement, in this case, the right answer to the task, was true or false, and to choose one out of four claims in supporting their choice. The task concerned the average speed when walking one way at a speed of 3 km/h and back at 6 km/h. The students just had to confirm that the average speed for the entire journey was 4 km/h. However, the majority of the students, 63 of 70, claimed that it was not true. The test was followed up with clinical interviews revealing that the difficulty with the task was on the relations inherent in the concept of rate. Instead of considering the average speed as the relation between two quantities, the students treated the rate as a one quantity, a scalar, and calculated the average speed for the entire journey as an arithmetic average. The pattern repeated itself when we gave the task, without multiple choices, to 74 16-year-old upper secondary school students, in which 50 of them calculated the arithmetic average of the two speeds and wrongly answered 4.5 km/h.

Speed is a function of distance divided by time. Therefore, distance is a bilinear function of time and speed, $d(v,t) = vt$, where distance is proportional to both time and speed; this is why in situations in which either time or speed is kept constant, students can rely on linearity. However, speed is inversely proportional to time when the distance is kept constant. When a situation involves the distance being constant, linearity no longer applies. This idea is complicated, and, therefore, dealing with average speeds unsurprisingly constitutes an epistemological obstacle for students (Bachelard, 1933).

The epistemology of the rate concept has been extensively discussed in research. In a classification of word problems in algebra textbooks, around one third of the problems involved rate problems of different kinds (Meyer, 1981). In the comparison of students' solution strategies to problems involving speed, the more complex problems were solved more successfully with algebraically based strategies than with either arithmetic or heuristic strategies (Jiang, Hwang & Cai, 2014). Studies that classify problem types or compare solution strategies typically do not discuss the epistemological difficulties inherent in the speed concept. A series of articles by Thompson (Thompson, 1994; Thompson & Thompson, 1994) reporting teaching experiments makes one of the major contributions to the understanding of the epistemological problems inherent in the speed concept and in rates, in general. Particularly enlightening is the teaching experiment with JJ, a fifth grader, conducted by Thompson himself, in the form of a prolonged clinical interview and lasting for eight sessions distributed over two months. An interesting point is that JJ did not previously receive instruction on the speed concept, which gave Thompson some leeway in helping JJ develop a speed concept (Thompson, 1994). He concluded that her obstacles with the speed concept could be overcome with teaching that aims to create a powerful scheme for speed so that she recognizes more general rate situations as being largely the same as situations involving speed.

Students' difficulties with rate concepts are well documented by research, so conducting another empirical study to support this claim is not needed. Instead, in this theoretical study, we will use empirical examples to discuss the possible roots of older students' epistemological obstacles, with theoretical constructs from Vergnaud's work. Students' conceptualization of the speed concept is well known to be crucial to their actions dealing with speed situations. In contrast to earlier research, however, we emphasize that the semiotic representation in a given task plays a greater role in students' behavior than has been recognized before. We conclude the paper with comments on the practical implications of teaching the speed concept.

THEORETICAL FOUNDATIONS

It is more or less accepted today that Vergnaud provides a superb contribution to didactics with his theories. His theorization largely builds on Piagetian ideas. However, Piaget's theory lacked many central components for application to educational research because Piaget was never interested in didactics (Vergnaud, 1996). According to Vergnaud, Piaget did not focus closely on the importance of language, in contrast to Vygotsky. Among all of Vygotsky's works, the systematic treatment of how the use of signs, language, and other artefacts organizes the mind is his most seminal contribution to our understanding of knowing and learning. Vergnaud combined Vygotskian insights with semiotics and developed a theory on the role of representation in scheme theory (Vergnaud, 1998a). Many researchers maintain that Piagetian and Vygotskian ideas are incommensurable, but for relating the internal and external parts of an activity, both Piaget and Vygotsky use the idea of interiorization (or internalization) in

their respective works *La formation du symbole chez l'enfant* (Piaget, 1970) and *Thought and language* (Vygotsky, 1962).

Schemes are a fundamental idea in Vergnaud's work. The concept of scheme was first introduced by Kant and later embraced and revised by Piaget who said, "Whatever is repeatable and generalizable in an action is what I have called a scheme, and I maintain that there is a logic of schemes" (Piaget, 1970, p. 42). For both Kant and Piaget, the role of the scheme concept is to account for the relationship between individuals' mental activity and the empirical world. Vergnaud (1998) defines a scheme as *the invariant organisation of behaviour for a certain class of situations*. He characterizes these invariants in terms of (1) *theorems-in-action*, which are propositions explicitly or implicitly held to be true, and (2) *concepts-in-action*, which are predicatives or categories held to be relevant in the class of situations associated with a particular scheme. A theorem-in-action can be true or false. A concept-in-action can be relevant or not relevant for the situation. By identifying concepts-in-action and theorems-in-action as the content in schemes, Vergnaud creates a system for analyzing students' mental actions in a way that structurally resembles how mathematics is organized. Because as Vergnaud writes, "there is no way to reduce mathematical knowledge to any other conceptual framework" (Vergnaud, 1998a, p. 167).

In educational settings, most situations involve tasks for students to solve. A scheme associated with a situation concerns an organization of behavior but not necessarily a full solution process. Some schemes may be effective in that they involve a process that leads to a solution, but not all schemes are like this. In general, schemes only involve an action deemed appropriate in relation to the interpretation of the situation. A scheme may even be wrong, or it may not help an individual deal appropriately with the situation (Vergnaud, 1998b). A particular form of erroneous scheme that has not been extensively dealt with in mathematics education research is when a student applies an appropriate scheme for one situation to another situation for which it is inappropriate. This phenomenon is typical for the situation we describe in this article. We claim that semiotic representations have a distinct role as a catalyst for how students' schemes are chosen in relation to a given task (Ahl & Helenius, 2018).

The semiotic representation of knowledge is connected to a student's schemes, which are his or her mental representations. It is a concept-in-action that is the signified referenced by the signifier, for example in the form of a word or a symbol (Vergnaud, 1998). As signs only exist in signifier–signified pairs, at the individual level, a word or symbol only makes sense when it is connected to some mental idea. However, when the mind is viewed in society, an individual will obviously encounter words and symbols that are not yet associated with any idea in his or her mind or are associated with ideas that are not in line with the usual meaning of the symbol agreed upon. In the work of Ahl and Helenius (2018), we exemplified how the wording *average speed* triggers dual schemes in a student—one that is correctly associated with speed (as a quotient of distance by time) and another that is associated with the arithmetic average. We use this example to extend Vergnaud's theory of representations (1998) by claiming

that semiotic systems also connect directly to situations and not only to schemes (for more information, see Ahl & Helenius, 2018).

According to Vergnaud, acquired knowledge has two forms, operative and predicative. The operative form of knowledge consists of action in the physical and social world, or the use of schemes. The predicative form of knowledge consists of the linguistic and symbolic expressions of the operational form of knowledge. These two forms of knowledge are always intertwined. Schemes, the operative form of knowledge, are adaptable resources when students assimilate new situations and accommodate to them. When predicative forms of knowledge are acquired, the possibilities of adapting schemes to new situations through accommodation increase because enunciation plays an essential part in the conceptualization process (Vergnaud, 2009).

DIFFERENT SCHEMES APPLIED TO THE TASK

The situation depicted in this task involves two average speeds over the same distance, which is why a simplistic manipulation of the $d = vt$ formula fails as a scheme for calculating the answer:

There is a path up a quite steep hill in Athens. Richard, who is in good shape, is going up the hill in an average speed of 3 km per hour. He goes down in double speed. What is Richard's average speed for the whole walk? (adapted from Niss & Jankvist, 2013)

In the work of Ahl and Helenius (2018), a theoretical model suggesting that the semiotic representations in task formulation trigger students' schemes is proposed. In our case, the word "average" induces students to deal with the task by applying the inappropriate arithmetic average scheme for the situation. Students made the calculation $(3 \text{ km/h} + 6 \text{ km/h})/2 = 4.5 \text{ km/h}$ without considering the properties of the inherent relation in a rate. In our sample of upper secondary school students, about two thirds of the students presented this answer. Interestingly, though, many students started out writing the formula $v=d/t$, assuming some distance, and perhaps calculating some time–distance relations. Such operations shows that they apply predicative elements from a scheme associated with speed before giving up; they then apply the arithmetic average scheme. Why do they not persist? We make a comparison with students who correctly solved the task.

The most common appropriate scheme for the situation includes presenting and using the $v=d/t$ formula, making a simple mathematization, and assuming that the hill is 3 km. It will take Rickard one hour to go up the hill and half an hour to go down. Therefore, Rickard has walked 6 km in 1.5 hours, which makes the calculation of his average speed as $6 \text{ km}/1.5 \text{ h} = 4 \text{ km/h}$. The students who end up following through their speed scheme write down the appropriate formula and then work on it until they fill the formula with the necessary content, which is a total distance and a total time for the entire journey. The students who do some or all of these things, except following through with the final calculation (and then often going for the arithmetic average instead), seem to have similar knowledge of the speed concept. A possible explanation

may be that the students who follow through have greater confidence that the speed formula $v=d/t$ will lead them to the right solution.

A thin line seems to exist between success and failure in solving the task. What if a student's failure is not due to his or her lack of knowledge of the speed concept? Students' behavior may stem from the didactic contract, regulating the division of labor in the classroom (Brousseau, 1997). Students are well aware of their role to produce an answer to the given task, preferably with low effort and within the shortest possible time. What if we close the opportunity to use the fast and easy arithmetic average scheme? To test this hypothesis, we gave the same task to another class of first-year upper secondary school students. Half of the class was given the task without any additional information, whereas the other half was given the additional information that 4.5 km/h is **not** the correct answer. The result was really interesting, although the sample was far too small to make any generalizations. Only 2 students of 14 provided the correct answer among those students who did not receive any additional information. In the group of students who received the additional information, 6 of 11 students gave the correct answer of 4 km/h, indicating a notable increase in the ratio of correct to incorrect answers.

An alternative solution also exists. Students conclude that the slower 3 km/h uphill part of the journey will consume two equally large units of time, and the twice-as-fast 6 km/h downhill part of the journey will consume one of such a time unit. Instead of decomposing the total journey into two parts of equal distance, they divide the journey into three parts of equal time. Then, they calculate the average speed as $(3+3+6)/3 = 4$, the average of speeds associated with the same time unit. This solution is a completely different one that circumvents the need for the mathematization of any particular distance or time for the journey.

DISCUSSION

In contrast to earlier research dealing with young students' conceptualization of rate, we have stressed that the power of language representation in tasks has not been given enough attention (Ahl & Helenius, 2018). Both the clinical interviews and the students' written solutions prove that many students act on the word *average* and use an arithmetic average scheme that appears to be strongly consolidated by these students' experience from prior successful actions in situations involving the word *average*. This semiotic issue makes the assimilation of the relations in the speed concept more complex than it would have been had the students lacked experience with the arithmetic average (Bachelard, 1939). The attractiveness of the erroneous arithmetic average scheme is particularly evident in students who first both present the $v=d/t$ formula and draw pictures of the hill, as well as indicate distances and times, but then abandon this scheme and use the arithmetic average scheme. Distinguishing what it is that these students do not know and what the students who follow through with their speed schemes and who produce the correct answer know is difficult. Our claim is that students who hold on to their speed scheme do this by placing confidence in the predicative form of knowledge inherent in the formula $v=d/t$. It is, in fact, an important aspect

of mathematical work that relying on formal representations instead of thinking about the conceptual meaning is often both possible and effective.

The identified advantage of placing confidence in the predicative form is in line with the finding of Jiang, Hwang, and Cai (2014), in which students from Singapore showed a greater variation in the methods and representations applied, possibly indicating a more developed understanding of speed situations, whereas Chinese students tended to stick to algebraic models that more often led them to the correct answers. Similar to the case of our successful students, working in an algebraic setting requires operating on the predicative forms of knowledge and processes. In some sense, this contrasts with the work of Thompson (1994), which aims to build strong foundations for the speed concept so that students can generalize it to any rate situation. Our students would probably not have been so easily swayed to using the average scheme had they possessed extensive experiences in operations on various speed situations.

The students who provided a solution based on dividing the journey into three equal time units certainly show an operative understanding of the rate concept. These students seem to have a way of dealing with rates directly, which is what Thompson aims for in his teaching experiment. The scheme that these students use does not involve a mathematization of the uphill–downhill situation, but rather operates directly on the rate as a concept-in-action. None of these students explain their reasoning. However, in their operation on the rate concept, they evidently use a theorem-in-action corresponding to the understanding that a rate is a relation that cannot be treated as a scalar.

The growth of predicative knowledge is driven by putting words onto actions. However, telling students that they cannot successfully calculate the average of two different averages is inadequate. As Brousseau expressed it, “If the teacher says what he wants from the student, he can no longer obtain it!” (Brousseau, 1998, p. 41). Our small-scale test that involves giving students information that 4.5 km/h is not the correct answer indicates that closing the door to the arithmetic average scheme may overshadow the effect of semiotic representation (Ahl & Helenius, 2018). However, we need to keep in mind that this is not yet a generalizable result.

Students’ knowledge grows from formation and testing in action (Bachelard, 1938). According to Vergnaud (1998a), students’ conceptualization process develops through reference to a set of situations giving meaning to the concept; the content or the set of invariants on which the operability of the schemes is based; and the signifier, or the set of linguistic and non-linguistic forms allowing the concept, its properties, the situations, and processing procedures to be represented symbolically. On top of this, we claim that the semiotic representation in the description of the situation is directly connected not only to the operational invariants but also to the interpretation of the situation activated by the signifiers in the representation of the situation (Ahl & Helenius, 2018). This claim is supported by the arithmetic average scheme solution presented above.

However, in practice, how can this theoretical discussion assist teachers in helping students accommodate their scheme appropriately to speed situations? We claim that scheme theory, combined with an analysis of semiotic representation, can provide a better understanding of the conceptualizing process from operative to predicative knowledge and how these knowledge forms interact. A main epistemological obstacle in the situation is the language representation that triggers the arithmetic average mean scheme. Seemingly, it can be overcome by either developing confidence in the predicative way of expressing the situation or by learning to operate on the more abstract rate concept directly, similar to what Thompson aims for with JJ (Thompson, 1994), or by blocking the arithmetic average scheme by giving information that the answer corresponding to that scheme is incorrect. By observing the language representation and identifying the scheme activated in the situation, teachers can organize didactic situations in which students' arithmetic mean scheme makes no sense, and can therefore guide them in assimilating and accommodating to the situation.

By changing the context in the task, but not the operational invariants, to two very different speeds, such as riding a motorbike one way and walking back, we can create a situation in which the answer given by applying the arithmetic average scheme becomes absurd. When students come up with an answer produced by calculating the arithmetic mean, the teacher may ask the following: How can that be when most of the time is spent at walking speed? This prompt may encourage the students to reconsider their solution and recognize the relation between the two quantities in the relation.

For students' assimilation, highlighting the similarities in properties already known is important to enhance the development of new concepts. This task can be done by comparing the addition of relations in the speed situation with the addition of rational numbers with different denominators. Then, teachers can draw on students' prior experience to support the interiorization of the relation in the speed concept.

Acknowledgement

We thank Olof Dahl, Torbjörn Jansson, and Gunnar Rex for letting their student work with the average speed task and for providing us with the students' solutions.

References

- Ahl, L. M., & Helenius O. (2018). *The role of language representation for triggering student's schemes*. Retrieved from <http://formular.ncm.gu.se/madif-11/>
- Bachelard, G. (1938). *La formation de l'esprit scientifique*. Paris: Vrin.
- Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht: Kluwer.
- Jiang, C., Hwang, S., & Cai, J. (2014). Chinese and Singaporean sixth-grade students' strategies for solving problems about speed. *Educational Studies in Mathematics*, 87(1), 27-50.
- Mayer, R. E. (1981). Frequency norms and structural analysis of algebra story problems into families, categories, and templates. *Instructional Science*, 10(2), 135-175.

- Niss, M., & Jankvist, U.T. (2013). *13 Spørgsmål fra Professorene* (detektionstest 3). Materiale udleveret til matematikvejlederuddannelsen [13 Questions from the Professor (detection test 3). Material handed out at the maths counsellor programme]
- Piaget, J. (1970). [Translated by Eleanor Duckworth] *Genetic epistemology*. New York: Columbia University Press.
- Thompson, P. W. (1994). The development of the concept of speed and its relationship to concepts of rate. In G. Harel & J. Confrey (Eds.), *The development of multiplicative reasoning in the learning of mathematics* (pp. 181-234). Albany, NY: SUNY Press.
- Thompson, P. W., & Thompson, A. G. (1994). Talking about rates conceptually, Part I: A teacher's struggle. *Journal for Research in Mathematics Education*, 25(3), 279-303.
- Vergnaud, G. (1996). Some of Piaget's fundamental ideas concerning didactics. *Prospects*, 26(1), 183-194.
- Vergnaud, G. (1998a). A comprehensive theory of representation for mathematics education. *The Journal of Mathematical Behavior*, 17(2), 167-181.
- Vergnaud, G. (1998b). Toward a cognitive theory of practice. In A. Sierpiska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity* (pp. 227-241). Dordrecht: Kluwer.
- Vergnaud, G. (2009). The theory of conceptual fields. *Human Development*, 52(2), 83.
- Vygotsky, L. S. (1962). *Thought and language*. Cambridge: MIT Press.

ACCOUNTABILITY AND ASSESSMENT: GAPS AND GRIDS

Julie Alderton¹ and Nick Pratt²

¹University of Cambridge (UK), ²Plymouth University (UK)

A recent major policy change in England dismantled the use of National Curriculum levels for assessing pupil outcomes and progress. Our study analyses how primary teachers responded to these reforms, reorganising their practices in a manner constrained by a well-established accountability culture. We draw on Foucault's conceptualisations of power, truth, discourse and governmentality to understand our research focus: how technologies create particular language forms which teachers then used in reconstructing mathematics teaching and assessment. Through interview analysis we focus especially on the frequent use of the term 'gap'. Analysis shows how pedagogical language can originate in the use of technology, affect the way teachers reconstruct practice/policy and potentially alter pupils' mathematical experiences.

INTRODUCTION

Our starting point for this paper is that assessment is not, as policy tends to suggest, a neutral uncovering of pupils' ability for the purposes of teaching and accountability. We work with the premise that assessment, instead, takes place in social practices as a technology of government and the 'truth' about children's mathematical development is produced within power/knowledge relations (Foucault, 1977, 1997). This study sprang from an opportunity provided by the recent reform of assessment practices in England which presented us with the possibility of making mechanisms of power visible and thereby offering an analysis of their functioning.

In September 2015 a revised National Curriculum (NC) (DfE, 2013) was implemented across all year groups in English primary (5-11 years) schools. Curriculum reform was accompanied by a new assessment system. This involved removing the use of curricular objectives scaled into ten levels of performance to assess pupils' attainment in external tests and teacher assessment. Despite numerous curriculum and testing reforms, assessment with levels had remained unaltered since 1994. The removal of levels was thus a significant upheaval for schools and teachers.

We are interested in how policy unfolds on the ground. Morgan (2009, p. 50) contends that in order to understand the practices of individuals it is necessary to understand how they "relate to the social structures within which they are situated". We take a sociological perspective, building on previous work (Pratt, 2016a, 2016b) which explored the effects of market-orientated education policy on teachers' assessment practices. We draw on a Foucauldian conceptual framework to examine the ways in which teachers talk about their work at the classroom level. In analysing policy texts and teachers' accounts of how they use assessment tracking software, we identified a

discourse of *gaps* in mathematical knowledge. The effect on mathematics teaching of the use of the term gaps forms the focus of our analysis in this particular paper.

The English policy context

English education has become increasingly marketized and the production and use of data is claimed to be the most advanced in Europe. A mantra of driving up standards is presented by policy makers as justification for an emphasis on competition between schools through the mechanism of performance in national standardised tests. Tests in mathematics and English at the end of primary school mean that these two subjects become used as tools to measure school effectiveness. Progress against centrally defined levels of performance has become the key measure by which schools are judged and held to account to parents and the wider society (Pratt, 2016a). We understand these audit practices within a well-established accountability culture as a mode of regulation that transforms what it means to be a teacher of mathematics. These social practices also act to shape mathematics in the school context.

In brief, the key change to assessment practices is a shift from *connotative* to *denotative* measurement, that is, away from a best fit model of groups of levelled objectives to performance on individual objectives (Ruthven, 1995). In national tests at ages 7 and 11, pupils are now awarded a scaled score between 80 and 120, with 100 being the expected standard. As previously, schools are judged and compared in two ways: with publication of an absolute measure of achievement in test results; and a value added measure comparing pupils' test scores between ages 7 and 11. Associated changes to the primary NC (DfE, 2013) included specified year-by-year objectives with an increased expectation of attainment and the direction that the majority of pupils should move through the yearly programmes of study at broadly the same pace, replacing previous guidance to accelerate high achieving pupils. Progress *through* the curriculum, in a stepwise acquisition of levels, has thereby been replaced by progress *within* it in terms of depth.

THEORETICAL FRAMEWORK

To understand the effect of changes to the 'rules of the game' of assessment, we draw on Foucault's notions of governmentality (Foucault, 1977). Governmentality comprises two kinds of technologies: techniques of domination and power; and techniques of the self (Foucault, 1997). Technologies of domination act from the outside via classification and objectification, for example through testing regimes, inspections, performance related pay and league tables. (Note, Foucault uses the word 'technologies' in a general sense but our focus, below, is on information technology as one example of these more general technologies.) Walls (2006) demonstrates how, through standardized testing, the 'numerate child' and therefore the 'innumerate child' are established and become visible for teachers, parents and policy makers. "The power of normalization imposes homogeneity, but it individualizes by making it possible to measure gaps, to determine levels, to fix specialities, and to render the differences useful by fitting them one to another" (Foucault, 1977, p. 184). Foucault did not con-

ceptualise power as solely coercive. He reformulated power as both productive and repressive, as creative and constraining. Power is exercised rather than possessed, circulating within interactions and relationships. It operates at both the macro and micro level and is present, for example, in routine pedagogical practices of everyday classroom life. Technologies of the self are “an exercise of the self on the self by which one attempts to develop and transform oneself, and to attain to a certain mode of being” (Foucault, 1997, p. 282). Such technologies are exemplified by the neoliberal focus on continuous development and performance management, in which even the most experienced professionals must embrace ongoing action to ‘improve’ practice. Power operates through discourses in which knowledge, meaning and truth are produced. Foucault’s meaning of discourse goes beyond language. Discourses are structures of language and practice through which objects, such as ‘effective teaching’ or ‘mathematical progress’, come into being. Foucault’s concern was not *what* is true but how some things *come to count* as true (Foucault, 1980). It is through this theoretical lens that we examine our research question, namely: how do teachers make use of particular language forms in constructing mathematics teaching and assessment?

GAPS IN MATHEMATICS EDUCATION

From a Foucauldian perspective, language does not reflect reality but constructs it and shapes practice. It is, therefore, important to examine how language makes things look natural and unquestionable. Historically, the influential Cockcroft report (1982) made two mentions of the word gap. The most well-known referred to “a wide gap in understanding and skill which can exist between children of the same age” (p. 101) and the second related deepening students’ knowledge of mathematics to “a process which may involve filling some gaps” (p. 207). These reflect the two ways in which the word gap is predominantly used in policy and research. The first refers, like a race, to the gap between competitors, the space between leaders and those following on a linear scale of progress. Gutiérrez and Dixon-Román (2011) highlighted increasing preoccupation with this discourse, illustrated by the 137,000 hits that a search in Google Scholar with the words “achievement gap” + “mathematics” produced. They argued that neoliberal discourse constructs equity in mathematics education as the need to reduce the achievement gap, which requires systematic measurement and monitoring. The second sense in which gap is used refers, metaphorically, to missing pieces in some kind of stack/developmental tower which will collapse if the pieces are not (re)placed properly. It is this second usage that the teachers in our study mostly employed and which we focus on in the remainder of this paper.

In 2005 the UK government produced intervention materials (DfES, 2005) entitled ‘Wave 3 materials: supporting children with gaps in their mathematical understanding’. These were designed to be used with small groups of children who were achieving significantly below expectations. A correlation is constructed between low-achieving children and gaps in understanding. This discourse positions them as deficient in mathematical knowledge with gaps between what they know and what

they *ought* to know, according to some norm. In an initial analysis of documents and speeches over the last three years relating to the removal of levels for assessment, we have identified an increasing focus on the notion of gaps in pupils' knowledge and learning, citing that these gaps were exacerbated by the practice of levelling. For example, the final report of the government's Commission on Assessment without Levels (McIntosh, 2015, p.5) argued that levels had "had a profoundly negative impact on teaching" as they "encouraged undue pace and progression onto more difficult work while pupils still had gaps in their knowledge or understanding" (p. 17). Nick Gibb (2015), the Minister of State for Schools, argued that levels achieved by primary pupils were not trusted by secondary schools, "since pupils described as level 4 were supposed to meet a common descriptor, but frequently had serious gaps in their reading, or in their capability in maths". Tim Oates (no date), the Chair of the Expert Panel who advised the Secretary of State for Education on the review of the NC, criticised the best fit nature of assigning levels to pupils, contending that "teachers choose the level which looks most appropriate to what a pupil can do – they are 'level 3' even if they have significant gaps in their learning". We argue that this discourse of gaps in mathematical knowledge has become increasingly dominant with the change in policy for assessment, coupled with the intensification of the accountability agenda.

RESEARCH DESIGN

The project involved extended, semi-structured interviews with primary teachers in eight different schools (11 teachers in total) during the summer term the first year after the removal of levels. Teachers and schools were chosen purposively to reflect a range of ages, experience, school types and locations. For this paper we predominantly draw on interview data with four of the participating teachers – Jill, Mike, Ann and Becky, (pseudonyms used). Jill and Mike had been teaching for eight and seven years respectively, Ann had been teaching for 19 years and Becky for 29. All were generalist class teachers and at the time of the interviews taught years 2, 3, 5 and 6 respectively (ages 6 to 11). Interviews started by asking teachers to describe in detail the assessment tools, materials and routines used over the first year of the new system. They were then asked to reflect on their practices, the reform to the system and their school's response. Data were analysed thematically in relation to the substantive and theoretical framework – teachers' assessment practices, as we understood them in relation to power/knowledge and truths. Whilst we can only present a small amount of data we have selected this carefully, ensuring that teachers' views, though sometimes individual, are never contradictory of the data set as a whole. Our aim is not to claim that the specifics are generalizable to every teacher beyond, or even within, the data set. Rather, the analysis is of the system of governmentality and the dominant discourses that constitute it. We think it offers a trustworthy and useful analysis to understand a discourse of gaps in constructing mathematics teaching and assessment. All our work conformed to the ethical procedures of the British Educational Research Association and were approved by our employing institutions.

FINDINGS AND ANALYSIS

Measuring gaps

All schools in our sample used some form of technology to track pupil progress across the year. The majority had bought in commercial packages, modified in response to the recent changes. The software allowed teachers to make judgements about individual pupils in their class in response to the NC objectives indicating, in different ways, whether the pupils had begun to, partially or fully met them. Teachers described using a range of evidence from day-to-day assessment and testing to input their judgements which produced grids, with which to manage and display data. Nine of the 11 teachers used the word gap in the interview, many frequently. Mike, for example, used it 26 times. Both meanings, as a race and a tower were used but the latter predominated, for example Becky described how she worked with the software that her school utilised.

Becky: Pupil Tracker is really useful to assess how the children are working so you can print all the objectives off and it creates tick boxes and ... you know, things that are a useful tool while you are assessing children really. They've added all of the objectives so as I say it's helpful to help us to learn them because we don't know them very well, to help us to see the gap.

For Becky the grid becomes a tool to familiarise herself with the objectives in the new curriculum. It also allows her to literally see gaps, i.e. the boxes not yet ticked. It seems this visual organisation helps her to manage the substantial number of objectives. For example, the year 5 NC programme of study (DfE, 2013) comprises 49 statutory objectives for mathematics. (For a class of 30 this is 1470 boxes). The large numbers were commented on by many, including Ann.

Ann: In maths, the number of objectives is massive. It just seems to be a long, long, endless [list] of objectives to have to click on and you ... I thought that in September that having done some things that I would come back to them, and I didn't.

Ann attempted to use the electronic grid as a record to inform future teaching with the intention of revisiting the objectives but found the scale unmanageable. Despite this, she felt the objectives were not specific enough and suggested, contrarily, that they be split further so she could tick yet more off.

Gaps as a measurement

Within the accountability agenda, progress in mathematics is the way schools and teachers (via performance-related pay) are measured and normalised. The notions that progress in mathematics is predictable and controllable across time and that teaching directly correlates with learning, are 'truths' that position teachers as responsible for pupil progress (Pratt, 2016b). The software packages allow senior staff constant access to class teachers' ongoing tracking grids, a panopticon gaze (Foucault, 1977). In turn, teachers monitor children, and are caught up in an accountability regime that requires them then to monitor, label and assign within-child deficits (Alderton & Gifford, in press). Jill divides children into those who are *on track* and those who are not.

Jill: These children are where they should be and these children aren't so that then continuing on ... so that the gaps that they have got in place value are filled rather than, 'oh they didn't get place value. Never mind let's try and do division. Oops they didn't get that either'. So that you've got a lot of 'not quite theres' throughout the year but [one is] trying to fill the gaps.

Thus, the solution for the children who are the 'not quite theres' is to identify and then fill their gaps. Despite the large number of objectives already, for Jill this data needs to be more specific to enable her to teach more expertly.

Jill: What I would like but I don't know if it is possible. I would like that you can have your class and say, look here is the data that is actually quite specific about where their specific gaps are, where they are and what their next steps are.

Identifying gaps becomes a technology to hunt down the truth (Foucault et al., 2008) of children's progress and also a technology of the self (Foucault, 1997); a means to account for and manage achievement of both pupils and of oneself. This draws on an understanding of mathematics knowledge as objective and as *acquired* by the learner where "credit for its acquisition can be attributed to particular teachers through acts of teaching" (Pratt 2016a, p. 9). Mike was very positive about the changes to the assessment system.

Mike: I think we maybe got a little bit lost in that over the years of 'I think they are now 3a. I can move them up, thank goodness' or beating ourselves up about the fact that we can't move them yet. When actually all we are trying to do is fill their gaps and help them learn.

Teaching mathematics becomes defined purely as filling gaps. Crucially, achieving this requires the atomisation of mathematics into small, highly hierarchical, detailed chunks of knowledge. The consequences of this can be seen in Ann's description of teaching decimals to her year 5 class.

Ann: My class didn't have a very good understanding of decimals, so rather than teaching thousandths and all of what was in the year 5 curriculum, I've had to go right back to the start and doing tenths, and quite a lot of work on tenths to get that understanding and then linking it into fractions and then going to hundredths. And that is your year 3 and year 4 objectives. It got to that point, which you just know as a teacher, when they've had enough. So I've called it a halt at hundredths and said 'that's what they understand' and next year they need to do thousandths.

Another example of how the discourse of gaps in mathematics constructs teaching and mathematics is visible in a new initiative, described by Mike, that he had introduced into his school in his role as maths lead, "closing the gap week".

Mike: We actually did our half termly assessment test the penultimate week of term and then we marked them and then from there we found children that had particular gaps. We did what we called a closing the gap week where we did mix them and we had one class that did shape for a week, one that

did fractions for a week and one that did time for a week because that's where those children had those gaps.

Significantly, ten children, who were identified as "not having any gaps", spent the week doing rich problem solving tasks instead; an opportunity denied to the others.

DISCUSSION AND CONCLUSION

The publishers of commercial monitoring and tracking software packages, which are in use in the majority of English schools, have amended their programmes in response to changes to the NC and assessment policy. Electronic grids of pupils with yearly objectives produce a pictorial array of gaps that teachers must act on. These actions are more than clicks on boxes; the gaps are visually conspicuous and need to be filled, becoming pedagogical actions in the classroom and performance measures in staff appraisal. Gaps within mathematics slips, in an essentialising shift, to become children (literally) with gaps. Filling these gaps becomes the responsibility of the teacher, who is positioned as lacking their own professional expertise, and/or the child, who is cast as deficient; the responsibility, thereby, not lying with the system. Software packages, sold in a marketized education system and used within a dominant discourse of accountability, act as a technology of governmentality producing truths about teaching and assessment. The grids they produce and their associated technological discourses begin to construct how teachers think about teaching mathematics, as gap filling, meaning that the nature of mathematics itself tends to become an atomised collection of pieces of knowledge. Teachers attempt to master the new system using the tools and strategies that have sprung up around it. Monitoring grids, tracking data and pedagogies to fill gaps and meet targets are refined, improved, manipulated and resisted, as technologies of the self. Our analyses, so far, are at a preliminary stage and we make no claims that what we present here is representative of all primary teachers' responses to assessment policy change. However, we would argue that if the government wants children to experience mathematics as "a creative and highly inter-connected discipline", as suggested in the NC (DfE, 2013, p. 99), then close attention needs to be paid to the effects of correlating mathematics learning with gaps in knowledge, and to the power of professional and technological discourses in this area.

References

- Alderton, J. & Gifford, S. (in press) Teaching mathematics to lower attainers: dilemmas and discourses, *Research in Mathematics Education*
doi.org/10.1080/14794802.2017.1422010
- Cockcroft, W. (1982). *Mathematics Counts*. London: Her Majesty's Stationary Office.
- Department for Education. (2013). *The national curriculum in England Key stages 1 and 2 framework document*, London: Crown Copyright.
- Department for Education. (2016b). *Primary school accountability in 2016 A technical guide for primary maintained schools, academies and free schools*. London: Crown Copyright.

- Department for Education and Skills (2005). *Supporting Children with Gaps in their Mathematical Understanding*. London: DfES.
- Foucault, M. (1977). *Discipline and Punish: The Birth of the Prison*. (Sheridan, Trans.) London: Penguin Books.
- Foucault, M. (1980). *Power/Knowledge: Selected Interviews and Other Writings, 1972-1977* (C. Gordon, L. Marshall, J. Mepham, & K. Soper). New York: Pantheon.
- Foucault, M. (1997) *Ethics: Subjectivity and Truth (Essential Works of Foucault, 1954-1984, Vol. 1)*. Edited by Paul Rainbow. New York: The New Press.
- Foucault, M., Lagrange, J., & Burchell, G. (2008). *Psychiatric power: Lectures at the college de France, 1973–1974* (Vol. 1): Macmillan.
- Gibb, N. (2015). Assessment after levels. Available from: <https://www.gov.uk/government/speeches/assessment-after-levels> [Accessed 2 Jan 2018].
- Gutiérrez, R, & Dixon-Román, E. (2011). Beyond gap gazing: How can thinking about education comprehensively help us (re)envision mathematics education? In B. Atweh, M. Graven, W. Secada, & P. Valero (Eds.), *Mapping equity and quality in mathematics education* (pp. 21-34). New York: Springer.
- McIntosh, J. (2015). *Final report of the Commission on Assessment without Levels*. London: Crown Copyright.
- Morgan, C. (2009). Understanding practices in mathematics education: structure and text. In M. Tzekaki, M. Kaldrimidou, & H. Sakonidis (Eds.), *Proc. 33rd Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 1, pp. 49-64). Thessaloniki, Greece: PME.
- Oates, T. (no date). Opening the door to deeper understanding. Available from: <http://www.cambridgeassessment.org.uk/insights/national-curriculum-tim-oates-on-assessment-insights/> [Accessed 2 Jan 2018].
- Pratt, N. (2016a). Neoliberalism and the (internal) marketisation of primary school assessment in England. *British Educational Research Journal*, 42(5), 890-905.
- Pratt, N. (2016b). Playing the levelling field: teachers' management of assessment in English primary schools. *Assessment in Education: Principles, Policy & Practice*. Online
- Ruthven, K. (1995). Beyond common sense: reconceptualising National Curriculum assessment, *The Curriculum Journal*, 6(1), 5-28.
- Walls, F. (2006). "The big test": A school community experiences standardized mathematics assessment. In J. Novotná, H. Moraová, M. Krátká, & N. Stehlíková (Eds.) *Proc. 30th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 5, pp. 4353-360). Prague, Czech Republic: PME.

PRESERVICE MATHEMATICS TEACHERS' CURRICULUM VISUALIZATION

Julie Amador and Darrell Earnest

University of Idaho, University of Massachusetts

The purpose was to investigate how preservice teachers draw upon curriculum materials to design an elementary mathematics lesson. We engaged preservice elementary licensure students in a four-part process of analyzing mathematics curriculum, planning a lesson based on the materials, demonstrating their visualization of enactment through an animation, and reflecting on the process. We present a case study of one focal pair to analyze their decision making with respect to curricular adaptations, and specifically the introductory launch portion of the lesson. Findings indicate that the pair modified the curricular materials to model mathematical aspects of fractions and introduced materials not mentioned in the curriculum with which they believed children would be familiar.

Translating a mathematics lesson from the printed or digital pages of curriculum into instruction is not straightforward. In fact, this process involves a complex interaction between those materials and the teacher (Remillard, 2012). Although research has identified some teachers at extremes who entirely ignore their curriculum or fully offload decision-making authority to textbook directives, the majority of teachers adapt materials for the purposes of instruction (Brown, 2009; Burkhauser & Lesaux, 2015). Such adaptations are at least in part due to necessity: curriculum typically offers a multitude of information and options in order to support classrooms and students across continents with varied backgrounds and who are members of various communities of practice.

In the present study, we engaged preservice teachers in a process termed Lesson Planimation (Earnest & Amador, 2017) to document their decision making as they translated a curriculum unit into a lesson, with an emphasis on the launch (Jackson et al., 2012). Given their status as novice teachers, preservice teachers are quite reasonably uninitiated into the ways in which curriculum supports practice, including the ways in which the lesson launch in particular sets a tone of collective activity that fosters student engagement in problem solving. In the present study, our purpose was to understand how preservice teachers would modify curriculum materials when planning and enacting the lesson; animations were used as a mechanism for preservice teachers to rehearse and enact complex decisions in a setting with reduced complexity, thereby serving as an approximation of practice (Grossman et al., 2009) to express their suggested modifications when translated into practice. We focused on the following research questions: How does one case study pair of preservice teachers translate curricular materials into a plan to launch a lesson as reflected in the Lesson

Planimation process? How do the same preservice teachers envision activity—particularly what they include, adapt or omit—in order to design a lesson from curriculum? The intent was to closely study the decision-making process of one case study pair of preservice teachers as they interact with materials and reflect on the lesson planning process.

METHOD

The four-part Lesson Planimation process engaged sixteen pairs of preservice teachers from two elementary mathematics methods courses at two different universities in completing a four-part task. For the task, the preservice teachers were given a copy of an introductory fractions lesson with all related teacher materials and asked to read the lesson and complete the Curriculum Spaces Analysis Tool (Drake et al., 2015). For the second portion of the assignment preservice teachers completed a written lesson plan template that had a predesigned area titled “launch”. Third, preservice teachers used an online software program to create an animation of the classroom that would incorporate sound and movement to express their visualization of only the launch portion of the lesson (see Figure 1, GoAnimate for Schools, 2015).



Figure 1: Screenshot of the animation generated by the focal preservice teacher pair

Finally, the fourth component asked the preservice teachers to reflect on the collaborative experience. To select the specific case study pair, we identified the pair from both institutions with the greatest amount of curricular adaptations across the first three elements of the Lesson Planimation assignment.

DATA ANALYSIS

Data analysis was consistent with case study research: the intent was to gain an understanding of how one pair adapted the curricular materials specific to the launch portion of the lesson (Yin, 2009). In Phase One of data analysis, we segmented the launch portion of the curriculum based on curricular headings into three parts (i.e. Target Mathematics Content, Academic Vocabulary, One Brownie to Share Activity). After identifying all data related to adaptations for the three categories, we mapped the

adaptations throughout the four-part assignment process to trace changes that were made and triangulated the data. In other words, we initially read the curricular materials for one of the three categories (e.g., Target Mathematics Content) and read the Curriculum Spaces Analysis Tool responses focused specifically on the Target Mathematics Content and mapped the changes from the curricular materials to the Curriculum Spaces Analysis Tool. Next, we read the lesson plan, again for the same category (e.g., Target Mathematics Content) and mapped all changes from what had been written with respect to adaptation in the Curriculum Spaces Analysis Tool to the lesson plan. Next, we compared the various data forms and noted differences and similarities. We considered this to be a cyclical process. We then completed this same process for the next two categories (i.e. Academic Vocabulary and One Brownie to Share Activity.)

In this analysis process, we mapped out data across the various components of the assignment (see Figure 2). Use of the data analysis maps for themes afforded deep analysis across the case and promoted consideration of multiple types of data to corroborate findings (Yin, 2009).

In Phase Two, we were focused on elements within the launch that the preservice teachers had *introduced* from their own thinking or from other resources or experiences. As a result, the data for the case were again read in entirety. Following this, the data were reread again for themes related to adaptations and memos were written to follow constant comparative methods for analysis (Corbin & Strauss, 2008). Four themes, all content related, were identified in the data: a) Bridging Everyday and Mathematical Ideas, b) Discussion of Wholes, c) Fraction Addition, and d) Naming Fractions.

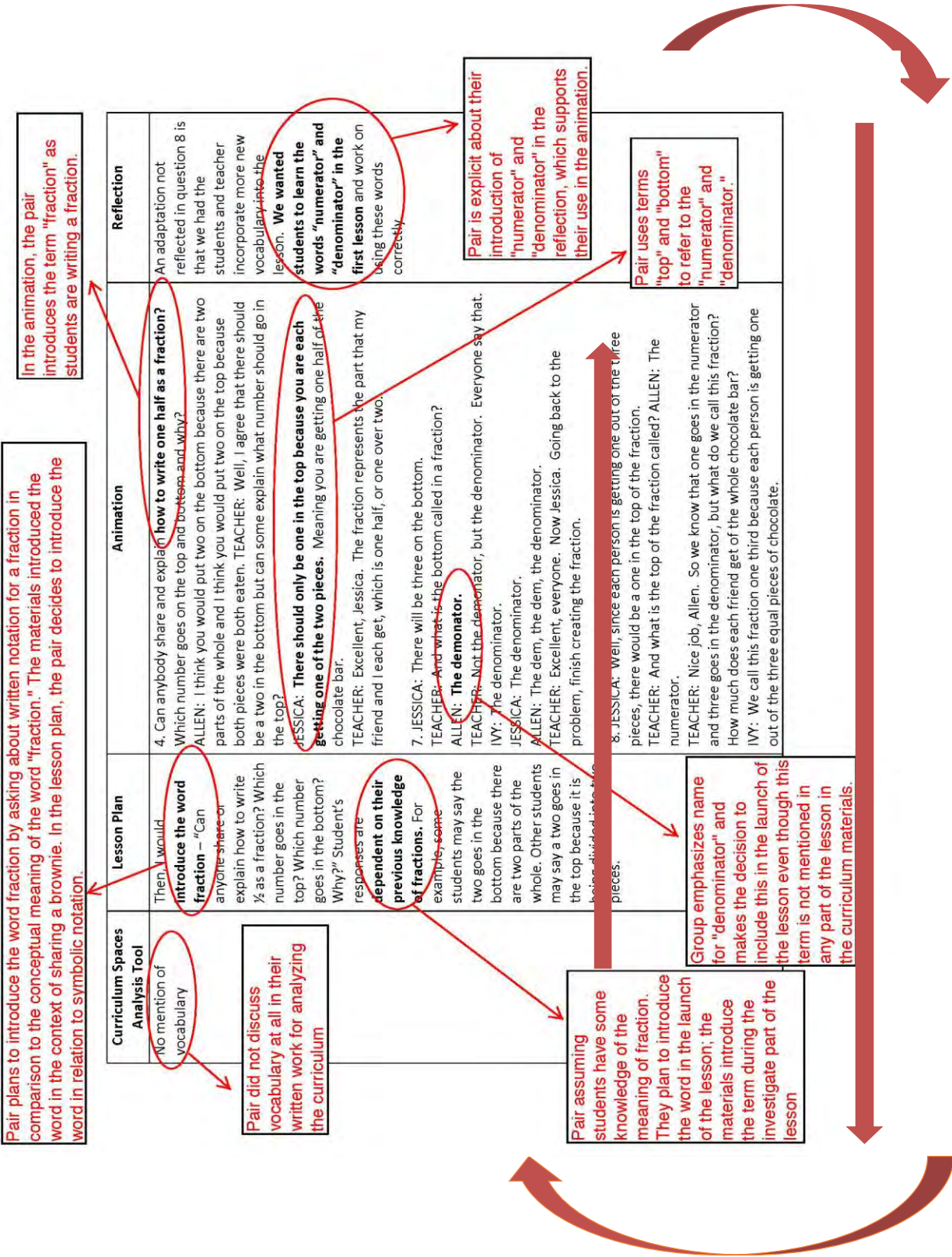


Figure 2: Example data analysis map

FINDINGS

First, we considered curriculum components *included* in the curricular materials (Curricular Components, Table 1) as well as the pair's introduced contributions that were not directly referenced in the curriculum materials (Introduced Components, Table 1). We further analyse two curricular components and two original contributions that were included in the reflection: target mathematics content and academic vocabulary as well as bridging everyday and mathematical ideas and discussion of wholes. We focus on these four themes in the data because they were the two most readily apparent within each of the two categories (i.e. Curricular Components and Introduced Components).

	Curriculum Spaces Analysis Tool	Lesson Plan	Animation	Reflection
<u>Curricular Components</u>				
Target Mathematics Content		X	X	X
Academic Vocabulary		X	X	X
One Brownie to Share Activity		X		
<u>Introduced Components</u>				
Bridging Everyday and Mathematical Ideas	X	X	X	X
Discussion of Wholes	X			X
Fraction Addition		X		
Naming Fractions			X	

Table 1: Curricular and original components identified across the Lesson Plan animation assignment.

CURRICULAR ADAPTATIONS

Throughout the lesson, the curriculum describes students dividing one brownie evenly for two people in ways that appear consistent with a productive lesson launch (Jackson et al., 2012). Eschewing the suggested brownies, the preservice teachers opted instead to bring in a different chocolate treat: chocolate bars featuring even partitions, noticing they were already split up into parts. The pair indicated that this adaptation provided a scaffolding visual that was relatable for students. In their own words, they noted the curriculum, “was truly lacking in real-life connections” and they, “wanted to imple-

ment as many opportunities for real-life connections as they could.” Consequently, the adaptations focused on bringing in materials (chocolate bars) with which students were familiar and having students physically break the chocolate bars as they discussed equal sharing of relatable objects.

The curriculum materials identify one key vocabulary term in the introduction to the lesson, *fraction*. Further, the curriculum features a final discussion in this same lesson introducing two additional terms, *numerator* and *denominator*, after children have the opportunity to create equal parts. In their plan and animation, the pair included the three vocabulary terms in the lesson launch featured across their lesson plan, animation, and reflection. In other words, they considered it important to familiarize students with all three academic vocabulary terms at the beginning of the lesson.

INTRODUCED COMPONENTS

The preservice teachers also introduced elements into their Lesson Planimation project that were not readily apparent in the curriculum materials.

On the Curriculum Spaces Analysis Tool, the preservice teachers wrote, “I would have students give examples of things they could share both verbally and also have them act out making/sharing equal pieces.” This introduction of discussion examples that were relatable to real life was an adaptation included in both lesson plan and animation. In the reflection, the preservice teachers provided a rationale for the decision:

I added a small section at the beginning of the lesson plan for a chance for students to share real-life examples. I incorporated this into both the written lesson plan and the animation because it is important to allow students the space to make real-word connections. This was something that was missing completely from the curriculum.

A concern for including real-world connections was the impetus for the adaptations for the preservice teacher pair, as opposed to including adaptations on the basis of students’ fractional understanding.

The preservice teachers also included a focus on discussing wholes and equivalent fractions, though without necessarily highlighting part-whole relations. On the Curriculum Spaces Analysis Tool, the preservice teachers wrote, ‘I would talk about wholes and how equal fractions represent pieces of a whole.’ In this instance, the adaptation centers on pieces of a whole without being explicit of part-whole relations; the distinction that unequal pieces would also represent pieces of a whole is not apparent.

DISCUSSION

For the purposes of discussion, we focus here on one component of the findings that permeated the overall lesson components: the decisions made around the target mathematics content with attention to the real-life connections provided in the curricular materials. Specifically, the pair was intentional in highlighting their adaptation from the recommended use of brownies in the materials to include chocolate bars in both their lesson plan and animation. Their rationale was to increase the opportunities

for real-life, which is similar to the findings of Nicol and Crespo (2006) who found that preservice teachers often adapt problems to include a familiar context; however, brownies were likely already a familiar contextual reference for students, making this shift to candy bars more notable. Of further interest is their preference for the pre-drawn lines on the chocolate bars versus brownies which would be a mathematical scaffold in the process of adapting materials; although such partitioning might enable access to the procedure of determining values for numerator and denominator (and in fact this appeared to be the pair's goal), it would deemphasize the mathematical concept of creating equivalent parts that underlies such a procedure and that the curriculum explicitly stated. Despite well-meaning changes proposed, the adaptation in essence reduced the mathematical rigor of the original materials (Stein, Smith, Henningsen, & Silver, 2000). Jackson et al. (2012) discuss the risk of reducing cognitive load in the launch through changes in lesson feature and emphasize the importance of planning complex tasks. This raises questions about the preservice teachers' interpretations of the materials, specific to the use of a food item for demonstration purposes, and indicates that their focus on real-life contexts likely superseded their concern for the mathematical rigor of the lesson—a notion that extends previous work on preservice teacher adaptations for real-world contexts (e.g. Aguirre et al., 2013; Nicol & Crespo, 2006). Essentially, their concern for real-life contexts lowered the cognitive demand of the task with which they were working (Jackson et al., 2012; Stein et al., 2000).

IMPLICATIONS

Novices of any profession are learning on the job and, as a result, their decisions are quite likely reflective of their novice status as compared to experts in the field (Goodwin, 1994). The case study pair's decisions, with their adaptations in the spirit of supporting the mathematics learning of their students, are perhaps expected and reasonable given their status as preservice teachers; they are novice educators. We contend that such coursework is a key site for engagement in the curriculum translation process to take place. The case presented illuminates for mathematics teacher educators areas in which preservice teachers need further support in retaining the mathematical rigor of tasks. In particular, we draw attention to the lesson launch as a critical component of collective activity in a mathematics lesson. An implication of the case above is that preservice teachers need further support in connecting intentions of the launch with translating curriculum into a lesson.

References

- Aguirre, J., Turner, E., Bartell, T., Kalinec-Craig, C., Foote, M., Roth McDuffie, A., & Drake, C. (2013). How prospective elementary teachers connect to children's mathematical thinking and community funds of knowledge in mathematics instruction. *Journal of Teacher Education*, 64, 178-192.

- Brown, M.W. (2009). The teacher-tool relationship: Theorizing the design and use of curriculum materials. In J. T. Remillard, B. Herbel-Eisenmann, & G. Lloyd (Eds.), *Mathematics teachers at work: Connecting curriculum materials and classroom instruction* (pp. 17 - 36). New York: Routledge.
- Burkhauser, M., & Lesaux, N. (2015). Exercising a bounded autonomy: novice and experienced teachers' adaptations to curriculum materials in an age of accountability. *Journal of Curriculum Studies*, online first, p. 1–22.
- Corbin, J., & Strauss, A. (2008). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (3rd ed.). Thousand Oaks, CA: Sage.
- Drake, C., Land, T. J., Bartell, T. G., Aguirre, J. M., Foote, M. Q., Roth McDuffie, A., Turner, E. E. (2015). Three strategies for opening curriculum spaces. *Teaching Children Mathematics*, 21(6), 346–353.
- Earnest, D., & Amador, J. (2017, online first). Lesson planimation: Preservice elementary teachers' interactions with mathematics curricula. *Journal of Mathematics Teacher Education*. <https://doi.org/10.1007/s10857-017-9374-2>
- GoAnimate For Schools. (2015). Make animated videos in the classroom [Software]. Available from <http://www.goanimate4schools.com/>
- Goodwin, C. (1994). Professional vision. *American Anthropologist*, 96(3), 606 – 633.
- Grossman, P., Compton, C., Igra, D., Ronfeldt, M., Shahan, E., & Williamson, P. W. (2009). Teaching practice: A cross-professional perspective. *Teachers College Record*, 111(9), 2055–2100.
- Jackson, K., Shahan, E., Gibbons, L., & Cobb, P. (2012). Launching complex tasks. *Mathematics Teaching in the Middle School*, 18(1), 24 - 29.
- Nicol, C., & Crespo, C. (2006). Learning to teach with mathematics textbooks: How pre-service teachers interpret and use curriculum materials. *Educational Studies in Mathematics*, 62, 331-355.
- Remillard, J.T. (2012). Modes of engagement: Understanding teachers' transactions with mathematics curriculum resources. In G. Gueudet, B. Pepin, & L. Trouche (Eds.), *From Text to 'Lived' Resources: Mathematics Curriculum Materials and Teacher Development* (pp. 105 - 122). New York: Springer.
- Stein, M.K., Smith, M.S., Henningsen, M.A., & Silver, E.A. (2000). *Implementing standards-based mathematics instruction: a casebook for professional development*. Teachers College Press: New York.
- Van de Walle, J., Karp, K. S., & Bay-Williams, J. M. (2012). *Elementary and Middle School Mathematics Methods: Teaching Developmentally* (8th ed.). New York: Allyn and Bacon.
- Yin, R. K. (2009). *Case study research: Design and methods* (4th ed). Los Angeles: Sage.

LINKS BETWEEN TEACHERS' PEDAGOGICAL TECHNOLOGICAL KNOWLEDGE AND THEIR PERSONAL CHARACTERISTICS

Ahlam Anabousy and Michal Tabach

Tel-Aviv University

The present study examines the relationship between the different components of pedagogical technological knowledge (PTK), as well as the effect of seniority, employment status, educational status and technological-integration level on PTK scores. We used the PTK questionnaire introduced by Thomas and Palmer. Forty-two middle school mathematics teachers participated in the study. The statistical analysis showed strong correlations between the different components of PTK and PTK itself with the exception of content knowledge component. Results showed also the significant effect of seniority and technology-integration level on PTK, the confidence and the technology instrumental genesis scores. Education status affected positively one of the PTK components: confidence.

LITERATURE REVIEW

Understanding the connections between teachers' knowledge components, which are necessary for integrating technology into classroom practices, are still challenging for the research community. We outline below various attempts to identify frameworks for teacher knowledge, in the following order: frameworks for teachers' knowledge, frameworks for teachers' knowledge connected to the affective domain.

Teachers' knowledge

Shulman (1987) proposed a framework for professional knowledge that includes seven domains of teaching knowledge: content knowledge; pedagogical knowledge; curriculum knowledge; pedagogical content knowledge; knowledge of learners; knowledge of educational contexts; and knowledge of educational purposes and values. The category that revolutionized researchers' thinking was pedagogical content knowledge (PCK), which links the knowledge bases of content and pedagogy.

Researchers continued to develop the field of PCK. For example, Ball and colleagues (Ball et al., 2008) noted that Shulman's categorization was theoretically rather than empirically based. They proposed a model that focuses on Mathematics Knowledge for Teaching (MKT), classified into six categories: common content knowledge, specialized content knowledge, knowledge of content and students, knowledge of content and teaching, knowledge at the mathematical horizon and knowledge of curriculum.

Shulman's PCK also influenced the theoretical frameworks proposed for teachers' knowledge concerning the integration of technology in classroom practice. One of the most important of these theoretical frameworks is the technological-pedagogical content knowledge framework (TPACK), defined as the coherent body of knowledge and skills required for the implementation of Information and Communication Technology (ICT) in teaching (Koehler et al., 2007), although not specific to mathematics education. This model describes types of teachers' knowledge, namely PCK, technological content knowledge (TCK), technological pedagogical knowledge (TPK), and especially TPACK. The acronym M-TPACK applies to the teaching of mathematics (e.g., see Guo & Cao, 2015).

A new theoretical framework parallel to the TPACK was proposed by Thomas and Palmer (2014) to describe teachers' knowledge of integrating technology in the mathematics classroom: the pedagogical technology knowledge (PTK) framework. Thomas and Palmer (2014) maintained that several teacher factors combine to produce PTK (Figure 1), including the MKT factor, which relates to pedagogical and mathematical content knowledge. The other factors are technology instrumental genesis, and personal orientations. The present study utilizes this framework.

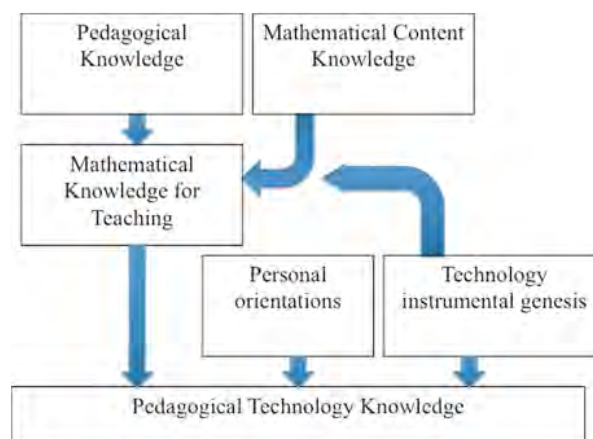


Figure 1: A model of the PTK framework (Thomas & Palmer, 2014)

The affective domain of teachers' knowledge

Researchers interested in mathematics teachers' knowledge required for teaching suggested that in addition to the cognitive knowledge of teaching, an examination of the affective domain of teaching is also needed (e.g., Tatto et al., 2008). A theoretical framework developed by Schoenfeld (2010) helped to explain why it is important to pay attention to the role of teachers' affective domain, especially their orientations towards their teaching practice. Therefore, researchers have explored the relationships between teachers' affective domain and their knowledge (e.g., Beswick, Callingham, & Watson, 2011). Doing so, Beswick et al. (2011) extended the framework proposed by Ball et al. (2008) to include teachers' confidence. Similarly, Barton (2009) suggested that knowing about mathematics teaching includes teachers' attitudes and orientations toward mathematics, which he described as the way teachers grasp mathematics and their own relationship with mathematics.

Some of the researches were concerned with the affective aspect in mathematics education in general, others with mathematics teachers' affective domain related to technology integration in teaching. Doing so, researchers were interested in examining teachers' orientations as a component of teachers' knowledge, especially beliefs about the value of technology and the nature of learning mathematical knowledge, and other affective aspects, such as confidence (Thomas & Palmer, 2014). Thomas and Palmer proposed a framework for pedagogical technological knowledge (PTK), based on Schoenfeld's framework, which includes teacher orientations as a main factor influencing teachers' PTK (Figure 1).

The present study utilizes the PTK framework to measure the initial PTK of practicing teachers in order to describe the relations between the PTK components empirically. It also examines the effect of seniority, employment status, education status and previous technology-integration level on PTK scores.

Research questions

1. Are there statistically significant differences in PTK level according to seniority, technology-integration level, employment status, and the education status?
2. Are there statistically significant relationships between the different components of PTK: pedagogical knowledge, content knowledge, technology instrumental genesis, confidence, beliefs about the value of technology and PTK itself?

METHOD

The participants were 42 mathematics middle school teachers from several schools of average socio-economic status in Israel. The research was conducted in the academic year 2017-2018. The participants differed in their education status, where twenty-three of them were participating in the course "technology in mathematics education" as part of their M.A degree in teaching mathematics. The rest were participants in a PD program aimed at increasing their technology-integration level in their classroom practices. The participants differed in their seniority. Ten of them were teaching for 0-5 years, eleven were teaching 6-10 years, eleven were teaching 11-15 years, and ten were teaching more than 15 years. Also, the participants differed in their employment status and previous technology-integration level. While eight of them reported low level of previous use, 17 with medium level and 17 with high level. Finally, nine of them were ICT coordinators.

We used a PTK questionnaire as data collection instrument. The questionnaire had two parts. The first part collected personal information as seniority, education status, employment status, and technology-integration degree. The second part was composed of the four scales: personal orientations (This scale measures two constructs - the teacher's beliefs about the value of technology (26 items) and teacher's confidence in using technology to teach mathematics (7 items)), pedagogical knowledge (10 items), technology instrumental genesis (5 items), and content knowledge (6 items). A portion of the scales (personal orientations, pedagogical knowledge, and technology

instrumental genesis) were borrowed from Thomas and Palmer (2014); the content knowledge scale was developed by Hill, Schilling, and Ball (2004). Note that originally the scales by Thomas and Palmer were intended to examine teachers' confidence in using graphing calculators. In the scales we used in the present study, "technology" replaced "graphing calculators". Responses are indicated on a 5-point Likert scale ranging from 1 = "Strongly disagree" to and 5 = "Strongly agree." The scales underwent face validity testing as they were translated from English into Arabic. In addition to the face validity procedure, each scale underwent reliability analysis by computing its Cronbach's alpha based on the scores of the teachers in the PTK questionnaire. The computations resulted in Cronbach alphas that ranged between .71 and .82, which are considered acceptable reliability scores.

Following Thomas and Palmer (2014), the PTK level for each teacher was computed as the average of the components: content knowledge, pedagogical knowledge, beliefs about the value of technology and technology instrumental genesis. In the present study confidence was included in computing PTK level. This was done according to Thomas and Palmer recommendation. PTK level was characterized into very low, low, normal, good, very good, using Daher and Saifi (2016) method. Question 1 was answered using independent sample t-test and ANOVA. Question 2 was answered using Pearson correlations and regressions analysis.

FINDINGS

For each of the participating teachers we computed his/her level of PTK. The mean of the group was 3.8 (0.37). Characterizing their level of PTK (following Daher & Saifi, 2016) indicated that six teachers had normal PTK levels, 30 had good PTK levels and six had very good PTK levels.

The effect of employment status on PTK level and on its components was examined using t-test. Results show no significant differences in the scores for the different components of PTK. Also, the effect of education status on PTK level was examined using t-test. Table 1 shows no significant differences in the scores of the different components of PTK, except in the confidence score ($t(40) = 2.154, p = .016$). The differences between PTK and its components scores according to seniority were examined using ANOVA. Results show significant effect of seniority on PTK level [$F(3,38) = 5.281, p = .004$]. Post hoc analysis using Scheffe showed that the PTK of teachers whose seniority is less than ten years was significantly more than that of teachers whose seniority is more than 15 years. ANOVA results also showed significant effect of seniority on confidence [$F(3,38) = 12.84, p < .000$]. Post hoc analysis using Scheffe showed that the confidence of teachers whose seniority is more than 15 years is significantly more than the rest of the teachers. Moreover, significant effect was found for seniority on technology instrumental genesis [$F(3,38) = 5.423, p = .003$]. Here, Scheffe's showed that the participants whose seniority is between 5-10 years had a significant more score of technology instrumental genesis than the participants whose seniority is more than 15 years.

Category	Mean (Sd)	Mean (Sd)	t value
	MA students	Training teachers	
Beliefs about the value of technology	3.62 (.58)	3.61 (.33)	.084
Confidence	4.13 (.55)	3.68 (.58)	2.514*
Mathematical content knowledge	4.05 (.50)	3.86 (.46)	1.280
Pedagogical knowledge	3.63 (.50)	3.78 (.48)	-1.014
Technology instrumental genesis	3.89 (.66)	3.54 (.53)	1.870
Pedagogical technological knowledge	3.86 (.38)	3.69 (.35)	1.458

*p<.05

Table 1: Means, standard deviations, T value of the scores of the components of PTK (N=42) according to learning status

The differences between PTK and its components scores according to technology-integration level (low, medium and high levels) were examined using ANOVA. Results showed significant effect of technology-integration level on PTK level [$F(2,39)=4.463$, $p=.018$]. Post hoc analysis using Scheffe showed that the PTK of teachers whose technology-integration level is high was significantly more than the teachers whose technology-integration level is low. ANOVA also showed significant effect of technology-integration level on confidence [$F(2,39)=8.101$, $p=.001$]. Post hoc analysis using Scheffe showed that the confidence of teachers whose technology-integration level is low was significantly less than the rest teachers. Moreover, significant effect was found for technology-integration level on technology instrumental genesis [$F(2,39)=6.958$, $p=.003$]. Here, Scheffe's showed that the participants whose technology-integration level is high had a significant more score of technology instrumental genesis than the rest participants.

A Pearson product-moment correlation coefficient was computed to assess the relationship between the PTK components: content knowledge, teachers' beliefs about the value of technology, teachers' confidence, pedagogical knowledge, technology instrumental genesis and PTK itself. There was a positive correlation between all the components except the content knowledge component, which was related only with PTK (see Figure 2). Note that r value indicates a weak correlation when its value is 0.3-0.5, moderate correlation when its value is 0.5-0.7 and strong correlation when its value is more than 0.7.

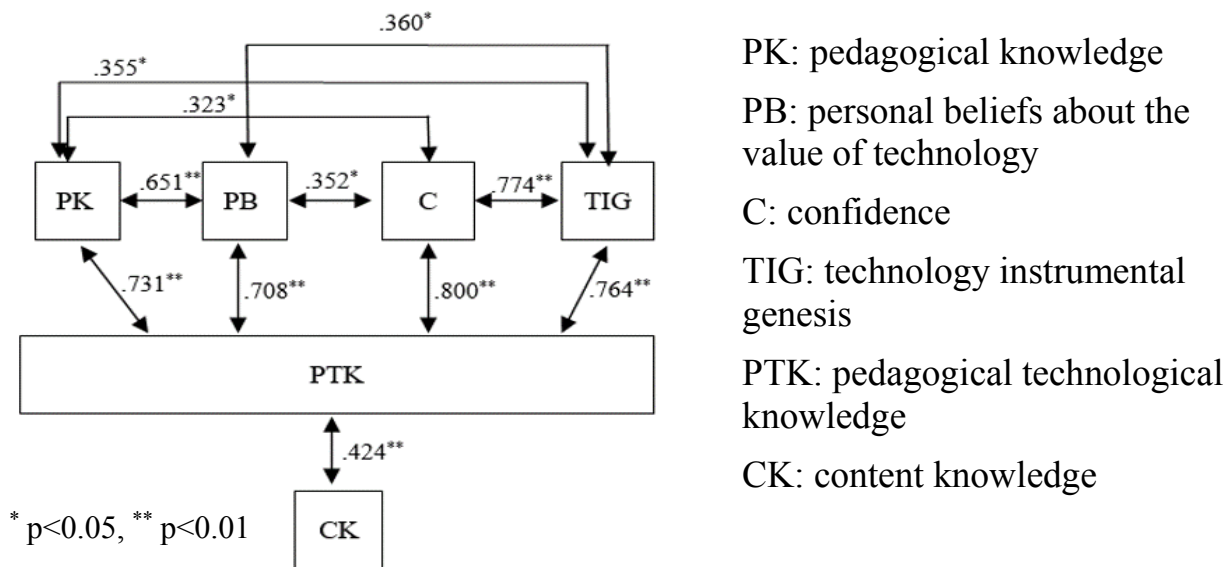


Figure 2: Correlation between the PTK components and PTK itself

Pearson's correlation showed strong positive correlations between the PTK as construct and the following components of PTK: beliefs about the value of technology, pedagogic knowledge, confidence in using technology and instrumental genesis. Regarding content knowledge component, Pearson's correlation showed a moderate positive correlation between it and PTK. It also showed weak to moderate correlations between the different components of PTK except for confidence and technology instrumental genesis which had a strong positive correlation.

Multiple linear regression analysis was used to develop a model for predicting teachers' PTK from their 'beliefs about the value of technology, pedagogic knowledge, confidence in using technology, content knowledge and instrumental genesis. Basic descriptive statistics and regression coefficients are shown in Table 2. Each of the predictor variables had a significant ($p < .01$) partial effects on PTK, where technology instrumental genesis and confidence had the strongest effect on PTK.

Model	Standardized Coefficients Beta	T	Sig.
(Constant)		.000	1.000
Beliefs about the value of technology	.255	77579057.833	.000
Confidence	.323	79376540.996	.000
Content knowledge	.259	96475620.825	.000
Pedagogical knowledge	.263	77829529.152	.000
Instrumental genesis	.331	79173177.171	.000

Table 2: Values of Beta appropriate for predicting teachers' PTK from its components

DISCUSSION

The present study intended to examine the relationship between the different components of pedagogical technological knowledge (PTK). The statistical analysis showed strong correlations between the different components of PTK and PTK itself, except for content knowledge (CK) which had a moderate correlation with PTK. This emphasizes the PTK framework proposed by Thomas and Palmer (2014). These results agree with those of Roig-Vila et al., (2015) who reported non-strong correlation between CK and teacher knowledge for integrating technology. Where, Roig-Vila et al., study used TPACK framework. Regarding the correlations between PTK components, results showed mostly weak to moderate correlation between the different components of PTK except for content knowledge, which had no correlation with the other PTK components. According to these results, it seems that the different components of PTK framework are correlated; an interesting finding that needs verification in a more extensive research.

The present study also examined the effect of several variables (seniority, employment status, educational status and technological-integration level) on PTK scores. Results showed significant differences in PTK, confidence and technology instrumental genesis that are related to seniority and technology-integration. From the other side, Pearson's correlations showed strong positive correlations between these components. It could be argued that these PTK components are flexible; meaning that they can be influenced by education or experience related to ICT integration. In addition, the effect of seniority on teacher's knowledge related to technology integration was also reported by other researchers, as Saltan and Arslan (2017) and Roig-Vila et al., (2015). This reveals the lack of PTK in teachers with the highest seniority.

The participants in this study differed in their education status. Research results showed that the education status affected positively confidence, in favor of teachers who were studying toward their M.A degree. They may have more confidence in their educational abilities and skills in general and in integrating technology in particular.

References

- Ball, D. L., Thames, M., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389-407.
- Barton, B. (2009). Being mathematical, holding mathematics: Further steps in mathematical knowledge for teaching. In R. Hunter, B. Bicknell, & T. Burgess (Eds.), *Crossing divides. Proceedings of the 32nd Annual Conference of the Mathematics Education Research Group of Australasia*, Vol.1 (pp. 4– 10). Palmerston North, NZ: MERGA.
- Beswick, K., Callingham, R., & Watson, J. (2011). The nature and development of middle school mathematics teachers' knowledge. *Journal of Mathematics Teacher Education*, 15(2), 131-157.

- Guo, K., & Cao, Y. (2015). Survey of mathematics teachers' technological pedagogical content knowledge and analysis of the influence factors. *Educational Science Research*, 3, 41–48.
- Hill, H.C., Schilling, S.G., & Ball, D.L. (2004) Developing measures of teachers' mathematics knowledge for teaching. *Elementary School Journal*, 105, 11-30.
- Koehler, M. J., Mishra, P., & Yahya, K. (2007). Tracing the development of teacher knowledge in a design seminar: Integrating content, pedagogy and technology. *Computers & Education*, 49(3), 740-762.
- Roig-Vila, R., Mengual-Andres, S., & Quinto-Medrano, P. (2015). Primary teachers' technological, pedagogical and content knowledge. *Comunicar* 45, 151–159.
- Schoenfeld, A. H. (2010). *How we think: A theory of goal-oriented decision making and its educational applications* (p. 2010). New York: Routledge.
- Shulman, L. (1987). Knowledge and teaching: Foundations of the new reform. *Harvard Educational Review*, 56(1), 1-22.
- Saltan, F., & Arslan, K. (2017). A comparison of in-service and pre-service teachers' technological pedagogical content knowledge self-confidence. *Cogent Education*, 4(1), 1311501.
- Tatto, M. T., Schwille, J., Senk, S., Ingvarson, L., Peck, R., & Rowley, G. (2008). *Teacher Education and Development Study in Mathematics (TEDS-M): Conceptual framework*. East Lansing: Teacher Education and Development International Study Center, College of Education, Michigan State University.
- Thomas, M.O.J. & Palmer, J.M. (2014). Teaching with digital technology: obstacles and opportunities. In A. Clark-Wilson, O. Robutti & N. Sinclair (Eds.), *The Mathematics Teacher in the Digital Era. An International Perspective on Technology Focused Professional Development* (pp. 71-89). Dordrecht: Springer.

MAKING MATHS MATTER: ENGAGING STUDENTS FROM LOW SES SCHOOLS THROUGH SOCIAL JUSTICE CONTEXTS

Judy Anderson and Alice Kriesler

The University of Sydney

Junior secondary students typically believe mathematics is challenging and has little connection to the real world. Consequently, Australian students are less engaged and fewer are choosing to study mathematics beyond grade 10, particularly in low socio-economic. Contextualising mathematics using tasks with a social justice perspective was investigated to ascertain the levels of behavioural, emotional and cognitive engagement of students in three grade 7 classrooms. Student surveys and teacher interviews indicated students were more engaged when the mathematics was perceived as relevant with connections to real world issues, enabled longer and deeper conversations about social justice perspectives, and was challenging with appropriate scaffolding when necessary.

INTRODUCTION

In Australia, there is a gap in achievement between students from high socio-economic status (HSES) and low socio-economic status (LSES) schools (Munns, Sawyer, & Cole, 2013; OECD, 2014). This gap is particularly evident in mathematics, with the 2012 Programme for International Student Assessment (PISA) revealing students from LSES schools scored, on average, 39 points lower than students from HSES schools (OECD, 2014), which is equivalent to nearly one year of schooling (Lamb, Jackson, Walstab, & Huo, 2015). The results also showed students from LSES deciles were less likely to be confident, interested, or have a sense of self-efficacy in mathematics than students from higher SES deciles (Lamb et al., 2015).

Since teachers determine how mathematics is taught, they have the potential to impact students' attitudes and perceptions of mathematics (Ernest, 2015) by choosing tasks which engage and interest students. Using mathematics as a tool for understanding the world and one's position within society allows students to have a voice and a sense of self-agency (Gonzalez, 2009; Leonard & Moore, 2014; Wright, 2016). Mathematics tasks with a social justice perspective (SJP) enable the exploration of issues of justice, power, oppression and agency and increase awareness of social injustice and the empowerment of marginalised communities (Gonzalez, 2009). Although previous studies have shown that implementing mathematics with a SJP empowers students, there is limited research on whether such approaches increases levels of behavioural, emotional and cognitive engagement of students from LSES settings (Gutstein, 2003; Wright, 2016), particularly in the Australian context. This paper reports on a preliminary investigation focused on whether the use of such tasks increased engagement of grade 7 students in three classrooms from LSES schools in Australia.

LITERATURE REVIEW

Current teaching practices in many secondary school classrooms contribute to student disengagement (Silver & Stein, 1996; Skilling, Bobis, Martin, Anderson, & Way, 2016). Mathematics teachers emphasise rote learning, memorisation and other conservative and controlling pedagogies (Munns, Zammit, & Woodward, 2008), with frequent use of lower-order, repetitive and tedious mathematics tasks (Sullivan, 2011). Furthermore, teaching practices are disconnected from students' lives and future aspirations, as teachers fail to contextualise the learning, leading to students feeling they have no voice (Munns et al., 2008). To engage with school, students from LSES schools, in all subject areas, need their learning to connect to their lives, allowing them to feel valued as their learning becomes meaningful (Lamb et al., 2015). Although research on mathematics education has shown that student engagement can be promoted through teaching practices which emphasise its relevance and connect to students' experiences (Martin, Anderson, Bobis, Vellar, & Way, 2012; Skilling et al., 2016; Sullivan, 2011), this finding is not specific to students from LSES schools, so further research is needed.

To explore student engagement of students in LSES schools, the framework developed by Fredricks, Blumenfeld and Paris (2004) is used – it describes three types of engagement. Behavioural engagement focusses on participation and the extent students are involved in their learning experience; emotional engagement encompasses interest, values and willingness to complete work; and cognitive engagement focusses on motivation and “willingness to exert the effort necessary to comprehend complex ideas and difficult skills” (p. 60). As Munns et al. (2013) suggest, a key component of cognitive engagement involves students completing intellectual work regardless of their level of schooling, but there is evidence that when students transition from primary to secondary school (from grade 6 to grade 7 in Australia), teachers reduce the level of challenge (Sullivan, 2011). This is accompanied by a decrease in student engagement in year 7 mathematics classrooms as well as a ‘dip’ in achievement (Martin et al., 2012) hence the focus on increasing engagement in the early secondary years is essential if such trends are to be reversed.

There is evidence that students from LSES settings become empowered when they learn mathematics through a SJP. While engagement focusses on students' behavioural, emotional and cognitive responses to their learning, empowerment focusses on students developing a voice and sense of agency (Leonard & Moore, 2014). Located in a working class low-income community, Gutstein's (2003) project focussed on changing students' orientation towards mathematics through real-world SJP projects. Critical pedagogies helped students develop critical consciousness and mathematics competencies (Gutstein, 2003), viewing mathematics as a tool to make sense of the world and their place in it, by inference students became engaged.

Rather than use longer projects, this study investigated the use of rich mathematics tasks with a SJP to address issues of student disengagement – such tasks involved higher-order thinking, were open-ended, connected to students' lives, promoted dis-

cussion and communication, involved cooperative learning, and allowed students to have a voice (Munns et al., 2013; Sullivan, 2011). They were based on tasks from Gutstein and Peterson's (2013) *Rethinking Mathematics*, containing a SJP which met the criteria for a rich task. Of importance was the need to choose tasks which were intellectually challenging but also potentially engaging so that students would persevere and be committed to working in a group to find solutions.

Using tasks which challenge and engage has been identified as a successful practice of exemplary teachers working in LSES schools. Munns and colleagues (2013) case-study research demonstrated that exemplary teachers gave students the opportunity to engage in learning experiences which were intellectually challenging and valued higher-order thinking skills including problem solving. Although their study was not specific to mathematics, the findings were like those of Silver and Stein's (1996) *QUASAR Project*, which focussed specifically on mathematics learning. It explored reformed teaching practices of mathematics teachers in economically disadvantaged communities, emphasising the importance of engaging students with intellectually challenging mathematics tasks. Although the QUASAR project demonstrated the importance of intellectual quality in engaging students with mathematics, it did not explore learning mathematics through a SJP.

Investigating the use of rich tasks with a SJP and finding evidence of their success for engaging students requires collecting data from several sources. While Wright (2016), Munns et al. (2013), and Silver and Stein (1996) focused on teacher responses, Gutstein (2003) used responses from his students. This study used data from both students and teachers to explore the research question: How does the use of rich mathematics tasks with a social justice perspective engage grade 7 students emotionally, behaviourally and cognitively in low socio-economic schools?

METHODOLOGY

To explore the research question, three government, co-educational, LSES schools in Australia were invited to participate in the study. The schools were diverse – in School A (metropolitan, 500 students) 90% of students had a language background other than English (LBOE), many of whom were recent refugee arrivals with more than 50% having lived in Australia for fewer than three years; in School B (regional, 1000 students) 9% of students were from LBOE, and 9% were Indigenous; School C (metropolitan, 1200 students) had 90% LBOE students and 1% were Indigenous. One grade 7 mathematics teacher from each school participated in the study and data were collected using researcher field notes, student surveys and teacher interviews.

Before data were collected, each teacher was given four rich mathematics tasks involving different SJP issues – this allowed teachers to select the task most suitable for their grade 7 students (each class had 24 students). Teachers A and B chose 'Living Algebra, Living Wage' whilst teacher C selected 'Sweatshop Math' (Gutstein & Peterson, 2013). One lesson was observed in each school with the researchers using a structured approach based on the *Motivation and Engagement Framework* used by

Munns and his colleagues (2013). Field notes were made during observations providing useful information about student engagement through high cognitive, high affective and high operative learning experiences. This was useful as it paralleled Fredrick et al.'s (2004) conception of behavioural, cognitive and emotional engagement, allowing predictions of engagement of each recognised type (Skilling et al., 2016).

Further data were collected after each lesson including student surveys and teacher interviews. The survey sought students' views about mathematics and their engagement during the lesson using both Likert-scale items and open-ended questions. Adapted from Fredricks et al.'s (2004) survey, the first section consisted of eleven statements for measuring the different types of school engagement. Informed by Gutstein's (2003) survey, three open-ended questions focussed on students' disposition towards mathematics, particularly its relevance, and their level of interest and engagement in the SJP lesson. Teachers participated in individual, semi-structured interviews to gain information about their students' response to the lesson compared to more typical mathematics lessons. Each interview was transcribed with transcripts coded for evidence of emotional, behavioural, or cognitive engagement.

Data were analysed in two phases - the first phase involved entering the quantitative data from the surveys into an Excel Spreadsheet to collate student responses. The agree/strongly agree responses and disagree/strongly disagree responses were combined to ascertain general agreement or disagreement with each item. The second phase involved triangulating the data from the field notes, interviews and surveys by reading and coding to identify evidence of behavioural, emotional and cognitive engagement in each class. Evidence for the three types of student engagement were identified separately although they are not necessarily discrete, so while the results and discussion of each are presented separately in the next section, there are overlaps.

RESULTS AND DISCUSSION

Behavioural engagement includes student conduct, work involvement, and participation (Fredricks et al., 2004). Most students in each class assessed their behaviour as being engaged, although slightly fewer than a half of students from School B reported not working as hard as they could. School B's responses were more negative than the other two schools, with field notes also indicating one-third of students were unfocused and off-task during the second-half of the lesson. Teacher B stated "I think that [at] the starting bit they were more engaged than a regular lesson ... They had to talk about something that related to them and they could all write something". In this class, the task introduction encouraged on-task behaviour and promoted engagement through student discussion. However, Teacher B believed the second part of the lesson, involving students individually completing graphs, was more disruptive than usual due to the content difficulty. In addition, about two thirds of students from School B indicated their behaviour was not different to normal, since they believed it was "just another regular maths lesson". This suggests some students off-task behaviour was a common response to challenging work. Students from School B who found the task

challenge appropriate, engaged behaviourally – e.g., one student stated he “felt tuned in and concentrated” and another reported she “was getting into the lesson and really learning because it was a good lesson”.

Teacher A, who used the same task realised the graphing component would be too difficult for her class so she adapted it to better support student engagement. Field notes recorded all students were deeply involved throughout the entire lesson except “two girls sitting in the back corner appeared very off-task”. Although students entered the classroom “chatting and unsettled” (field notes), there was silence as students completed the graphing activity. The teacher believed that most students “generally worked well” during the lesson, with students actively participating and completing the required work. Like the first part of teacher B’s lesson, the discussion allowed contribution from all students, suggesting they were behaviourally engaged when discussing real-world problems relating to their life experiences (Skilling et al., 2016; Wright, 2016). Slightly fewer than half of the students from School A believed their behaviour was different from normal, with four of these students stating that it improved because the lesson was “more interesting”. One student from School A wrote “Today’s lesson was very interesting and I learnt something I never thought of and so my whole behaviour was different” revealing the potential of such SJP tasks.

Similar findings on student behavioural engagement were evident from teacher C’s interview response when asked if most students actively participated in the lesson – he stated “the ones that were interested did. And I was surprised some were interested, like there were a few girls who normally don’t do very much”. This is reminiscent of findings from Schools A and B, where students were engaged behaviourally when they were interested in the issue under discussion. Furthermore, when asked how student behaviour compared to a normal lesson, teacher C responded “they are usually more off-task”. Both tasks provided opportunities for students to engage behaviourally through the discussion of real-world issues that connected to their lives, resulting in active participation and on-task behaviour. However, when the mathematics required to complete the task was too difficult, some students still became disruptive and consequently, teachers needed to ensure the task was scaffolded appropriately.

Emotional engagement involves students’ interest, values and affective reactions in the classroom and is mostly assessed through self-reporting measures. All students from School A and three quarters from School C enjoyed their lesson, whereas at School B, just over a half of the students enjoyed the lesson. This discrepancy might have been due to the difficulties school B experienced with the task. When students from School B were asked if they would prefer to do more mathematics activities like the Living Algebra, Living Wage lesson, half disagreed - the school with the highest proportion of negative responses for this question. Their reasons included not liking mathematics in general, finding mathematics confusing and boring, and not liking the topic of graphing. The other half of the class said they would prefer more mathematics lessons like this, because they enjoyed the opportunity to work with peers to discuss the issues raised in the task. Two students from School B stated:

I would prefer to do more activities like this because we discussed real world problems.

[The task] didn't just talk about maths, it talked about important issues happening in the world.

This aligned with teacher B's perception that all students enjoyed the first part of the task, involving the discussion:

The first student who put his hand up ... never puts his hand up ... normally he doesn't write very well, doesn't have much to say, and doesn't want to talk. For him to get really involved in that, it's just awesome. I think they really enjoyed the discussion questions. They all wanted to have their input, which was great.

Three quarters of the students from Schools A and C reported they would prefer more of these types of mathematics lessons because they were "fun", "interesting", and:

[The lesson] was very helpful to make me understand so many things.

It is interesting and we learn more about the world and poverty.

Not only are we learning maths but also learning other things.

During the discussion, student responses to the teacher's questions in School C were enthusiastic, displaying emotional reactions to the findings of other countries' wages, with two students remarking "it's sad", and other students agreeing. When asked whether he would use the task again, teacher C responded:

Its engaging for the kids. It gets them asking questions not just linking within the numeracy topic but also society and social implication ... Just their expressions when they found out [some people only earn] sixty-four dollars a month.

Teacher C recognised students' affective reaction to the task, showing they were emotionally affected by the social justice context. The findings from each school suggest that the use of mathematics tasks which contain a SJP encourages students to engage on an emotional level – most students enjoyed the lessons, found them interesting, and appreciated the opportunity to use mathematics to talk about 'real-world' issues.

Cognitive engagement focusses on student involvement in intellectual work (Munns et al., 2013) so for students to cognitively engage, the tasks needed to be intellectually challenging. All teachers believed that the tasks gave students this opportunity, with teacher A elaborating:

In a normal lesson ... you don't have a lot of talking and thinking about issues... Today we have covered a whole life experience in this lesson ... the students [were] able to connect the classroom to the real-world issues.

Although, teacher B thought the task involved intellectual work, he found this problematic:

Because of its difficulty level.... it involved them doing a lot more than what they are used to in terms of their own thinking, of them producing stuff from their own knowledge. I think that part was more disruptive than usual.

To further assess cognitive engagement, students responded to survey items assessing self-regulation, independent work styles and attempts to relate the lesson to prior learning (Fredricks et al., 2004). Most students from Schools A and C felt they did not let other people complete the ‘hard parts’ in the lesson. In School B, it was evident from the field notes and teacher interview that some students were not willing to persevere when they faced difficulties. The field notes reported two students exclaiming “I can’t even do anything” and “I don’t get it”, during the graphing stage of the lesson, whilst another student exclaimed “this is so confusing to me”, and then became off-task and disruptive. Teacher B commented on this in the interview, stating “when they are having heaps of trouble they turn to me for help. And if they don’t get help straight away then they go off-task ... they don’t persevere”.

The tasks were designed to be implemented using cooperative learning, which Wright (2016) identified as an important practice of social justice mathematics as it promotes discussion and collaboration. Student preference for cooperative learning was endorsed by almost all students from Schools A and C. The main utilisation of group work in the tasks was through discussion, which was designed to help students connect the mathematics with their own experiences. The discussions allowed students to engage in substantive communication by asking questions which demonstrated higher-order thinking and connecting the lesson with their prior learning. The ability to make connections demonstrated deep understanding, an element of Munn’s et al.’s (2013) intellectual challenge, indicating cognitive engagement.

CONCLUSION

While Gutstein (2003) investigated how teaching mathematics through a SJP helps students from LSES schools develop mathematical power and change their disposition towards mathematics, this study went further and sought to investigate how social justice mathematics also promotes behavioural, emotional and cognitive engagement. The findings suggest that to engage students from LSES schools with social justice mathematics tasks, teachers need to: ensure the mathematics is relevant to students and help them recognise that mathematics can be used to understand the world around them; provide opportunities to use mathematics to critique social practices; and needs to be appropriately intellectually challenging. Although each school, teacher and classroom learning environment was different, data from student surveys, teacher interviews and field notes reinforced that when students are given a context that relates to their real-life experiences, they can recognise the usefulness of mathematics (Gutstein, 2003; Munns et al., 2013; Wright, 2016). Teachers need to provide students with opportunities to use mathematics to analyse the world critically (Frankenstein, 1990; Gutstein, 2003; Wright, 2016). The tasks provided students with this opportunity, and the data illustrated that students found this interesting and engaging with discussion allowing students to share their own experiences, ask further questions and become emotionally invested in their learning.

References

- Ernest, P. (2015). The social outcomes of learning mathematics: Standard, unintended or visionary? *International Journal of Education in Mathematics, Science and Technology*, 3(3), 187-192.
- Frankenstein, M. (1990). Incorporating race, gender, and class issues into a critical mathematics literacy curriculum. *Journal of Negro Education*, 59(3), 336-347.
- Fredricks, J. A., Blumenfeld, P. C., & Paris, A. H. (2004). School engagement: Potential of the concept, state of the evidence. *Review of Educational Research*, 74, 59-109.
- Gonzalez, L. (2009). Teaching mathematics for social justice: reflections on a community of practice for urban high school mathematics teachers. *Journal of Urban Mathematics Education*, 2, 22-51.
- Gutstein, E. (2003). Teaching and learning mathematics for social justice in an urban, Latino school. *Journal for Research in Mathematics Education*, 34, 37-73.
- Gutstein, E. & Peterson, B. (Eds.) (2013). *Rethinking mathematics: teaching social justice by the numbers*. Wisconsin: Rethinking Schools Ltd.
- Lamb, S., Jackson, J., Walstab, A., & Huo, S. (2015). *Educational opportunity in Australia 2015: Who succeeds and who misses out*. CIRES, Melbourne: Mitchell Institute.
- Leonard, J., & Moore, C. M. (2014). Learning to enact social justice pedagogy in mathematics classrooms. *Action in Teacher Education*, 36, 76-95.
- Martin, A. J., Anderson, J., Bobis, J., Vellar, R., & Way, J. (2012). Switching on and switching off in mathematics: An ecological study of future intent and disengagement among middle school students. *Journal of Educational Psychology*, 104, 1-18.
- Munns, G., Sawyer, W., & Cole, B. (2013). *Exemplary teachers of students in poverty*. London: Routledge.
- Munns, G., Zammit, K., & Woodward, H. (2008). Reflections from the riot zone: the Fair Go Project and student engagement in a besieged community. *Journal of Children and Poverty*, 14, 157-171.
- OECD. (2014). *PISA 2012 results in focus: What 15-year-olds know and what they can do with what they know*. Paris, France: Author. Retrieved from <http://www.oecd.org/pisa/keyfindings/pisa-2012-results-overview.pdf>.
- Silver, E. A., & Stein, M. K. (1996). The QUASAR project: The “revolution of the possible” in mathematics instructional reform in urban middle schools. *Urban Education*, 30(4), 476-521.
- Skilling, K., Bobis, J., Martin, A. J., Anderson, J., & Way, J. (2016). What secondary teachers think and do about student engagement in mathematics. *Mathematics Education Research Journal*, 28(4), 545-566.
- Sullivan, P. (2011). *Teaching mathematics: Using research-informed strategies*. Victoria: Australian Council for Educational Research (ACER).
- Wright, P. (2016). Social justice in the mathematics classroom. *London Review of Education*, 14(2)104-118.

CLASSROOM DIALOGUE AS A FRENCH BRAID: A CASE STUDY FROM TRIGONOMETRY

Mette Andresen

University of Bergen, Norway

Bettina Dahl

Aalborg University, Denmark

Studies have shown that dialogues in teaching that follow the model of IRE (initiation, response, evaluation) are widely used in teaching despite it having some deficiencies. Based on the theories of collective learning and the social nature of thought, knowledge creation and learning, we argue in this paper that a ‘captivating’ dialogue which at a first glance seemed to follow the model of IRE can be understood as part of a process that initiates the students into the practices of school mathematics including learning to speak mathematically. We illustrate this point with an example from an upper secondary Norwegian class in trigonometry. Here the teacher combined asking open-ended problems to the students with IRE questions during a joint review of problem-solving using the principles of Polya.

INTRODUCTION

It is not uncommon to hear experienced teachers value the classroom interaction that develops between the teacher and the class while the class is introduced to and guided through mathematical reasoning or problem-solving. On the other hand, the literature on mathematics teaching tells that classroom interaction between the teacher and the students in general is dominated by obstacles for learning and pseudo discussions in the form of IRE (Initiation-Response-Evaluation), the Topaz effect, and ‘Guess what the teacher has in mind’ for instance seen in a state-of-the-art report by the Norwegian National Center for Mathematics Education (Nosrati & Wæge, 2014, p. 7).

This mismatch of decades of research and advice from the literature and what teachers do in practice motivated us to analyse examples of such dialogues. As Kilpatrick (1988) stated in a similar discussion of a discrepancy between the advice of educational researchers and the teachers’ actions: “What is it that teachers know that others do not?”

This paper therefore aims to characterise and discuss a ‘captivating’ dialogue observed in a Norwegian upper secondary mathematics classroom in trigonometry, which at a first glance appeared to follow the IRE model. The concept of captivating dialogue is meant to capture a particular element of what is often referred to in classroom teaching as a joint review of mathematical reasoning or problem-solving, different from a teacher’s lecturing. The captivating dialogue is guided by the teacher, focuses on the collective learning, and the utterances of each student become pieces of the class’s shared knowledge.

THEORETICAL FRAMEWORK

Initiation-Response-Evaluation, IRE

IRE has multiple times been identified as the most frequently type of interaction in classrooms. Often it consists of at least 90% of the teacher's questions (eg. Heritage & Heritage, 2013; Hogan et al., 2014). It is regarded as an effective way for a teacher to check factual knowledge or recall but it may be criticized for not promoting understanding or higher order thinking because it gives no room for the individual student to elaborate on their answers or articulate their opinions or doubts, neither does it support classroom discussions.

However, sometimes what at first appears to be an IRE interaction can in fact be a dynamic teaching and learning encounter (Forman, 1989). Burbules and Bruce (2001) also argue that IRE interactions are beneficial to learning as part of a review and rehearsal. For example, some students can feel motivated by being able to answer a direct question and be rewarded for it; which can lead to greater confidence and motivation. If used in a skillful manner in the right context, it may even become more than plain rehearsal. Tainio and Laine (2015) shows example of Finnish teachers' emotion work through IRE classroom interactions. Other studies point to the importance of IRE as forming a predictable classroom routine and that the IRE interactions serves the purpose of aiding the students in learning the knowledge that previous generations have built up (Roth & Gardener, 2012). Hogan et al. (2014) also refers to a number of studies arguing that IRE is neither good nor bad in itself, it depends on the purposes and the particular occasions.

It therefore appears that although IRE interactions clearly have shortcomings, many studies also point to benefits of IRE interactions. Understanding the benefits of IRE depends on the understanding of learning, which we will discuss below.

Discursive approach and collective learning

Cobb et al. (2011) write about the collective learning of the classroom community in terms of the evolution of classroom mathematical practices. Lerman (1994) also argues for a shift in focus from the individual's 'understanding' to the social nature of thought, knowledge-creation, and learning. Later he writes: "In the mathematics classroom, interactions should not be seen as windows on the mind but as discursive contributions that may pull others forward into their increasing participation in mathematical speaking/thinking, in their zones of proximal development" (Lerman, 2002, p. 89). Thus, in a cultural, discursive psychological view, the students' answers should not be interpreted in terms of their grasping or understanding certain concepts, explanations or relations, but rather that the answers are interpreted as acts of participation. This is in line with Sfard (1998) who argues for perceiving learning as a combination between acquisition and participation. Participating indicates learning as a process of becoming a member of a community, thus taking part and being part of the conversation. In the Discursive Approach to mathematics education by Sierpiska (2005), the teachers' role in classroom conversations is characterised by an obligation to lead the discussion

in the direction of relevant mathematical ideas, themes, and issues. Apparently, this leading and direction of the discussion may sometimes be observed and misinterpreted as cases of teachers shaping the ‘Topaz effect’ (Brousseau, 1997).

In Lerman (2002) the principles of a cultural, discursive psychology are outlined and operationalised as tools of research with illustrating examples. Research in cultural, discursive psychology in mathematics teaching and learning includes the following elements: i) Intersubjectivity and internalisation, ii) The zone of proximal development and semiotic mediation, iii) Positioning and voice in classroom mathematical practices, iv) Social relationships, v) Mathematical artefacts, and vi) Development as a process of thinking/speaking mathematics. For the purpose of this paper, vi) is of particular interest as the learning of school mathematics is seen as an initiation into the practices of school mathematics including learning to speak mathematically. The teacher therefore has a vital role in showing what is approved within the discourse. In this view, the captivating dialogue may be interpreted as the teacher’s introduction and development of elements of acceptable constructions by means of collective inclusion of the class’ students into the teacher’s review of mathematical reasoning or problem-solving etc.

Research Question

How can a teacher initiate the students into the practices of school mathematics in a captivating dialogue, which at first appears to include series of IRE interactions?

METHODOLOGY

The data consisted of video recordings of teaching in eight Norwegian upper secondary classrooms as part of the EU research project KeyCoMath (<http://keycomath.eu/>) about students’ strategies for creative problem-solving (Andresen, 2015, 2017). The aim of the project was to develop and study teaching that encourages students’ activity, inquiry, and autonomy. The recordings (around 30 hours in all) were done during the autumn of 2013 by one of the authors and translated below by the other.

This paper focuses on one sequence (15 minutes) and discusses the interactions between the students and the teacher (Tom). The discussions are illustrated with excerpts from the transcription of the sequence. The micro-contexts of these interactions are analyzed using Conversation Analysis (Have, 2007) where the focus is the “one phenomenon, the in-situ organization of conduct, and especially talk in interaction” (p. 27). The choice of just one sequence is made to be able to have some level of detail.

DATA AND ANALYSIS

Tom teaches a science class with 13 students at a private upper secondary school in Norway. Polya’s (1985) scheme for problem-solving was introduced in the previous lesson. This lesson was spent on introduction of the use of trigonometry for solving problems. The next lessons were planned to be group work on a larger problem-solving project: modelling a Ferris wheel. Due to constraints of the length of the paper, we will

only show two pieces of the classroom dialogue, one from the beginning and one towards the end of the sequence where Tom reviews the students' work with a problem from a task picked from the National written examination the previous year. In the problem discussed in the excerpts, the students were supposed to find the length x where the angle α would be the biggest (see Figure 1, left). As part of the problem, they were asked to use the formulas for $\sin(u-v)$ and $\cos(u-v)$ to verify a given expression of $\tan(u-v)$ (see Figure 1, right).

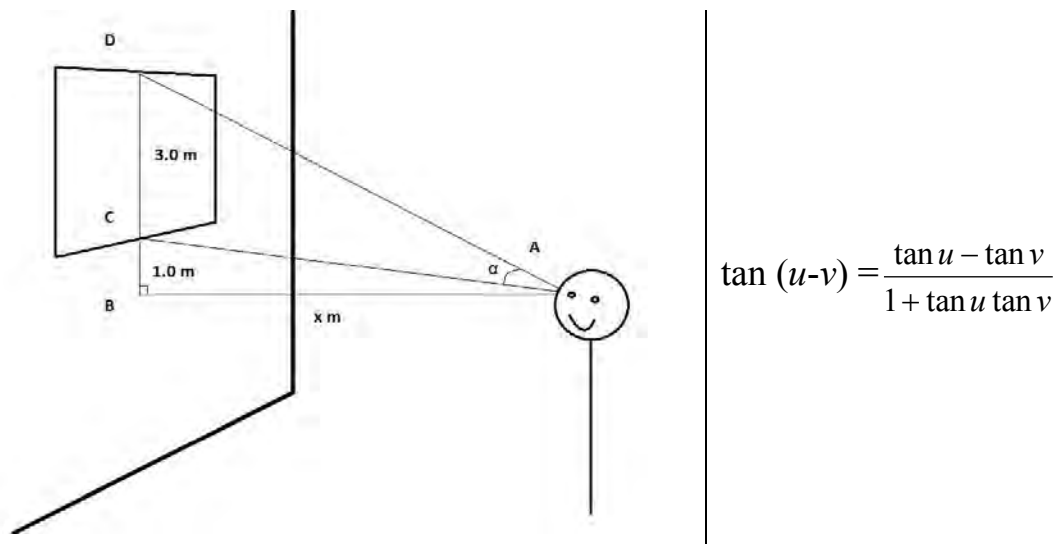


Figure 1: Figure accompanying the problem (redrawn by the authors due to copyright) and the expression of $\tan(u-v)$.

During the sequence Tom writes on the blackboard, frequently turns around and moves forward when talking with the class and the turns back, writes etc. The atmosphere in the classroom is calm and friendly without tensions, the students concentrate on the teacher's review which aims to support their thinking more deeply about how to solve a problem in the actual case as well as in general.

Two minutes into the recording Tom talks while writing on the board and frequently turning around facing the students:

Tom: Did any of you have the need to clean up what you have been doing? ... Polya's four steps. ... Sometimes we do not have time to do this ... to clean up the problem-solving and not just be happy with the answer and clear the thinking towards the end, it is a good thing to do. Think about, what did I actually use? The first thing I used was as Student 1 said an argument that $u-v$ is actually two angles in play ... The difference between two angles becomes a new angle. And then you can use that \tan to the angle is equal to \sin to \cos to the angle. There were no cheap points in this problem, you see. [laughing] How was it with \sin ? [a little silence] \sin is the difference...?

Student 2: $\sin u$, $\cos v$, plus [Tom writes a formula on the board while two students discuss minus or plus for 10 seconds].

Tom: Minus or plus?

Student 3: Minus.

Tom: Yes. ... [Another student says Yes] What is most interesting here is not to remember this. It is not what mathematics is.

Here, the last three lines of the teacher–student exchange apparently follow the IRE model. However, seen in context this exchange serves to constructively conclude the longer discussion between two students about minus or plus. Tom redirects focus towards the principles of cleaning up problem-solving. Far from neglected, the two students' discussion become a minor part of the sequence's overall goal, which is emphasised in the lines after "Yes".

In the second excerpt towards the end of the sequence, Tom is aiming to understand the students' thinking:

Student 4: Thinking that, the basic that tan is sin to cos. You can try to divide with cos and see how this goes. This is very well.

Tom: So actually a kind of trial and error?

Student 4: Yes actually.

Student 5: Take cos, pick out.

Tom: Aha, so you recognised that one of this - so it looks like this is minus and this is minus this is plus. And if it was the one we have to divide by the same as what we have. This was what you were thinking?

Student 5: Yes.

Tom: Aha. Then here we have minus and minus and plus and plus. Here we have $\sin u$ divided by $\cos u$, and $\sin v$ divided by $\cos v$, how does this look, I think that this is actually what you discovered. But can you actually just take an expression and divide it by something? Is that allowed? [Some seconds silence] You don't know. Can you just divide by $\cos u$ or $\cos v$? Or how? [some hands showing].

Student 2: [Mumbling] numerator and denominator, it is ok to divide by the same.

Tom: So actually, I multiply by one over $\cos u$ times $\cos v$ and then I multiply $\cos u$ times $\cos v$. Is this what you are saying.

Student 2: Hmmm.

Tom: Are we allowed to do this?

Student 6: It is usual calculation with fractions.

Tom: Yes, it is usual calculation with fractions. Is it allowed?
[Several silent seconds].

Students: [Mumbling] denominator.

Tom: Yes, the denominator. What about this?

- Students: [Mumbling; then several silent seconds].
- Tom: Does the problem state any limitation?
- Students: [Mumbling; then several silent seconds].
- Tom: What is it that we cannot take tan to?
- Student 7: 90 degrees.
- Tom: 90 degrees. Or 270 degrees. Can $u-v$ be 90 or 270? If u is 270 and v is 90 for instance, they could be like this, right? You can take tan to 180, look here, if this is 270 and this is 90. It is working.

In this part several students participate in the interaction with Tom. First Tom meets the students in what they have done, then he challenges them. The students draw on their pre-understanding about fraction and contribute to the joint construction of the class' knowledge. The 'mumbling' seems to be part of the student-teacher interaction and might be interpreted as a sign of the students' work in progress. Like in the first excerpt, Tom interchanges between open and closed questions and deliberately uses brief and clear IRE-like exchanges to constructively conclude the review's single steps. The conclusions are added to the class' shared knowledgebase.

DISCUSSION

In the first excerpt, Tom used an IRE interaction to constructively conclude a discussion between two students about if a minus or a plus should be used in the formula. Alternatively, the students, who already had spent some time on the task before the sequence might have spent further time on settling this detail. This would result in getting side-tracked from the original purpose of the sequence. The individual student's understanding of all details is not important at that moment. Each student can later check it out or ask the ones that do know. By now, collective learning (Cobb et al., 2010) has taken place and *the class* knows that it is minus. Tom focuses on the class as a whole and makes sure that the class is on the path he has decided for this sequence. This is in line with Sierpinska's (2005) views of the role of the teacher as someone who has the responsibility of leading a class in a relevant direction.

One main characteristic of this captivating dialogue in the classroom is the students' joint focus on the review of mathematical reasoning. Focus is not on individual students or on single answers; the teacher directs the review and keeps it intentionally proceeding on path, in accordance with the planned learning trajectory. In line with this, the classroom dialogues will not necessarily reveal whether each single student has learnt what the teacher had set as the learning goal since the teacher-student exchanges do not serve as windows to the students' minds. In line with Lerman (2002), the students' contributions are not necessarily a window into the student's mind, but an act of participation. However, following Sfard (1998), though, the students who attended the review have learnt from it since learning may (also) be interpreted as progressive participation in existing and joint activity. It is worth mentioning that the

teaching sequences were aimed at developing problem-solving skills – it was not a drill-sequence.

The concept of ‘captivating dialogue’ is distinct from what is commonly understood as an IRE dialogue in important aspects: i) The questions do not serve as a means for checking the student’s mind in the form of checking knowledge or ways of thinking ii) Since focus is on the review, not on the individual student, the teacher will not necessarily pick the respondent. iii) The response to a student’s answer should be interpreted as inclusion of the student into the joint activity, rather than assessment of the student’s performance. The term ‘captivating’ was vindicated by the students’ engagement in attempts to answer the questions and by their participation, sometimes only through mumbling.

CONCLUSIONS

In this paper we wish to argue for seeing teacher–student interactions in the classroom in context with their aims and goals, content, and the roles they play for the development of new interactions. We have discussed a case of classroom interactions that at first appeared to have a lot in common with IRE interactions; they indeed showed IRE-pattern but when analysed within a framework of collective learning, we could interpret them in terms of an initiation into school mathematics practice and communication. One question the analysis raises is, if the interaction could be characterised as following the IRE model or not. From one perspective it was IRE, as we observed a teacher centred dialogue which many times where short questions that were then answered, evaluated by the teacher, then followed by a new question. A simple count of percentage of different types of question would however never suffice to catch the context of these questions, the function that each question and answer has in the class interaction and the development of them.

In the title of the paper we use the term ‘French braid’ which is a certain type of way to weave/braid hair into a tail where gradually more and more of the hair is braided into the rest. This is a metaphor which we find describes well how the teacher braids the input of the different students into the shared knowledge of the class in this collective learning process.

References

- Andresen, M. (2015). Students’ creativity in problem solving. *Acta Mathematica Nitriensia*, 1(1), 1–10.
- Andresen, M. (2017). Glimpses of students’ mathematical creativity, which occurred during a study of students’ strategies for problem solving in upper secondary mathematics classes. In P. Błaszczyk, & B. Pieronkiewicz (Eds.), *Mathematical Transgressions 2015* (10 pp). Kraków: Taiwpu Universitas. In print.
- Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht: Kluwer.

- Burbules, N. C., & Bruce, B. C. (2001). Theory and research on teaching as dialogue. In V. Richardson (Ed.), *Handbook of research on teaching*, 4th edn (pp. 1102–1121), Washington, DC: American Educational Research Association.
- Cobb, P., Stephan M., McClain, K., & Gravemeijer, K. (2010) Participating in Classroom Mathematical Practices. In A. Sfard, K. Gravemeijer, & E. Yackel (Eds), *A Journey in Mathematics Education Research. Mathematics Education Library, Vol 48* (pp. 117–163). Dordrecht: Springer.
- Forman, E. A. (1989). The role of peer interaction in the social construction of mathematical knowledge. *International Journal of Educational Research*, 13, 55–70.
- Have, P. T. (2007). *Doing Conversation Analysis: A Practical Guide*, 2nd edn. Los Angeles, CA: Sage.
- Heritage, M., & Heritage, J. (2013). Teacher questioning: The epicenter of instruction and assessment. *Applied Measurement in Education*, 26(3), 176–190.
- Hogan, D., Rahim, R. A., Chan, M., Kwek, D., & Towndrow, P. (2014). Understanding Classroom Talk in Secondary Three Mathematics Classes in Singapore. In B. Kaur & T. L. Toh (Eds.), *Reasoning, Communication and Connections in Mathematics: Yearbook 2012, Association of Mathematics Educators* (pp. 169–197). Singapore: World Scientific.
- Kilpatrick, J. (1988). Editorial. *Journal for Research in Mathematics Education*, 19(4), 274.
- Lerman, S. (1994) Changing focus in the mathematics classroom. In S. Lerman (Ed.), *Cultural perspectives on the mathematics classroom* (pp. 191–213). Dordrecht: Kluwer.
- Lerman, S. (2002). Cultural, discursive psychology: A sociocultural approach to studying the teaching and learning of mathematics. In C. Kieran, E. Forman, E. & A. Sfard (Eds.), *Learning discourse. Discursive approaches to research in mathematics education* (pp. 87–113). Dordrecht: Kluwer.
- Nosrati, M., & Wæge, K. (2014). *En oppsummering av status for forskning på hva som kjennetegner god læring og undervisning innenfor matematikk* (State of the art on research in mathematics education). Trondheim: Nasjonalt senter for matematikk i opplæringen.
- Polya, G. (1985). *How to solve it. New aspects of mathematical method*. Princeton, NJ: Princeton University Press.
- Roth, W. M., & Gardener, R. (2012). ‘They gonna explain us what makes a cube a cube’ Geometrical properties as contingent achievement of sequentially ordered child-centered mathematics lessons. *Mathematics Education Research Journal*, 24, 323–346.
- Sfard, A. (1998). On Two Metaphors for Learning and the Dangers of Choosing Just One. *Educational Researcher*, 27(2), 4–13.
- Sierpinska, A. (2005). Discoursing mathematics away. In J. Kilpatrick, C. Hoyles, & O. Skovsmose (Eds.), *Meaning in mathematics education* (pp. 205–230). New York, NY: Springer.
- Taino, L., & Laine, A. (2015). Emotion work and affective stance in the mathematics classroom: the case of IRE sequences in Finnish classroom interaction. *Educational Studies of Mathematics*, 89, 67–87.

USING SELF-VIDEO ANALYSIS TO PROMOTE TEACHER GROWTH

Nick Andrews, Jenni Ingram, and Andrea Pitt

University of Oxford, UK

In this paper we examine one teacher's professional growth as she participates in a self-video based professional development project. Using the Interconnected Model of Professional Growth we explore how one teacher identified areas of her teaching she wanted to change and the nature and extent of the change for one of these areas over the two years of the project. We particularly address the question of the nature of teacher professional growth in video studies with goals that are responsive to the emerging concerns of the teachers involved.

INTRODUCTION

In this paper we draw on work by Clarke and Hollingsworth (2002) to analyse examples of teacher change and longer-term growth during a two year project where mathematics teachers shared clips of their teaching with their colleagues and researchers in order to explore ways of developing their students' mathematical talk in lessons. We focus on one particular teacher, Anna, who was part of the project for the full two years. Looking across the project we examine her professional learning including focusing in on instances of change that were more nuanced or fleeting. We also consider the areas in which there was no evidence of change.

We begin this paper by reviewing the research into the role of video in teacher professional development and connecting this to the specific approach we used in the project based on Mason's discipline of noticing (2002). Deliberately we choose to use a different lens through which to view professional learning across the project to the model that shaped our approach to working collaboratively with the teachers. This analytical lens is Clarke and Hollingsworth's (2002) Interconnected Model of Professional Growth. We outline this model and exemplify its application to the case of Anna and an area of interest she herself identified and spoke of changing.

VIDEO AND PROFESSIONAL LEARNING

Video is now widely used in mathematics teacher education in a variety of ways and in a variety of contexts. Videos can be of exemplary or problematic teaching (e.g., Borko, Koellner, Jacobs, & Seago, 2011), familiar classroom settings (e.g., Ingram, 2014), unfamiliar classrooms from other countries, or of a teacher's own teaching (e.g., Hollingsworth & Clarke, 2017). Teachers can work in a long established group, such as mathematics departments within a school (Coles, 2013), groups specifically formed to discuss the video clips (e.g., van Es & Sherin, 2010), in online communities (e.g.,

Borko, Koellner, Jacobs, & Seago, 2011) or as individual teachers working with an expert (Hollingsworth & Clarke, 2017). The video clips discussed in the group meetings can be chosen by an expert such as a researcher (e.g., van Es & Sherin, 2010) or by teachers themselves (e.g., Hollingsworth & Clarke, 2017) and the discussions can also be guided by a variety of principles or frameworks developed by researchers such as the distinction between accounts of an accounting for (Coles, 2013; Mason, 2002) and the learning to notice framework (e.g., van Es, 2011), and which can be facilitated by the researcher or by the teachers themselves.

Two key aspects of working with teachers and videos of classroom teaching are who chooses the clip to be discussed and how the clip is chosen. Issues around teacher agency research have led to a current interest in teachers themselves selecting the clips to be discussed (Hollingsworth & Clarke, 2017; Sherin & Dyer, 2017). Sherin and Dyer (2017) propose that opportunities for teacher learning arise at each stage from teachers videoing their own teaching, through selecting clips to share with the group, to the discussions of the clips. The choice of clip is shaped by the aims and focus of the chooser, who might be looking for clips that include specific features of teaching, those that exemplify particularly good teaching or clips that have the potential to generate meaningful discussion amongst teachers (Sherin, Linsenmeier, & van Es, 2009).

What teachers notice when watching videos of classroom teaching has also been the focus of research (Ingram, 2014; van Es, 2011). Attending to specific features of classroom teaching can help teachers to prepare themselves to both notice these features in their own classroom and to make decisions about how to adapt their teaching in light of what they notice and the outcomes they seek to achieve (Mason, 1998). Mason (1998) speaks of teachers' awareness of what they are stressing and ignoring when teaching and video provides opportunities for this awareness to be heightened. Indeed, Mason's discipline of noticing, by which he means "arranging to alert oneself in the future so as to act freshly rather than automatically out of habit" (Mason, 2012, p. 37), underpins the way we work with teachers and video. Changes in teaching are associated with teachers developing sensitivity to noticing opportunities in the moment when they are in the classroom, "becoming aware of possibilities which were not previously available" (Mason, 2002, p. 144) and acting in light of the possibilities that come to mind in this moment. Yet this is not an introspective task, as Mason proposes that it "requires the support of a compatible group of people whose presence can sustain individuals through difficult patches, and who provide both a sounding board and a source of challenge for observations, conjectures and theories" (2002, p. 144).

Common to Mason (2002) and other models of professional learning (e.g., Korthagen & Kessels, 1999) is the principle that the origins of professional growth are in the concerns of teachers. Thus, facilitation of groups of teachers working together needs to either shift to the teachers' goals driving the discussions or needs to begin from this stance in the first place. Yet in many studies of video as a tool for professional learning the structure and goals are predetermined and remain unchanged throughout the duration of the study (e.g., Hollingsworth & Clarke, 2017) or become more guiding as the

project develops (e.g., Santagata, 2008). Mason (2002) reminds us that teachers rarely change their teaching in any sustained way when told or guided to. This raises the question of the nature of teacher professional growth in video studies with goals that are less predetermine and more responsive to the emerging concerns of the teachers involved.

TEACHER PROFESSIONAL GROWTH

The Interconnected Model of Professional Growth (Clarke & Hollingsworth, 2002) has affordances as a predictive, interrogatory and—most significantly for the current study—analytical tool. The model features four distinct ‘change domains’: the *personal domain* (change in teachers’ knowledge, beliefs and attitudes), the *domain of practice*, the *domain of consequence* (change in salient outcomes) and the *external domain*. The four domains “encompass the teacher’s world” (p. 950) alongside the mediating processes of enactment and reflection (see figure 1). As an analytical tool the model allows for an examination of teachers’ professional learning, recognising the individuality of how teachers’ practice, beliefs, and values change and acknowledging teachers as active learners who shape “their professional growth through reflective participation in professional development programs and in practice” (p. 948).

The context within which an individual teacher is situated is represented both by the external domain and by the *change environment*, the latter capturing the extent to which the context affords or constrains change.

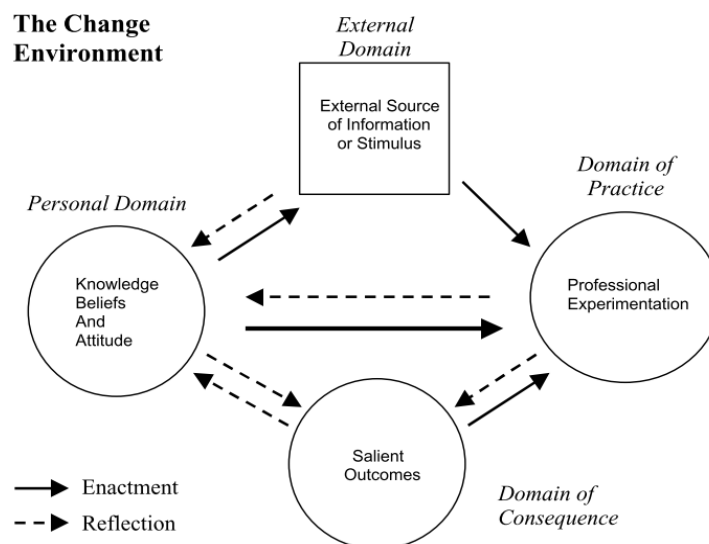


Figure 1: The Interconnected Model of Professional Growth.

This model is compatible with Mason’s (2002) discipline of noticing. What is available to be noticed in the moment by the teacher is captured by the dynamic relationship between the personal and external domains. Possibilities that come to mind in the moment (personal domain) and the outcomes the teacher seeks to achieve (domain of consequence) shape what comes to action (domain of practice). The “compatible group

of people” to which Mason refers (2002, p. 144) contribute both to the change environment and the external domain.

The Interconnected Model enables analysis of how change in one domain can lead to a change in another and give rise to change sequences. Clarke and Hollingsworth (2002) recognise that such changes may be “fleeting” (p. 958) and so distinguish between these unstable change sequences and stable growth networks, which are associated with enduring professional growth.

METHODS

The Talk in Mathematics project ran for two years with two different schools and had a broad focus on whole class discussions. Within each school a group of mathematics teachers worked together videoing themselves teaching and sharing clips of these videos for discussion when the group and the researchers met. The groups had experience working together to develop their teaching. Both schools were secondary schools in the UK taking students between ages 11 and 18. The teachers chose which classes to video, and when to video them, and some teachers chose to video more than one of their classes. The teachers also chose how to video their lessons and which clips to share.

Ahead of each meeting one or two teachers would volunteer to share a short (approximately 3-minute) clip from their lesson, and this teacher would choose the clip and the focus of the discussion that would follow. At the beginning of the project the researchers were given the clip and the focus in advance of the meeting but by the end of the project these were first shared during the meeting. The teacher discussed in this paper, Anna, took part in a total of 10 meetings over the two years.

The meetings began with the teacher sharing the clip from their lesson. Some of the presenting teachers provided background information or details of their choice before sharing the clip whilst others offered this information after showing the clip. The presenting teacher then talked through what they noticed in the clip and this was then followed by questions by the other teachers and the researchers that prompted the re-watching of the clip, often to identify more accurately what was said by students. The discussion of the clip then ended with an exploration of possibilities for acting in similar situations with different classes or within different mathematical topics. Both researchers involved share Mason’s belief that there is no “‘best way to teach’, nor even a ‘best action to choose with particular learners at a particular time and place’” (Mason, 2016, p. 224) and the intention of the discussions was to dwell on what the presenting teacher noticed, and then to draw upon what the others in the group noticed or were sensitised to in order to broaden what the teachers (and the researchers) were noticing and to open up possibilities for action.

The shared video clips, the video of the full lesson from which they were taken and audio recordings of the meetings are the main sources of data on which we draw in this paper. These were transcribed and the conversations around the video clips were coded

for the nine change sequences arising from the Interconnected Model consisting two change domains and the reflective or enactive link connecting them. Growth networks were identified through what the presenting teachers in the meetings reported about their learning triangulated by qualitative changes in teaching seen over time through the lesson videos.

ANNA AND PAUSING

In this paper we focus on two clips that Anna shared with the group that illustrate Anna's growth in relation to one aspect of her teaching that she was consciously working on: pausing during classroom interaction. Amongst the cases available to choose, the case of Anna and pausing was selected for the current paper because we wanted to make sense of noticed differences between some of her espoused intentions and her actions in the classroom and in meetings.

Anna had attended each meeting and had shared several video clips over this time. We focus our analysis on the first and the last clip she shared, in order to illustrate her professional growth over the course of the project. The sharing of each clip was followed by a group discussion, and Anna led both discussions and these drew on key features of classroom interaction that other teachers also chose when sharing their clips. We focus specifically on the feature of pausing in the analysis below. In doing so we offer an example of a growth network that arises from a teacher working with video clips and a focus of their own choosing.

Working on pausing

For the first clip, Anna reported that she had been working on adding pauses after her questions, but also after her students had given their responses (described as wait-time by Ingram & Elliott, 2016). She framed the purpose of this in terms of allowing her students time to give answers in full. In this clip Anna was seen to use deliberate pauses, intently looking at the student who was being given time to speak. In the discussions later in the project she expressed concern at how 'staring' had made her students, and herself, feel uncomfortable and 'put on the spot', and the other teachers in the group also made jokes about 'Anna's glare' both in this meeting and the other meetings that followed. Later in the video of the lesson, which was not shared with the group but was made available to the researchers, Anna had stopped pausing deliberately but did use continuation markers (Sidnell & Stivers, 2012) in the places where she was previously pausing. These continuation markers also encourage students to add to and continue their turns but Anna made no reference to them in her descriptions of her teaching.

For the second clip, Anna expressed her aim as improving students' communication skills. This aim was enacted again through offering students the time, through pausing, to articulate their responses, but this time she used 'displacement activities' rather than stares. She described displacement activities as busying herself 'doing something on the board' so that students could 'buy a bit more time'.

“I picked that bit because I was trying to improve their communication skills and you know that previous one with the 6th form when we had the big pause, ((laughs)) and it was just a slight difference in that, that there were still the pauses, but I was doing displacement activities. You know what people do when you don’t like pauses” (Anna, meeting 8)

Anna’s growth network across the project in relation to pausing

An analysis at the project level provides evidence of Anna’s professional growth in relation to pausing, which we represent as a growth network using the Interconnected Model. Following the initial project meeting she enacted the external impulse of 'try out pausing in your classroom' through professional experimentation with pauses and stares (arrow 1 in figure 2). There is some evidence to suggest that Anna had not fully thought through the purposes of pausing in this way before the first clip she shared in the project, as in this clip she paused after every student turn rather than pausing more strategically. Further, both Anna and the other teachers at the meeting described the pauses as uncomfortable, and that opportunities in the meeting to describe her intentions as richly as the other teachers who shared clips of themselves pausing were not taken up. Nevertheless, in that first lesson clip there was evidence of Anna reflecting-in-action (arrow 2 in figure 2), in that she was becoming more aware of the emotional consequences of the enacted pausing strategy within the lesson—a change in the domain of consequence—and adapting her pausing behaviour as the lesson unfolded. We conjecture that this change was associated with Anna returning to a habitual questioning behaviour.

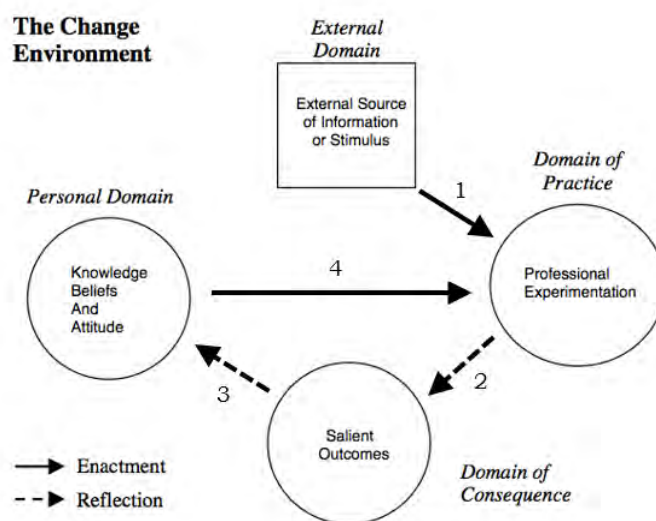


Figure 2: Anna’s growth network across the project with respect to pausing.

There was however evidence of Anna continuing to reflect in the early meeting and over the course of the project on the outcomes of her pausing (arrow 3 in figure 2), refining her pausing strategy—a change in the personal domain—and enacting this through engaging in further professional experimentation in the later lesson (arrow 4 in figure 2). From our available data, these changes in pausing behaviour were associated

with an on-going outcome of avoiding uncomfortableness. Although Anna discussed the affordances of pausing for developing student explanations with others in the meetings and when introducing clips included references to ‘improving communication skills’, we could not link these affordances to what she noticed in the clips and was observed doing in her lessons.

We therefore had no evidence for longer-lasting changes in Anna’s personal domain associated with reflection on the external domain, just in the domain of practice, and so figure 2 represents through the Interconnected Model the whole of Anna’s growth network across the project based on our analysis. Interestingly, this underlying growth network was also identified in a case reported by Clarke and Hollingsworth (2002, p.960) and the loop in figure 2 consisting of arrows 2, 3 and 4 has the same structure as an action research cycle “in the absence of in-service activity” (p. 961).

DISCUSSION

Anna reported on changes in her teaching in each of the meetings and in a post-project interview. Our analysis of videos of her teaching and what she said in meetings supports the notion of change, but this was largely confined to changes in the domain of practice. We had little evidence of Anna “becoming aware of possibilities that were not previously available” (Mason, 2002, p. 144) with respect to pausing. While we could have offered an account for this based on Anna’s considerable experience as a classroom teacher, our analysis around the Interconnected Model highlights the significant role the external domain played for Anna at the very start of the project and the absence of links between it and the personal domain as the project proceeded. The self-videoing approach to the project design afforded flexibility in the shape of any participating teacher’s growth network, and so this included the possibility of growth following a traditional intervention pattern originating in the external domain. And while possibilities seen in a video may be remarked upon, similar possibilities in the classroom may not be noticed and acted upon. Our analysis therefore adds to current thinking on self-video use for the professional learning of teachers and raises questions of how its affordances might be realised.

References

- Borko, H., Koellner, K., Jacobs, J., & Seago, N. (2011). Using video representations of teaching in practice-based professional development programs. *ZDM - International Journal on Mathematics Education*, 43(1), 175–187.
- Clarke, D., & Hollingsworth, H. (2002). Elaborating a model of teacher professional growth. *Teaching and Teacher Education*, 18, 947–967.
- Coles, A. (2013). Using video for professional development: The role of the discussion facilitator. *Journal of Mathematics Teacher Education*, 16(3), 165–184.

- Hollingsworth, H., & Clarke, D. (2017). Video as a tool for focusing teacher self-reflection: supporting and provoking teacher learning. *Journal of Mathematics Teacher Education*, 20(5), 457–475.
- Ingram, J. (2014). Supporting student teachers in developing and applying professional knowledge with videoed events. *European Journal of Teacher Education*, 37(1), 51–62.
- Ingram, J., & Elliott, V. (2016). A critical analysis of the role of wait time in classroom interactions and the effects on student and teacher interactional behaviours. *Cambridge Journal of Education*, 46(1), 1–17.
- Korthagen, F. a. J., & Kessels, J. P. a. M. (1999). Linking Theory and Practice: Changing the Pedagogy of Teacher Education. *Educational Researcher*, 28(4), 4–17.
- Mason, J. (1998). Enabling teachers to be real teachers: Necessary levels of awareness and structure of attention. *Journal of Mathematics Teacher Education*, 1, 243–267.
- Mason, J. (2002). *Researching your own practice: The discipline of noticing*. Routledge.
- Mason, J. (2012). Noticing: Roots and branches. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 35–50). Mahwah, New Jersey: Erlbaum.
- Mason, J. (2016). Perception, interpretation and decision making: understanding gaps between competence and performance—a commentary. *ZDM - Mathematics Education*, 48(1–2), 219–226.
- Santagata, R. (2008). Designing Video-Based Professional Development for Mathematics Teachers in Low-Performing Schools. *Journal of Teacher Education*, 60(1), 38–51.
- Sherin, M. G., & Dyer, E. B. (2017). Mathematics teachers' self-captured video and opportunities for learning. *Journal of Mathematics Teacher Education*, 20(5), 1–19.
- Sherin, M. G., Linsenmeier, K. A., & van Es, E. A. (2009). Selecting Video Clips to Promote Mathematics Teachers' Discussion of Student Thinking. *Journal of Teacher Education*, 60(3), 213–230.
- Sidnell, J., & Stivers, T. (2012). *The handbook of conversation analysis*. Wiley-Blackwell.
- van Es, E. A. (2011). A framework for learning to notice student thinking. In M. G. Sherin, V. R. Jacobs, & R. A. Philipp (Eds.), *Mathematics teacher noticing: Seeing through teachers' eyes* (pp. 134–151). New York: Routledge.
- van Es, E. A., & Sherin, M. G. (2010). The influence of video clubs on teachers' thinking and practice. *Journal of Mathematics Teacher Education*, 13(2), 155–176.

DEVELOPING PROSPECTIVE TEACHERS' MATHEMATICS ORIENTATIONS IN THE CONTENT COURSES

Aina Appova

The Ohio State University

We examine the work of expert mathematics teacher educators (MTEs) in mathematics “content” courses, specifically MTEs’ reflections and insights (from interviews) on the mathematics-related “orientations” that they promote and develop with K-8 prospective teachers (PTs). We found that expert MTEs, during content courses, offer PTs opportunities to develop “conceptual” orientations towards teaching mathematics (Thompson et al. 1994) and “problem-solving” orientations towards mathematics as a discipline (Ernest 1991). Implications from this work offer recommendations for teacher education, research, and practice.

INTRODUCTION

Thompson, Philipp, Thompson, and Boyd (1994) argued that teachers have different *orientations* toward mathematics and pedagogy and that these *orientations* have serious consequences for the teaching and learning that occurs in their classrooms. Research extensively documents that prospective teachers (PTs) enter teacher education programs with a variety of orientations and beliefs about mathematics and what it means to learn and teach mathematics (Cobb et al. 1991). Recent studies also strongly suggest that much work of the university faculty is necessary in providing opportunities for PTs to experience effective models of mathematics teaching and learning in order to develop critical elements of practice valued by the profession (Ghousseini & Herbst 2016; Grossman et al. 2009). These opportunities have been primarily identified in mathematics “methods” courses (Kazemi et al. 2009; Windschitl et al. 2012), however very little is known about the mathematics “content” courses, particularly the types of mathematics-related “orientations” promoted in these courses (Bergsten & Grevholm 2008; Even 2008). In this study, we begin to address this gap in the literature by examining the work of *expert* mathematics teacher educators (expert MTEs) who specifically teach mathematics “content” courses. We provide MTEs’ reflections and insights (from individual interviews) on mathematics “orientations” that they foster and promote with K-8 (age 5-14) PTs during mathematics content courses. The overarching research question of this study was: *What orientations toward mathematics and mathematics teaching do expert MTEs foster and promote when teaching undergraduate K-8 mathematics content courses?* We define MTEs as “professionals who work with practicing and/or prospective teachers to develop and improve the teaching of mathematics” (Jaworski 2008, p. 1). We also recognize that mathematics “content” courses vary across different settings, and that many teacher education programs do not include (or do not separate) mathematics “content” and “methods” courses

(Blömeke et al. 2014; Kaur et al. 2017). However, in the United States, the majority (80%) of K-8 teacher preparation programs include a minimum of two one-semester mathematics content courses. Nearly all (90%) of these courses are taught by the mathematics department faculty/staff, including mathematicians, adjuncts, and graduate students (Greenberg & Walsh 2008; Lutzer et al. 2007; Masingila et al. 2012), who often do not have formal training in mathematics education or teacher preparation, nor do they have experience teaching mathematics to schoolchildren (Bass 2005; Sztajn et al. 2006).

THEORETICAL FRAMEWORK

The construct *orientations* was formally introduced in late 1980s, when Grossman (1987), in her analyses of novice English teachers, found that despite very little differences in their content knowledge, teachers' orientations toward literature differed markedly and, as a result, influenced their decisions regarding the selection of content for instruction and how to teach that content to students. Grossman, Wilson, & Shulman (1989) further articulated this concept as teachers' *orientations toward the subject matter*, encompassing beliefs about subject matter and discipline rooted in teachers' conceptions of the content "mirroring competing substantive structures" of the discipline and subject matter, which teachers typically acquire in their "undergraduate and graduate coursework" (Grossman et al. 1989, pp. 29-31). Anderson and Smith (1985) additionally noticed "general patterns of thought and behaviour" that reflected teachers' *orientations toward teaching the subject*, which is different from *orientations toward the subject matter* (Anderson & Smith 1985, p. 99). Kuhs and Ball (1986) articulated that *orientations toward teaching mathematics* involve "ideal" images of mathematics teaching and learning. For example, teachers may adopt a "calculational" orientation and perceive mathematics as the activity of getting answers, but they may have difficulty focusing instruction around students' reasoning and "may consider such a focus as being irrelevant to what mathematics is about" (Thompson et al. 1994, p. 9). In contrast, teachers who adopt a "conceptual" orientation are more likely to engage learners in extended explorations and deliberations about mathematical concepts, and view such a focus as an integral part of mathematics teaching (Thompson et al. 1994). Based on these literature recommendations, we explored "orientations" within two domains: *orientations toward the subject matter* and *orientations toward teaching the subject*. We defined *orientations toward the subject matter* as learning opportunities that support PTs' development of orientations toward mathematics, specifically the nature of mathematics as a discipline and orientations about mathematics rooted in their conceptions of the content (e.g., Grossman 1987; Grossman et al. 1989). We defined the learning opportunities that support PTs' development of specific instructional images, characteristics, and strategies for teaching mathematics as *orientations toward teaching the subject* (Anderson & Smith 1985; Kuhs & Ball 1986).

METHODOLOGY

We conducted this study from the phenomenography perspective (Marton 1981).

Phenomenography, adopts an empirical orientation to investigate the perceptions, conceptions, and experiences of others regarding the phenomenon, and suggests utilizing interviews as primary data collection procedures for qualitative studies involving smaller participant samples (Marton 1981). Participants included a group of ten *expert* MTEs (6 males; 4 females), from five different institutions in the Eastern portion of the United States. We define *expert* as: a) having at least a Master's degree; b) having at least 20 years of combined K-12 teaching experience and teaching mathematics content to PTs; and c) being professionally active by attending/presenting at local, state, and national professional meetings. All participant names were replaced by pseudonyms. Data for the project was gathered through two one-hour semi-structured interviews (one at the beginning and one at the end of the semester). Interviews were audio-recorded, transcribed, and coded using constant comparison analysis (Corbin & Strauss, 2008). During the first interview, participants were asked about their educational background, and intended goals for PT's learning (including goals not included in the syllabi). In the process of coding the initial interviews (and preparing for the second interview), we found a number of common "orientations" codes across the entire participant sample. We used the second interview to follow-up on these common "orientations" codes to ask MTEs to elaborate on these codes by providing examples from their classrooms. At no point during the study (or data collection) was the term "orientations" specifically referenced or explicitly used with participants.

RESULTS

Data analyses across the entire sample of participants revealed six overarching coding themes (and 36 total codes) for both categories: orientations towards mathematics and orientations towards teaching mathematics. Due to limited page space (and qualitative nature of our study), we briefly elaborate on these findings, including direct representative quotes from the data. We also provide detailed summaries of the codes and themes for each category in Table 1 and Table 2.

Orientations Towards Mathematics

One of the central goals, articulated by the expert MTEs in our study, was to develop deep, profound, and conceptual mathematical knowledge to help PTs "understand mathematics more deeply and get far beyond the procedural knowledge that they tend to have" (as described by Ethan). Similarly, Vance explained that his goal was to not only to promote deeper understanding but also to help PTs make meaningful connections between procedures, formulas, and concepts by "building on their basic skills to develop deeper understanding and richer connections between math topics". Expert MTEs mentioned that they wanted PTs to learn to explain mathematics more conceptually, using precise mathematical language, including their ability to generate conjectures, pose thoughtful questions, and be more curious. Ingrid shared an example from her classroom on how she fosters inquisitiveness in her PTs during geometry:

Getting them [PTs] to form conjectures about things. I ask, "What do you think might be happening? Why?" Getting them to think rather than just to accept. I also want them to

"see" the [mathematical] language. For example, in geometry, I give them directions and characteristics. I use different terms and language.

MTEs indicated that most PTs come to a realization (and appreciation) that fundamental mathematics topics might not be as easy as they thought, and that PTs need a profound understanding of these topics, "I want them to come to terms with the math they know, the math they think they know, and the math they need to know" (as described by Vance). Expert MTEs also indicated that it was very important for them to help PTs develop specific orientations toward the nature of mathematics as a subject. For example, many of them explicitly mentioned that they want PTs to leave the course "with a deeper understanding and a greater appreciation for mathematics" (as described by Odessa). Also help PTs "see" mathematics "as a way of thinking" (as described by Trina) and regard mathematics as more of an "active and engaging subject, as opposed to one that is presented and practiced" (as described by Vance). Furthermore, expert MTEs shared that they often try to find ways to provide support for PTs in helping them gain more confidence about their mathematical skills, abilities, competence, as well develop more positive attitudes towards mathematics. We noticed a common data pattern indicating that expert MTEs are familiar with their learners' population, their backgrounds, and are cognizant of PTs' mathematical anxieties and uncertainties. For example, Ethan mentioned, "Most of them have had bad experiences in mathematics. I even take a little poll the first day - almost always, overwhelming majority. A goal of mine is to change that". Similarly, Ingrid shared that it's a "good day" when "someone, who came in this class nervous and worried, all of a sudden blossomed. When that happens it's a good day". Overall, when coding data for orientations towards mathematics, three overarching themes and 15 codes were commonly identified across the entire sample of participants. These results are summarized in Table 1: Teacher Educators' Goals for Developing PTs' Orientations Towards Mathematics

Orientations Towards Teaching Mathematics

MTEs shared one of the main goals was to help PTs better prepare for teaching mathematics to students. MTEs specifically articulated that they wanted PTs to develop a strong knowledge of the mathematics they will teach, particularly understanding these topics more conceptually and pedagogically, learning the physical and concrete representations for these topics. For example, Trina stated: We use manipulatives to explore mathematical representations, multiple solutions, and multiple methods. I really emphasize the visual power of mathematics and how it can be derived from hands-on materials. While describing the use of manipulatives, expert MTEs also provided insights into other learning processes and activities that they embed into their content courses, including specific ways the MTEs want PTs to view mathematics learning. For example, MTEs articulated that they want PTs to recognize that "there could be many different ways to get to the solution" (Ella) and that "solutions don't come up right off, and it may go into the second day [of class]" (Trina), and that MTEs

require PTs to constantly explain and share their thinking during class, verbalize their solutions, learn to take insightful notes, and develop meaningful questions.

Overarching Themes:	Common Codes:
MTEs want PTs to know more mathematics than what PTs will teach, and understand math in a connected, conceptual way	<ul style="list-style-type: none"> • know more mathematics than what they will teach • know more than what they came in with [to class] • understand connections across mathematics • have a good grasp on fundamental concepts, be able to explain them • be able to generate conjectures, questions, and inquiries • learn and develop mathematical language • come to terms with the math they [PTs] know, math they think they know, and math they need to know
MTEs want PTs to appreciate mathematics, “see” its nature differently and distinctly	<ul style="list-style-type: none"> • math is useful and meaningful - it is more than computations, manipulations, procedures • appreciate that math is useful and meaningful • math is a way of thinking • math is problem-solving involving flexible-thinking and reasoning • math as an active and engaging subject
MTEs want PTs to gain more confidence and positive attitude toward mathematics as a subject	<ul style="list-style-type: none"> • get excited and have a better attitude about math • enjoy and gain confidence for doing mathematics • feel better about own math skills, competence, and understanding

Table 1: Summary of Results for Orientations Towards Mathematics

One of the emerged data patterns also indicated that MTEs deliberately challenge PTs’ thinking and embed opportunities that require PTs to persevere in problem-solving and learn to be more patient during mathematical learning. For instance, Ian noted, “They [PTs] are unwilling to persevere in thinking through a mathematics problem. If they get frustrated they don’t ask follow-up questions”. In describing PTs’ learning, MTEs also elaborated on the teaching strategies they employ during class and the specific views about teaching mathematics that they want PTs to develop as part of their own practice. MTEs articulated that they actually insert challenging problems and cognitive dissonance opportunities for PTs to experience during their content courses, and they encourage PTs to do the same with their future school students. For example, Ethan called these “teachable moments” and shared that he “always assign problems that have a lot of thought provoking non-routine problems”. Similarly, Odessa stated, “I give them problems that push their limits of understanding of the math, having them struggle with it and explore with each other, share how they got through that struggle”. MTEs also shared that they want PTs to learn to teach mathematics from a problem-solving perspective, using small-groups and allowing students to collaborate.

Overall, three overarching themes and 21 codes commonly emerged across the entire sample of participants for orientations towards teaching mathematics. These results are

summarised in Table 2. Teacher Educators' Goals for Developing PTs' Orientations Toward Teaching Mathematics.

Overarching Themes:	Common Codes:
MTEs want PTs to know the mathematics PTs will teach in a more pedagogical and coherent way	<ul style="list-style-type: none"> • understand mathematics in a way that's more effective for teaching – be able to explain “why”, provide physical and concrete models • understand the content deeper. more conceptually, than their school experiences • gain confidence in teaching math conceptually • understand K-8 “big ideas” and learn to teach them in ways that connect with children’s thinking
MTEs want PTs to develop specific views about mathematics learning	<ul style="list-style-type: none"> • accept that there are multiple ways to get to the solution • accept that solutions don’t come up right off, and may take another day of class • persevere in thinking and problem-solving • ask questions when frustrated • verbalize and write down ideas • learn to explain and what it means to explain • use math language
MTEs want PTs to develop specific views about mathematics teaching	<ul style="list-style-type: none"> • teach math from a problem-solving perspective • use small-groups for problem solving, whole-groups to share • use manipulatives and models • create environment for students to collaborate, discuss, and build on their thinking • don’t let your dislike for math rub off on your students • Demonstrate how one is a learner and doer of math by showing problem-solving and critical thinking skills • pose good questions, give good explanations, use multiple explanations, correct language • move beyond what you know as 20+ year old, think about the learning that children need • implant challenges and cognitive dissonance

Table 2: Summary of Results for Orientations Towards Teaching Mathematics

CONCLUDING REMARKS

Our findings indicate that expert MTEs provided ample opportunities for PTs to develop mathematics-related orientations. Specifically, expert MTEs offered PTs opportunities to develop *conceptual orientations towards teaching mathematics*, focusing on reasoning, explorations, and discussions, and considering such a focus as an integral part of mathematics instruction (Thompson et al. 1994). Additionally, expert MTEs provided PTs with opportunities to develop specific *orientations towards mathematics as a discipline*, primarily reflecting the problem-solving conception, embodying mathematics as a continuous process of inquiry that always remains open

to revision and improvement (Ernest 1991, p. 250). Research shows that (in the United States) the majority of content courses are taught by the mathematics faculty/staff, who mainly engage PTs in lectures, with only 13% of the institutions reporting inquiry-based instruction (Masingila et al. 2012; Lutzer et al. 2007). Our findings show that classroom practices of expert MTEs differ from these (documented) practices. Therefore, given that “too often, the person assigned to teach mathematics to elementary teacher candidates is not professionally equipped to do so” (Greenberg and Walsh 2008, p. 46), we provide Table 1 and Table 2, as practitioner tools, for other MTEs to use to examine their practice. Furthermore, current theoretical models for “orientations” are scarce (Appova & Taylor 2017). Thus, our classifications (Table 1 and Table 2), although may not be exhaustive, provides the foundation for further research on the nature of mathematics content courses, the work and practices of MTEs, and the learning opportunities afforded to PTs that directly translate to their mathematics-related orientations.

References

- Appova, A. & Taylor, C. (2017). Expert mathematics teacher educators' purposes and practices for providing prospective teachers opportunities to develop pedagogical content knowledge in content courses. *Journal of Mathematics Teacher Education*. Published online: doi.org/10.1007/s10857-017-9385-z
- Anderson, C. W. & Smith, E. L. (1985). Teaching science. In J. Koehler (Ed.), *The educator's handbook: A research perspective* (pp. 84-111). New York: Longman.
- Bass, H. (2005). Mathematics, mathematicians, and mathematics education. *Bulletin of the American Mathematical Society*, 42(4), 417–430.
- Bergsten, C., & Grevholm, B. (2008). Knowledgeable teacher educators and linking practices. In B. Jaworski & T. Wood (Eds.), *The international handbook of mathematics teacher education* (Vol. 4, pp. 223-246). Rotterdam, The Netherlands: Sense Publishers.
- Blömeke, S., Buchholtz, N., Suhl, U., & Kaiser, G. (2014). Resolving the chicken-or-egg causality dilemma: The longitudinal interplay of teacher knowledge and teacher beliefs. *Teaching and Teacher Education*, 37, 130-139.
- Cobb, P., Wood, T., & Yackel, E. (1991). A constructivist approach to second grade mathematics. In *Radical constructivism in mathematics education* (pp. 157-176). Springer Netherlands.
- Corbin, J., & Strauss, A. (2008). *Basics of qualitative research: Techniques and procedures for developing grounded theory*. Sage Publications: Thousand Oaks, CA.
- Ernest, P. (1991). Mathematics teacher education and quality. *Assessment and Evaluation in Higher Education*, 16(1), 56-65.
- Even, R. (2008). Facing the challenge of educating educators to work with practicing mathematics teachers. In B. Jaworski & T. Wood (Eds.), *The international handbook of mathematics teacher education* (Vol. 4, pp. 57-73). Sense Publishers.
- Ghousseini, H., & Herbst, P. (2016). Pedagogies of practice and opportunities to learn about classroom mathematics discussions. *Mathematics Teacher Education*, 19(1), 79-103.

- Greenberg, J., & Walsh, K. (2008). *No common denominator: The preparation of elementary teachers in mathematics by America's education schools*. Washington, DC: National Council on Teacher Quality.
- Grossman, P. L. (1987). *A tale of two teachers: The role of subject matter orientation in teaching*. Stanford University.
- Grossman, P. L., Wilson, S.M., & Shulman, L.S. (1989). Teachers of substance: Subject matter knowledge for teaching. In M. Reynolds (Ed.), *Knowledge base for the beginning teacher* (pp. 23–36). Oxford, England: Pergammon Press.
- Grossman, P., Hammerness, K., & McDonald, M. (2009). Redefining teaching, re-imagining teacher education. *Teachers and Teaching*, 15(2), 273-289.
- Jaworski, B. (2008). Mathematics teacher educator learning and development: An introduction. In B. Jaworski & T. Wood (Eds.), *The international handbook of mathematics teacher education* (Vol. 4, pp. 1-13). Sense Publishers.
- Kaur B., Kwon O., Leong Y. (2017). *Professional Development of Mathematics Teachers. Mathematics Education – An Asian Perspective*. Springer, Singapore.
- Kazemi, E., Lampert, M., & Franke, M. (2009). Developing pedagogies in teacher education to support novice teacher's ability to enact ambitious instruction. In R. Hunter, B. Bicknell, & T. Burgess (Eds.), *Crossing divides: Proceedings of the 32nd annual conference of the Mathematics Education Research Group of Australasia* (Vol. 1, pp. 12-30). Palmerston North, NZ: MERGA.
- Kuhs, T. M., & Ball, D. L. (1986). Approaches to teaching mathematics: Mapping the domains of knowledge, skills, and dispositions. *East Lansing: Michigan State University, Center on Teacher Education*.
- Lutzer, D. J., Rodi, S. B., Kirkman, E. E., & Maxwell, J. W. (2007). *Statistical abstract of undergraduate programs in the mathematical sciences in the United States*. Washington, DC: American Mathematical Society.
- Marton, F. (1981). Phenomenography—describing conceptions of the world around us. *Instructional Science*, 10(2), 177-200.
- Masingila, J. O., Olanoff, D. E., & Kwaka, D. K. (2012). Who teaches mathematics content courses for prospective elementary teachers in the United States? Results of a national survey. *Journal of Mathematics Teacher Education*, 15(5), 347-358.
- Sztajn, P., Ball, D. L., & McMahon, T. A. (2006). Designing learning opportunities for mathematics teacher developers. In K. Lynch-Davis & R. L. Rider (Eds.), *The work of mathematics teacher educators: Continuing the conversation* (pp. 149–162).
- Thompson, A. G., et.al. (1994). Computational and conceptual orientations in teaching mathematics. In A. Coxford (Ed.), *Professional Development for Teachers of Mathematics. Yearbook of the NCTM*. (pp. 79-92). Reston, VA: NCTM.
- Windschitl, M., Thompson, J., Braaten, M., & Stroupe, D. (2012). Proposing a core set of instructional practices and tools for teachers of science, *Science Education*, 96(5), 878-903.

BIG BLOCKS OF PROOF

Amir Hossein Asghari^{1,2}, Sharareh Taghi Dastjerdi²,
and Fatemeh Ahmadpour Mobarakeh^{2,3}

¹Liverpool John Moores University, ²Isfahan Mathematics House,

³Shahid Bahonar University of Kerman

In the present study, 193 groups of three students in grades 4 to 6 were assigned a proof-based problem in the field of number theory. The written responses were analysed. Not surprisingly, the analysis showed that the majority of them relied on examples to ‘prove’ the given statements. However, there was some variation in the ways that examples had been used. Considering the observed variation, 18 students whose proofs were somehow different from each other were invited and interviewed individually for finding more details about their performances. None of them were able to produce accurate formal proofs. However, their performances had an important similarity to mathematicians’.

INTRODUCTION

In the last two decades, there has been a growing and widespread consensus on the importance of learning proof among mathematics educators. Concurrently, educational policy makers and curriculum developers has put an emphasis on a certain level of proof in all students’ mathematical experiences, even in the elementary schools (Mariotti, 2006, cited in Stylianides, Bieda & Morselli, 2016; National Council of Teachers of Mathematics, 2000, 2014). Proof can be understood as a process of justifying a general statement by any possible mean. As such, the verification by examining a few cases is frequently seen among children and even adults (Reid & Knipping, 2010; Biehler & Kempen, 2013; Lynch & Lockwood, 2017). This is also common in mathematicians (Lynch & Lockwood, 2107), although unlike most students, they are aware of the temporal nature of verification by examples (Weber & Mejia-Ramos, 2011; Mejia-Ramos Fuller, Weber, Rhoads & Samkoff, 2012; Lynch & Lockwood, 2017).

This study addresses the differences and similarities in the students’ and mathematicians’ uses of examples in the process of proving. Such similarities, if there is any, can be brought to the fore and emphasized in teaching proof. The research questions are:

How do elementary school students use examples in proving? And does there exist any similarities in the ways that students and mathematicians argue?

LITERATURE REVIEW

A proof is “a connected sequence of assertions for or against a mathematical claim” (Stylianides, 2007, p. 291). Albeit in the most cases, the process of proving does not

begin by presenting a chain of propositions, but by examining examples, which satisfy or disprove the claim. Research has shown that these examples play a rather different role for mathematicians than the role they play for most students. Mathematicians apply examples for:

1. Attaining intuition about what they want to prove (or refute) (Michener, 1978).
2. Verifying the middle assertions, temporarily, in a long proof for gaining a holistic understanding of the proof (Weber & Mejia-Ramos, 2011; Mejia-Ramos et al., 2012).
3. Finding or seeing the structure of proof, occasionally (Sandefur, Mason, Stylianides & Watson, 2013).

In contrast, the way of using examples by students and even student teachers is ‘simpler’. They apply examples in order to attain the ultimate justification of the given statement (Biehler & Kempen, 2013; Knuth, Choppin & Bieda, 2009). Different names given to this usage of example indicate the extent of its use: ‘empirical arguments’, ‘naïve empiricism’, ‘experimental proofs’, ‘empirical proof scheme’, ‘inductive reasoning’ (Reid & Knipping, 2010).

In general, it seems that the known differences in how and why students and mathematicians use examples in proving is much more than their similarities. The main goal of this study is to find probable similarities.

METHODOLOGY

The present study adopted a phenomenographic approach (Marton & Booth, 1997) to examine the elementary school students’ conceptions of examples in the process of proving. The focus of the study was mainly on the overall variation in the students’ conceptions, rather than the conception of individual students (Asghari, 2007). The research was organized in two phases. In the first phase, written responses of 193 groups of three students were analysed. The students were 4th to 6th grader, and answered one of following two problems:

Ali says if anyone gives me three whole numbers, I can add two of them and get an even number. How do you prove Ali’s claim? (64 groups of three students in grade 4 answered to this question).

Ali says if anyone gives me three whole numbers, I can choose two of them such that their sum is divisible by 2. How do you prove Ali’s claim? (89 and 40 groups of three students respectively in grades 5 and 6 answered to this question).

A multi-level model was developed based on coding students’ written responses (see Figure 1 in the next section). In the second phase of the study, 18 students who also participated in the first phase were interviewed. The aim was to find different types of conceptions; therefore, they were selected in maximum variation sampling. Samples were from different grades and their responses belonged to different levels of the model. After about two to three months, they individually answered the same problem,

also were asked to think loudly about how they thought. Then they were interviewed to explain their responses. The audiotaped semi-structured interviews were transcribed, and then coded according to theoretical coding (Strauss and Corbin, 1998).

FINDINGS

The developed model does not only show the different roles of examples in the process of proving, but also reveals the student's understanding about the nature of proof (Figure 1).

Then, 18 students were interviewed with the aim of finding more details about the levels of the model and the role of examples in proving. The results confirmed the model and revealed more details about the students' way of thinking as well as similarities between mathematicians and students when they use examples in proving.

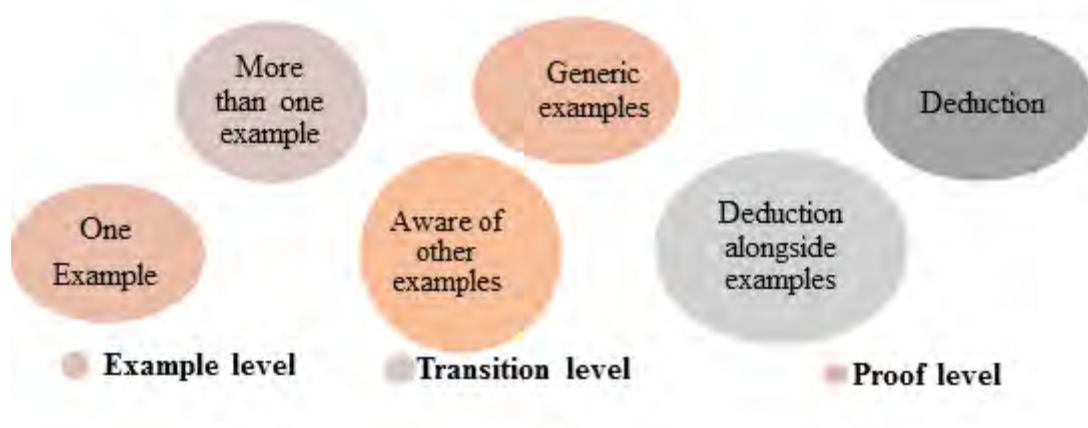


Figure 1: A model for the statuses of examples in proving. The figures are schematics.

THE ROLE OF EXAMPLES IN THE PROVING PROCESS

Figure 1 shows example-based 'proofs' on one side of the spectrum, and deductive proofs on the other side. However, unlike what Figure 1 might imply the two ends are not separated from each other. In fact, as the following descriptions of the levels shows, the levels are our attempt to understand the complexity of students' understanding of the status of examples in proving.

Example level

In this level, students verify the claim by examining a few cases (We, and in fact, the literature, have already pointed out the spread of this level).

Transition level

The students' answers are categorized in two groups.

Aware of other examples: students, in this level, are aware that the examples they have argued with are not the only possible examples. In a way, they are aware that others might choose some other examples; nevertheless, they are happy with their own choice as a means of proof (and not the means of proof). For instance, a group of 3 students in

six grade gave 6 examples and then wrote “there exists other numbers, but we somewhat prove the claim of Ali”. These elementary school students had not received any formal teaching in regard to the notion of proof. They also did not have the algebraic experience that could be handy for tackling such unfamiliar problems. Thus, it was predictable that most of the students would not give a fully-fledged mathematical proof. However, it is worth emphasising that these participants realized that their own examples are not the only examples that might be used. From a teacher perspective, such awareness might be considered as one of the first steps towards understanding proofs. Thus, we have called it the transition level to emphasis its importance rather than to claim that these group of students moved, with no return, to something that might be considered as a higher level than the example level.

Generic examples: In this category, the other examples are not just any examples; the fall into some classes represented by some other examples. As an example, we have chosen a fifth grader, Helia. She wrote 4 groups of numbers: 8-7-6, 8-12-6, 5-14-17, 13-15-17 and for each group, she showed that there are two numbers whose sums are even. On the surface Helia’s answer may be considered in the example level, but interview with her showed that she had categorized numbers in two distinct sets (even and odd numbers); and she had considered 4 possible types of choosing numbers from these two sets: even-even-odd, even-even-even, odd-even-odd, and odd-odd-odd.

Helia: In these 4 [sets of] cases, I calculated even numbers, odd numbers and also even and odd numbers; and it was divisible by 2.

Then, Helia added that: “if my numbers change, this is still true”. It shows that her choices did not depend on any specific numbers, and regarding the evenness or the oddness of her 3 numbers, she could consider them in one of the 4 types of classes. After a few questions and answers, she said: “because the sum of two even [numbers], is an even [number], and the sum of two odd [numbers] is also an even [number]”.

Though it might seem repetitive, again we should emphasise that the students who used generic examples did not necessarily move, with no return, to a higher level. (Follow Helia below.)

Proof level

At this level, students are able to produce a proof. Nonetheless, examples more or less have the same status with deductive proofs as a means to justify the claims. For instance, the answer of a group of 3 students in grade six was:

There are 4 types for choosing these 3 numbers. First type: even, even, & even. In this case, Ali chooses 2 evens that give an even. Second type: odd, odd, and odd. Ali obtains an even by choosing 2 odds. Third type: even, even, and odd. Ali obtains an even by choosing 2 evens. Forth type: odd, odd, and even. Ali obtains an even by choosing 2 odds. All even answers are divisible by 2, and in this way the [claim of] problem was proved.

Deduction alongside example(s): As Knuth et al. (2009) mention, some secondary school students think examples are substantial for better understanding of their proof.

This was also the case for some of our elementary school students. For example, Atria, a student in the fourth grade, in her interview said: “It is better if there exist an example the meaning of this sentence could be understood better”. Some others corresponded a proof to the “final answer” for a problem. For instance, Zahra, a student in the fifth grade, told that she must write some examples along her proof because otherwise “my teacher thinks I cheated, because she knows that if I know the answer, I must write the solution”.

The deduction: Some students think their proof is adequate and they do not need to give an example. However, it is not necessarily because they think giving examples would be inadequate. For instance, Aria, a student in the fifth grade, thought that it is optional to give or not to give an example along the proof, and when the interviewer asked him to evaluate the answers given by two other students, he gave the complete score to both, one was with an example and the other without, saying that: “because they have proved their claims”.

The instances presented above show different ways of example usage in the process of proving. The individual examples given was to clarify the points made, not to distinguish between students. Of course, some of them created better arguments occasionally. But, all of the 18 interviewees, only relied on examples at times. And, more importantly, none fully distinguished the status of examples and proofs as a means of justifying the claims. Yet, surprisingly, some of them showed a kind of performance quite similar to mathematicians.

SIMILARITIES BETWEEN MATHEMATICIANS’ AND STUDENTS’ USE OF EXAMPLES IN PROVING

As far as the differences are concerned, the results of this research are in agreement with previous studies: unlike the mathematicians, most students cannot distinguish between the generality obtained by example(s) and the generality obtained by a deductive reasoning (Lynch & Lockwood, 2017). However, as far as the similarities are concerned, one of the findings of these research is quite distinctive.

As we mentioned above, mathematicians sometimes apply examples for temporary justification of middle propositions in the process of a long proof and for a better understanding of the general structure of the proof. In written proofs, these middle propositions are often called *lemma*. The major purpose of them is to reveal the structure of the proof. Mathematicians are well aware that each of them needs a deductive proof on its own. Students do not necessarily have such an awareness. But, it does not stop them to use certain middle propositions which are like **big blocks of proof**, in the process of their reasoning. As an example, let us see another part of the interview with Helia. In her proof, she used the assumption that “the sum of two even numbers is even, and also the sum of two odd numbers is even”. When the interviewer asked her to argue for the correctness of her assumption, she verified the first part (the sum of two even numbers is an even number) by examples. Although, Helia used examples for the sum of two

even numbers, for demonstrating the truth of the other part (the sum of two odd numbers is an even number) tried to argue deductively:

Helia: Odd numbers well... if we want to divide them by 2, we obtain a number with a 0.5, and when we get two 0.5, by adding these two numbers, we get a complete number.

In fact, Helia used deductive reasoning as far as possible, and whenever she did not have proper resources for convincing herself or others she used examples. Although Helia could not necessarily distinguish between ‘proof’ by examples and deductive proof, she completed her proof with the aforementioned proposition, as a big block of the structure of her proof. In the interview, the interviewer allowed Helia to complete her proof with the assumption that the statement is true, instead of pushing her to argue more.

As another example, let us consider Romina, a fifth-grade student. After examining some examples, Romina concluded:

Romina: Well, certainly, in [three] numbers, if we add two of them, two of them are even or the answer is even. For example, I choose 55, 93, and 1. I add these two [55 and 93] numbers, which are not even ... it equals 148. 148 is an even number ... it happens when the addition is even or two of our three numbers are even numbers.

When the interviewer questioned her “how do you know the addition of two odd numbers is an even number”, she said: “well, we checked it, and we saw that these things I said are true”. She, then, was confronted with the question “how do you guarantee your claim is true for other odd numbers”. Romina responded that:

Romina: Because these numbers that I choose are random numbers somewhere between 1 to 100, and we saw that it was true.

Even, the answers placed in the proof level in the first stage of the study were not exempt from examples. For instance, the answer of Matin, Moein, and their teammate (who were in the grade six) seems to be a complete proof, but in the individual interviews with Matin and Moein, both of them, utilized examples to indicate the truth of statements such as “the sum of two even numbers is an even number”.

Matin: I choose 1, 2, and 3. Well, 1 plus 3 is 4. 4 divided by 2 is 2. Now, if the numbers were 1, 3, 5, again 1 plus 3 is 4. 4 divided by 2 is 2. Now, if our three numbers were even, 2, 4, 6, then, 4 plus 2 is 6. 6 is divisible by 2... I understand that the sum of two odd numbers is even, and the sum of two even numbers, is also even.

Helia, Romina, Matin, Moein and some other students showed that they often use such statements in their argument chain, without knowing the exact proof of these statements or even being aware of the necessity of proving them. But, they could use these big blocks of proof in a proper way. For our second problem, these blocks are shown in Figure 2.

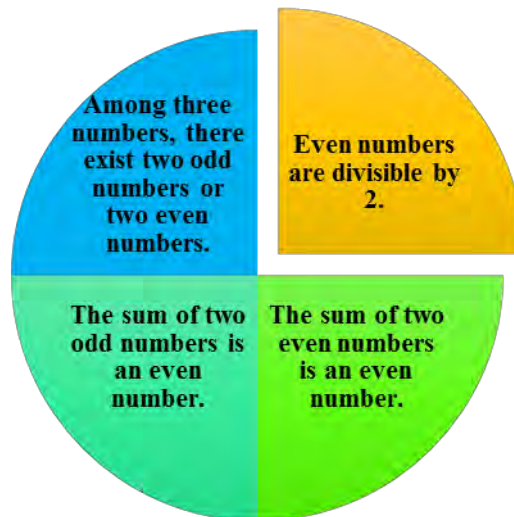


Figure 2: The big blocks of proof for Ali's claim.

CONCLUSION

The current study confirmed the previous research on the differences between students' and mathematicians' uses of examples in the process of proof. However, our study also revealed an important and potentially useful similarity between the two. Students often apply big blocks (lemmas in the sense of mathematicians) for developing a chain of arguments. This does not necessarily mean that they are able to prove these blocks or are even aware that they need to be proved. Therefore, instead of asking students to prove each of these big blocks from the beginning, it is better to postpone them for an appropriate time. Depending on the type of problem and the student's proving ability, the appropriate time can be right after presenting the proof of the problem at hand, or in the subsequent years, when the proper resources are available. So, it is not necessary to postpone elementary school students' learning of argumentation and proof skills until secondary school. True, they are not able to provide accurate, complete deductive proofs. Yet, by using the proper blocks, a certain level of proof could be accessible to the elementary school students.

References

- Asghari, A. H. (2007). Examples, a missing link. In Woo, Jeong-Ho (ed.) et al., *Proc. 31st conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 1-4 pp. 25-32). Seoul, Korea: PME.
- Biehler, R., & Kempen, L. (2013). Students' use of variables and examples in their transition from generic proof to formal proof, In: B. Ubuz, C. Haser & M. A. Mariotti (Eds.), *Proc. 8th Cong. of the European Society for Research in Mathematics Education*, Ankara: Middle East Technical University, pp. 86-95.

- Knuth, E. J., Choppin, J., & Bieda, K. (2009). Middle school students' production of mathematical justifications. In D. Stylianou, M. L. Blanton & E. J. Knuth (Eds.), *Teaching and learning proof across the grades: A K-16 perspective* (pp. 153-170). New York: Routledge.
- Lynch, A. G., & Lockwood, E. (2017). A comparison between mathematicians' and students' use of examples for conjecturing and proving. *Journal of Mathematical Behavior*, <http://dx.doi.org/10.1016/j.jmathb.2017.07.004>.
- Marton, F., & Booth, S. (1997). *Learning and Awareness*. Mahwah: LEA.
- Michener, E. R. (1978). Understanding understanding mathematics. *Cognitive Science*, 2, 361–383.
- Mejia-Ramos, J. P., Fuller, E., Weber, K., Rhoads, K., & Samkoff, A. (2012). An assessment model for proof comprehension in undergraduate mathematics. *Educational Studies in Mathematics*, 79(1), 3-18.
- National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.
- National Council of Teachers of Mathematics (NCTM). (2014). *Principles to actions NCTM, Ensuring Mathematical Success for All*. Reston, VA: NCTM.
- Reid, D., Knipping, C. (2010). *Proof in mathematics education: Research, learning and teaching*, Rotterdam: Sense.
- Sandefur, J., Mason, J., Stylianides, G. J., & Watson, A. (2013). Generating and using examples in the proving process. *Educational Studies in Mathematics*, 83, 323-340.
- Strauss, A., & Corbin, J. (1998). *Basics of qualitative research: techniques and procedures for developing grounded theory*. Thousand Oaks, CA: Sage. Translated into Persian.
- Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education*. 38: 289–321.
- Stylianides, A. J. Bieda, K. N., & Morselli, F., (2016). Proof and argumentation in mathematics education research, In Á. Gutiérrez, G. C. Leder & P. Boero (Eds.), *The Second Handbook of Research on the Psychology of Mathematics Education* (pp. 315-351). Sense Publisher, pp. 315-351.
- Weber, K., & Mejia-Ramos, J. P. (2011). Why and how mathematicians read proofs. *Educational Studies in Mathematics*, 76, 329–344.

MIDDLE GRADE STUDENTS' PERFORMANCE ON ARITHMETIC CALCULATIONS PRESENTED AS WORD PROBLEMS OR NUMERIC PROBLEMS

Mike Askew and Hamsa Venkat

Wits School of Education, University of the Witwatersrand

This paper examines South African middle grade students' performance on problems set either within a word problem format or a purely numeric format. Students across Grades 4, 5 and 6 in suburban and township schools answered, as part of a written assessment, four word problems – two involving additive relations and two multiplicative – and also four matched numeric calculations. Analysis of differences in performance on items and formats indicate that, for these students, there was little evidence to support the claim that students find it more difficult to solve problems when presented in word problem format. The findings show that while both formats presented challenges, students displayed a preference for one or other of the formats.

INTRODUCTION

Are calculations easier to carry out when presented in a context (word problems) or in a purely symbolic format (numeric problems)? Popular opinion has it that students find word problems harder to solve than numeric problems. In our professional development, teachers often comment: 'Ah, learners can do a calculation, but putting it into context confuses them'. This perception is held not only by teachers: a survey of 35 mathematics education researchers found that the majority expected word problems to be more difficult than matched equations (Nathan & Koedinger, 2000). In South Africa, working with word problems is writ large within the curriculum but evidence shows that students at all levels find them difficult. Other evidence, however, indicates that word problems are not necessarily more difficult than numeric problems and that there are, in the words of Koedinger, Alibali and Nathan (2008), 'trade-offs' between representations whether grounded (word problems) or abstract (purely numeric).

Much of the research on word problems has focused either on the early years of primary or on secondary students. This paper begins to bridge the gap between these by examining South African upper primary students' performance on problems presented either in context or numerically. We also examine whether the type of mathematical relation in the problem – additive or multiplicative – has any effect on performance. The students involved in the study came from schools serving historically disadvantaged learner populations, with a history of low attainment. The findings not only have significance for raising standards in South Africa but also for contributing to a more nuanced understanding of the role of word problems in teaching and learning mathematics.

LITERATURE REVIEW AND THEORETICAL BACKGROUND

Much empirical data indicates that word problems are more difficult than numeric problems. For example, Cummins and colleagues (1988) found first graders' performance on story problems was considerably poorer than on matched numeric problems, concluding that 'word problems are notoriously difficult to solve' (p. 405). One argument for word problems being the more difficult is based in examining the linguistic and modelling demands that such problems present including the challenges of processing word problems' syntactic and semantic structures – difficulties exacerbated for learners for whom English is not their first language (Setati & Barwell, 2006), as is the case here. Sepeng (2014) researching in South Africa argues that 'computational errors made by learners, in particular with regard to number skills, appear to stem from the inability to use language(s) (home and/or language of learning and teaching) effectively in order to resolve problems in realistic situations' (p. 22).

Having surmounted the linguistic demands of word problems, students need to create an accurate model of the problem context, that they can then use as the basis for an appropriate mathematical model from which the answer can be calculated. Some refer to these as the comprehension and solution phases or, in the terms used by the Dutch Realistic Mathematics Education (RME) approach, the former is horizontal mathematizing, the latter vertical mathematizing (Treffers, 1991). While the expert problem solver may conflate or interweave these stages, rather than treat them sequentially, given that the solution stage is, if only tacitly, contingent upon the comprehension stage, then word problems present a double jeopardy – an incorrect horizontal model will lead to an incorrect vertical model. Further, as Hickendorff (2014) argues, even if a correct mathematical model is set up then pure computational skills need to be employed. Such arguments conclude that word problems will necessarily be harder than the same calculation presented in a pure numerical format.

A further argument for the difficulty of word problems arises from the socio-cultural expectations of students about the role of word problems (see Greer, Verschaffel, & Mukhopadhyay, 2007). The argument is that word problems are unique to the classroom, and students make assumptions about them on the basis of classroom socio-cultural norms, rather than on the mathematical content of the problems: students come to regard the problem context as unimportant and only attend to the numbers, selecting what they think is an appropriate calculation based on the numbers.

What is less clear is whether the difficulties outlined above continue as students' progress through primary school. While Cummins et al. (1988) claim that the challenges they identified in G1 would continue as students move up through school, not all studies corroborate this claim. Koedinger and Nathan (2004), for example, in their study of algebra problems found that students had more success on problems presented in story form than in the equivalent purely symbolic form. One aim of this paper is to fill the gap in the literature about middle school mathematics and whether or not the difficulties found in the early primary years do continue through.

There is also an argument that word problems can present advantages. Koedinger, Alibali and Nathan (2008) argue that problem contexts, if familiar to students, makes them easier to solve and less prone to errors. Equally they argue, that pure problems have the advantage of placing fewer demands on working memory. Thus, they argue, there is a representation-complexity trade off. Finally, much of the research into the differences focuses on only one mathematical domain at a time, for example additive relations problems or algebra: here we examine whether the type of arithmetical relation has an effect on performance. In the light of the literature, and our experience of working in schools, this paper thus sets out to address the following two research questions:

- For South African Middle Grade (Grades 5–7) historically disadvantaged students, what is the effect on performance of presenting problems in a word problem format compared to a pure numeric format?
- Do additive or multiplicative reasoning problems make any difference?

RESEARCH METHODS

The data discussed here is taken from a project – Multiplicative Reasoning in Intermediate Phase (MRIP) – that investigated the impact of a targeted intervention. The initiative was located within the Wits Maths Connect Primary project, a ten-year research and development project aimed at raising standards of teaching and learning in government primary schools. The MRIP project included student pre- and post-tests and our focus here is not on the intervention, but on data collected from the pre-test as this provides insights into student performance under typical classroom conditions.

The test comprised 19 items: 15 word problems (12 involving multiplicative relations (MR) and three additive relations (AR)) and 4 numeric problems. In the design and setting of the test we minimised, as far as possible, linguistic demands, as, while English was the medium of instruction in the middle grades in all the schools, it was a second or third language for most learners. Word problems were therefore written to keep syntactical demands as low as possible and in administering the test each problem was read out, to reduce reading demands, and any words likely to be at all unfamiliar explained to reduce non-mathematical semantic demands. This paper is focuses on four problems common across Grades 5, 6 and 7, two based in additive relations (AR) and two in multiplicative relations (MR):

AR: Pies: Corin puts out 81 pies on one tray and 19 pies on another tray.
How many pies does Corin put out all together?

Cycle: Sam cycles 112 km. Sameera cycles 99 km.
How much further does Sam cycle than Sameera?

MR: Tiles: Hamsa is counting how many tiles cover the bathroom floor.
She counts 12 rows of tiles. There are 11 tiles in each row.
How many tiles cover the floor?

Box: A company packs pencils into boxes. Each box contains 15 pencils.
How many boxes are needed to pack 195 pencils?

The matched calculations were presented as the final four calculations. Each calculation was presented horizontally, with space below for working:

$$81 + 19 = \quad 112 - 99 = \quad 12 \times 11 = \quad 195 \div 15 =$$

Setting and participants

The research participants were the students from one class in each of Grades 5 and 6 in 10 primary schools in South Africa, and 6 Grade 7 classes drawn from six of the same set of schools. All students consented to take the test, giving sample sizes of G5 – 376; G6 – 313; G7 – 221. As noted, language of instruction in all the classes was English.

Data collection and analysis

Members of the research team administered the test to ensure consistency of delivery. As noted the word problems were read out loud twice with time made available to work on finding an answer before proceeding (the four numeric problems were not read out). Students were encouraged to show their working, or, if they knew how to work out the answer mentally, to record the mathematical equation.

Student scripts were marked and coded for: correct interpretation and correct answer (including correct answer only), correct interpretation and incorrect answer, incorrect interpretation, and omitted answers. On the few instances where only an answer was provided with no working or equation, then an answer close to the correct one was interpreted as a correct interpretation. All results were entered into a spreadsheet for quantitative analysis. To compare performance on different formats and across AR and MR, students were also assigned 4 combined scores: total score on the pairs of problems in each format (i.e. two AR problems in word format; two AR problems in numeric format, and similarly for MR problems): 2 for correct answer, 1 for correct interpretation (on the word problems) but incorrect answer and 0 for incorrect interpretation, incorrect answer only or omitted.

RESULTS

Table 1 presents percentages of Grade 5, 6 and 7 students correctly solving the four word problems alongside each matched numeric problem. We observe two important results. First, performances on each problem were not dramatically different across the two formats. In 7 out of the 12 pairings (four in each grade) performance was higher when the problem was in word format compared to numeric, with this reversed in the other five pairs. Whichever was easier, differences were modest: the maximum difference was in G6 with $195 \div 15$ at 7 percentage points higher when in word format. Thus, with respect to the first research question, these results show little difference in performance whether a calculation was presented in word or numeric format.

This first finding has, however, to be interpreted in the light of the second finding that, the $81+19$ problems aside, overall facilities on items across the grades are not large. Although in every instance the results show that there were increasing facilities across the grades, even in G7 there was still considerable room for improvement. To examine

the reasons behind this we look first at the performance on the word problems and then on the numeric format problems. We then examine the relationships between these.

	81 + 19		112 – 99		12 x 11		195 ÷ 15	
	WP %	NP %	WP %	NP %	WP %	NP %	WP %	NP %
G5 n=376	75	73	34	29	18	6	6	5
G6 n=313	80	79	40	34	27	28	16	9
G7 n=221	83	84	50	52	34	38	19	22

Table 1: Percentages of students in each grade correctly answering each question as word problem (WP) or numeric problem (NP)

Students' interpretations and approaches to the word problems

Table 2 presents the data on students' interpretations of the word problems, whether their workings indicated that they were attempting to solve a mathematically appropriate model of the context, and if they were, if they got the correct answer. Not directly shown in the table are the small numbers of students omitting a question (this figure is the difference between the sum of the three figures shown and 100%).

	Pies: 81 + 19			Cycle: 112 – 99			Tiles: 12 x 11			Box: 195 ÷ 15		
	CICA %	CIIC %	II %	CICA %	CIIC %	II %	CICA %	CIIC %	II %	CICA %	CIIC %	II %
G5	75	11	13	34	11	53	18	32	49	6	23	67
G6	80	10	11	40	11	49	27	25	48	16	21	60
G7	83	8	8	50	8	36	34	18	46	19	15	57

Table 2: Percentage of students correctly interpreting the context and correct answer (CICA), correct interpretation and incorrect answer (CIIC), incorrect interpretation (II)

The first finding of note from Table 2 is the marked difference in the pattern of performance across the four problems. Across both the AR problems and all three grades the figures show that between 8 and 11% of learners could correctly interpret each context but did not follow through this interpretation to a correct answer. Thus, by Grade 7 over 90% of learners were able to correctly comprehend the addition situation, and almost 60% correctly interpret the subtraction. The two MR problems, however, present a different picture, with higher percentages of students able to correctly mathematically interpret the context but not follow through to a correct answer. When these correct interpretations are taken into consideration, the pattern of performance on Tiles is very similar to that on Cycles; around half of students in Grades 5 and 6 correctly interpreting the context and a similar proportion not interpreting it correctly. For the division problem, considerably more G5 and G6 students were able to interpret the context but not able to produce a correct answer than those who were able to do both these steps.

Students' interpretations and approaches to the numeric problems

With respect to the AR numeric pairing, even though performance on $81 + 19$ was relatively good, there was little evidence from the scripts of students recognizing that this pair is a bond of 100 or of using a compensation strategy to create the equivalent calculation of $80 + 20$. The majority chose to use the standard algorithm. Worrisome is the performance on $112 - 99$. The numbers here were chosen to be amenable to a mental strategy, yet, even in G7 only just over half the students answered this correctly. Again, the scripts reveal that the vast majority of students – whether answering this correctly or not – rewrote the calculation to be able to use the vertical algorithm.

The two numeric MR problems – 12×11 and $195 \div 15$ – are both well within the curriculum grade level expectations, the former being an expectation for attainment by the end of G4 and the latter by the end of G5, the curriculum document specifying:

G4: Multiplication of at least whole 2-digit by 2-digit numbers

G5: Division of at least whole 3-digit by 2-digit numbers (DBE, 2011)

Even taking these as end of grade level expectations, the fact that by G7 only 38% of students could accurately calculate 12×11 and 22% accurately calculate $195 \div 15$ demonstrates that performance here was well below expectations. One might expect that number sense would enable students to see that partitioning 195 into $150 + 45$ makes easy the division of this by 15. Again, students' scripts indicated that the majority attempted to use a division algorithm and did so erroneously.

Comparing performance across formats and arithmetical relations

The findings above indicate that patterns of performance differed depending on whether the problem relation was additive or multiplicative: the former being easier than the latter. In each case, the addition and multiplication problems showed better results than their inverse counterparts. To examine the effect of the type of relation—additive or multiplicative—Pearson product-moment correlation coefficients (r) were calculated for various combinations of problem types. For example, the first column of results in Table 3 presents the r values for the correlation between performance on the two additive reasoning calculations ($81 + 19$ and $112 - 99$) when in word problem format, with the two multiplicative calculations (12×11 and $195 \div 15$), in the same format. Taking as a benchmark that correlation coefficients with magnitudes above 0.7 indicate variables which can be considered as highly correlated, then the results in Table 2 show high correlations in Grades 6 and 7 between successfully answering the word problems, whether AR or MR (G6, $r = 0.77$; G7, $r = 0.74$). It also shows high correlations in these grades between successfully answering the numeric problems, whether AR or MR (G6, $r = 0.76$; G7, $r = 0.74$).

In contrast, comparing the pairs of problems within AR across both formats, there was low correlation between success on the word format problems and success on the numeric problems: (Grade 6 $r = 0.39$; Grade 7 $r = 0.41$). Similarly, correlations were low between success on the pair of MR problems in word or a numeric format (G6 $r =$

0.39; G7 $r = 0.42$). In Grade 5, correlations were low or very low across all pairings, a result requiring further investigation.

Correlations	W v W AR v MR	N v N AR v MR	W v N AR v AR	W v N MR v MR
G5 n=376	0.41	0.11	0.44	0.33
G6 n=313	0.77	0.76	0.39	0.42
G7 n=221	0.74	0.75	0.41	0.41

Table 3: Correlation coefficients (r) for combinations of word (W) and numeric (N) format and additive reasoning (AR) and multiplicative reasoning (MR)

Taken together these observations reveal in Grades 6 and 7 high correlations within formats and low correlations between formats, whether AR or MR. In other words, the format of the problem – word or numeric – has more effect on an individual's performance than whether the problem involves AR or MR.

DISCUSSION

Our evidence leads to three major findings. Firstly, the results do not support the claim that context problems would be more difficult for these students than matched numerical problems. Equally, however, the hypothesis that word problems are easier than numerical problems is also not consistent with our results. Each format presented challenges for these South African students, although the challenges differed across the formats. In the word problems, students arriving at a correct model of the context produced predominantly iconic or indexical models that remained close to being literal models of the situation. Rather than compressing these situational, horizontal, models into a mathematical model, they often worked directly on that initial model.

Our second major finding is that students successfully interpreting the word problems only made up about half of the total (the exception being the additive problem). In the case of subtraction and multiplication around half the students did not correctly set up a model of the problem context, this rising to around two-thirds of students in the case of division. Performance was no better on the numeric problems: the numbers answering these correctly were similar to those for the word problems. Given that there was no need for interpretation of which operation to apply in the numeric problems, the reasons for poor performance are different. Virtually all students re-formatted the numeric problems into vertical calculations but made errors in carrying these out. There was scant evidence of learners bringing number sense to bear on these problems.

A third, and unexpected, finding in Grades 6 and 7 at least, given the overall similar levels of performance across word and numeric formats, is that the students fall into two groups: those successful on the word problems and those successful on the numeric problems with correlations pointing to limited overlap between these two

groups. This in turn points to word problems and equivalent numeric problems being worked with, largely, as different problems. The findings here point to the need for more research not simply on whether or not the format of a problem affects performance but also on how students can be encouraged to understand that there are links to be made between these formats.

In terms of practice, the findings suggest that teaching needs to be directed towards helping students develop number sense and effective strategic means of calculating, both formally and informally, whether or not a problem is presented in a word or numeric form. They also suggest that students need to be exposed to interpreting a range of problem formats and helped to understand that the same mathematics can be represented in different formats.

References

- Cummins, D. D., Kintsch, W., Reusser, K., & Weimer, R. (1988). The role of understanding in solving word problems. *Cognitive Psychology*, 20, 403–438.
- DBE. (2011). *Curriculum and Assessment Policy Statement (CAPS): Intermediate Phase Mathematics, Grade 4-6*. Pretoria: Department of Basic Education.
- Greer, B., Verschaffel, L., & Mukhopadhyay, S. (2007). Modelling for life: Mathematics and children's experience. In W. Blum, P. Galbraith, H.-W. Henn, & M. Niss (Eds.), *Modelling and applications in mathematics education*. New York: Springer.
- Hickendorff, M. (2014). The Effects of Presenting Multidigit Mathematics Problems in a Realistic Context on Sixth Graders' Problem Solving. *Cognition and Instruction*, 31(3), 314–344.
- Koedinger, K. R., Alibali, M. W., & Nathan, M. J. (2008). Trade-Offs Between Grounded and Abstract Representations: Evidence from Algebra Problem Solving. *Cognitive Science*, 32, 366–397.
- Koedinger, K. R., & Nathan, M. J. (2004). The Real Story behind Story Problems: Effects of Representations on Quantitative Reasoning. *The Journal of the Learning Sciences*, 13(2), 129–164.
- Nathan, M. J., & Koedinger, K. R. (2000). Moving beyond teachers' intuitive beliefs about algebra learning. *Mathematics Teacher*, 93, 218–223.
- Sepeng, P. (2014). Use of Common-sense Knowledge, Language and Reality in Mathematical Word Problem Solving. *African Journal of Research in Mathematics, Science and Technology Education*, 18(1), 14–24.
- Setati, M., & Barwell, R. (2006). Discursive practices in two multilingual mathematics classrooms: An international comparison. *African Journal of Research in Mathematics, Science and Technology Education*, 10(2), 27–38.
- Treffers, A. (1991). Didactical background of a mathematics program for primary education. In *Realistic mathematics education in primary schools*. Utrecht: Freudenthal Institute, Utrecht University.

MOTIVATION AND CANING IN GHANAIAN SECONDARY SCHOOL: EVIDENCE FROM A SURVEY AND INTERVIEWS

Florence Christianah Awoniyi

University of Cape Coast, Ghana

A total of 3,342 eleventh graders from ten public Senior High Schools were engaged in an investigation of motivation for the learning of mathematics. Likert survey revealed both intrinsic and extrinsic motivation, while in the interview students mainly provided extrinsic motivation explanations. The interviewed students mentioned fear for corporal punishment as the most important demotivating issue. Interviewees' responses, observation of the lessons and the verification of the students' exercises by the researcher confirmed that some teachers either do not provide constructive feedback or provided quite a bit of harsh feedback and high stake national examination remains the impetus for the learning of mathematics in Ghana.

INTRODUCTION

Motivation has been conceptualised by Andrew (2003, p. 88) as students' energy and drive to learn, work effectively, and achieve to their potential at school and the behaviours that follow from this energy and drive. Different researchers have investigated motivation differently. According to Deci and Ryan (1985), some researchers have treated motivation as a singular construct despite the fact that people are moved to act by very different types of factors, with highly varied experiences and consequences.

Although some researchers have broadened their investigations to include various sub-constructs of motivation (e.g. Deci & Ryan, 1985; Lim & Chapman, 2014), others focus on either intrinsic motivation as the most important (e.g. Deci & Ryan, 2000a) or emphasize extrinsic motivation (Wigfield & Eccles, 2000), or a combination of both intrinsic and extrinsic motivation (Lepper, Corpus, & Iyengar, 2005). According to TIMSS (Mullis, Martin, Foy, & Arora, 2012), grade eight students in Ghana are both intrinsically and extrinsically motivated in mathematics. Their average score for liking learning mathematics (intrinsic motivation) is one of the highest in the study and their average score for utility value of mathematics is higher than any other participating country. The current study aims to explore in more detail the nature of student motivation in Ghana, targeting at upper secondary level.

Deci and Ryan (1985) classified motivation into three major categories: intrinsic, extrinsic, and amotivation but further divided extrinsic motivation into four sub-constructs: (i) external regulation, (ii) introjection, (iii) identification, (iv) integrated regulation motivation. Studies have reported that each sub-construct measures distinct

aspects of motivation and exist on a continuum, according to the level of control that individuals have over their actions (e.g. Deci & Ryan, 2000a).

When students take up challenges simply because it interests them or because they think it is good or enjoyable, their motivation is said to be intrinsic. Ryan and Deci (2000a, p. 56) define intrinsic motivation as “the doing of an activity for its inherent satisfaction rather than for some separable consequence”. Intrinsic motivation has been linked to individual interest, enjoyment and liking (e.g. Gaspard, Dicke, Flunger, Schreier, and Hafner (2015).

However, extrinsic motivation is the force behind doing something for some consequence separate from the immediate action (Wigfield & Cambria, 2010). It is driven by external forces such as rewards, punishments, praises and approval by peers, which can end in the absence of a reward or a gift. External regulation is performed because of external demand or possible reward (Deci & Ryan, 1985). Individuals would only take action in order to obtain a reward or avoid punishment and not just for fun. Extrinsic motivation that is driven by introjected regulation is driven by ego, and it is meant to maintain self-esteem or pride or to avoid guilt (Deci & Ryan, 1985).

Deci and Ryan (1985) describe regulation through identification as involving consciously valuing a goal or regulation so that the said action is accepted as personally important and strongly associated with the student’s personal goals. The student makes the benefit of the object his/her own, understands its rationale and experiences a sense of self-determination in acting in line with it. They (i.e. Deci & Ryan, 1985) define integrated regulation as that which occurs when regulations are fully assimilated with self so they are included in a person's self-evaluations and beliefs on personal needs. Because of this, integrated motivations, according to them, share qualities with intrinsic motivation but are still classified as extrinsic because the goals that are to be achieved are for reasons extrinsic to the self, rather than the inherent enjoyment or interest in the task.

Amotivated is defined as lacking sense of purpose and expectation of reward or of the possibility of changing the course of events (Vallerand & Bissonnette, 1992). According to them, amotivated individual is said to experience feelings of incompetence and expectancies of uncontrollability.

Some countries, including New Zealand, Australia, the United Kingdom and some states in the United States have recognised the deleterious effects of corporal punishment and thus have abolished it (Agbenyega, 2006). Unfortunately, corporal punishment in the form of the use of physical force with the intention of causing a child to experience pain so as to correct their misbehaviour (as cited by Straus, 2001 in Gershoff & Font, 2016), is still widely used in schools all over the world, despite being banned in national legislation in most countries (Morrow & Singh, 2014). According to them, there is national and international concern about the effects of corporal punishment upon children and its implications for their capacity to benefit from school.

In African context, it was acceptable in the homes as well as in the schools during the earliest days of formal education to use cane. Christian sayings of ‘spare the rod and spoil the child’ (Proverb 13:24), was taken to mean an approval of the practice. In Ghana the cane has been the sanction in instances of minor breaches of school and home rules. However, in recognition of Article 19 of the UN Convention on the Rights of the Child, many African countries including Ghana have put in place policies that either prohibit or regulate the use of physical discipline against children in schools (Ministry of Education, 2008). In Ghana, caning is restricted to be a punishment for severe cases of breaking school rules and it should be supervised and approved by the school principal. Nonetheless, despite the regulation of the practice, teachers still use cane routinely in schools all across the country because they believe without cane their schools will experience reduced academic standards (Agbenyega, 2006). Thus, caning as an external coercion is used to 'motivate' students to learn.

METHODOLOGY

With the approval of the school principals, the study was conducted in 10 schools in a metropolitan area in Ghana. These schools are mutually exclusive and collectively exhaustive. The metropolis is made up of 5 high performing schools, 2 average performing schools and 3 extremely poor performing schools. The 5 high performing schools originally belonged to the missionaries before government took over the schools. The population comprises 6,317 eleventh graders. From this population I randomly selected intact classes with altogether 3,342 students to participate in the study. Full quantitative data was received from 2,575 students representing 77% of the sample. Moreover, 240 students participated in the interviews. Majority of the sampled students, about 88% were between the ages of 16 to 18 years, out of which 54% were male while 46% were female. All the conventional programmes were involved in the study, with the highest representations of 34% and 28% from Science and General Arts respectively and the lowest 3% from the Technical.

The research instruments used for the study were questionnaire, interview guide and observation schedule. The self-constructed students' focus group interview guide contained five items. Observation schedule was designed based on the “University of Cape Coast College of Education Studies Teaching Practice Unit' Teaching Practice Assessment Form A”. The schedule which contained 20 items was adapted to reflect motivation concept.

The Academic Motivation Scale (AMS) questionnaire by Lim and Chapman (2014) originally contained 21 items. The researcher adapted the instrument to the Ghanaian context: new items were added and the language was made simpler to understand. Special effort was made to preserve questionnaire's underlying constructs. The final instrument had 35 close-ended items (Likert scale: ‘1’ for ‘*strongly disagree*’, ‘2’ for ‘*disagree*’, ‘3’ for ‘*neutral*’, ‘4’ for ‘*agree*’ and ‘5’ for ‘*strongly agree*’). All these instruments were reviewed by four senior colleagues and then pilot-tested. The administration of the questionnaire took place between January and April 2016, obser-

vation of lessons took place in May/June, 2016, and interviews were conducted in June 2016.

RESULTS AND DISCUSSION

A principal component analysis was conducted on the 35 items, (KMO = .953, Bartlett tests = 35549.013) and five factors had eigenvalues over Kaiser's criterion of 1 and in combination explained 51.53% of the variance. The scree plot showed inflexions that would justify retaining the five factors: 'amotivation' subscale (14 items, $\alpha = .841$), 'identified regulation' subscale (10 items, $\alpha = .885$), 'introjection regulation' subscale (5 items, $\alpha = .763$), 'external regulation' subscale (3 items, $\alpha = .666$) and 'intrinsic' subscale (4 items, $\alpha = .661$). An item 'I am doing my best in mathematics so that I can have the best grade at the national examination', factored in 3 places (external, identified and intrinsic) but was retained for external and identified subscales based on interview explanations.

For the amotivation scale, majority of the respondents (92%) strongly disagreed with the opinion that it is a waste of time studying mathematics while as many as 89% strongly disagreed that mathematics will not be important for the rest of their lives and about 76% claimed that mathematics is useful to them. This seemed to indicate that the students are generally motivated to learn mathematics. However, with about 52% respondents in strong agreement to the idea that they were studying mathematics because it is a compulsory subject in senior high school, connotes amotivation. They lack the intent to study mathematics and would be pleased to discard of it at any opportune time.

For the external regulation subscale, about 70% of the respondents believed they would need a firm mastery of mathematics in the future, while as many as 58% planned to major in a mathematics-related programme at the university and 95% agreed that they were learning mathematics in order to get a best grade in the national examination ($M = 4.58$, $SD = 0.74$). This suggests that students seemed to be guided by external regulation as well as identified regulation motivations since they favoured items from both sub-constructs. This suggests an overlap between the two thematic areas.

For the introjected regulation subscale, as many as 54% respondents revealed that they are studying mathematics because doing well in mathematics makes them feel important. Similarly, 58% respondents agreed that, they work very hard in mathematics because they want to be respected as intelligent students. Majority of the respondents portrayed great feeling of self-importance and ability as the main reason for studying mathematics ($M = 3.4$, $SD = 1.3$).

For the identified regulation subscale, it was observed that 91% respondents claimed that they were studying mathematics because they believe it will improve their work competence in future. It was also observed that 83% respondents strongly agreed that they were studying mathematics because what they learnt in mathematics now would be useful for the course of their choices in the university. Almost all the respondents in

this study are learning mathematics because of its future usefulness ($M = 4.2$, $SD = 0.7$). They are studying mathematics because it identified with their personal goals. This is good news because students seem to agree that mathematics is useful for their future career and daily lives.

For the intrinsic motivation subscale, as many as 56% respondents agreed that they were studying mathematics for the pleasure that they experienced when they were able to solve questions while majority (75%) of the respondents agreed that they were studying mathematics because they want to feel the personal satisfaction of understanding mathematics and 66% of respondents indicated that they were studying mathematics for the pleasure that they experienced when they learnt how things in life worked because of mathematics. In other words, respondents claimed a full sense of volition and choice in studying mathematics ($M = 3.8$, $SD = 0.8$).

The findings of the survey which suggested students to be intrinsically and extrinsically motivated necessitated the need for the interviews with the view to understanding the students' choice of both sub-constructs. Thus, students who made a choice of both sub-constructs were shortlisted and 240 of them were interviewed.

Interview

The interviews with 10 to 14 students at a time were conducted (20 groups in all) to provide causes of effects that were apparent from the survey. The following were the interview questions: (1) What do you understand by motivation in relation to the learning of mathematics? (2) Why are you studying mathematics? (3) How are you motivated to learn mathematics? (4) What can prevent you from studying mathematics? (5) What can motivate you to study mathematics better?

The data collected in response to these questions were analysed qualitatively using common themes. The interview data revealed that about 50% of the students understood motivation as being supported, 21% as being encouraged and 8% as being appreciated. It was evidence that majority of the students sought for motivation from sources that were external to self. For instance, one student claimed 'I am learning math because I want good grade or mark, nothing else matter to me...' Another student stated 'I just must do well in mathematics to make my parents proud...' In both cases, the students imposed their own rewards or constraints in order to protect their ego. Likewise, some of the students were not motivated. For example a student declared 'the day I complete my final exam would be the day I bid farewell to math, give me three or four reading subjects in place of math, I would gladly accept, ...' Another student stated 'I don't want to learn math in its entirety, but I am being forced. It's not funny'. This group of students seemed to lack intent to learn mathematics.

About 38% of the students indicated they were learning mathematics because it was a compulsory subject, 33% for admission into higher institutions, and 17% for the flexibility of mathematics that enhanced the learner to branch into other fields. The reasons given by the students were all within the domain of extrinsic motivation, with the exception of about 2% that explained they learn mathematics in order to 'improve

their thinking ability'. Extrinsic motivation is also necessary in the educational setting (Reeve, Deci, & Ryan, 2004), but if the stimulation to learn mathematics depends on external incentives rather than the inherent enjoyment or interest in the task, its continuity cannot be guaranteed. For example, in the interview one student stated: 'I have no option than to learn math for me to gain admission... although I would not like to study mathematics related programme in my life again'.

About 67% of the students claimed that the only way their teacher is motivating them is through the use of cane while about 5% students affirmed that some teachers gave money as a reward to encourage them to learn mathematics. Classroom observations indicated that caning was used on students for failing to turn in homework or for receiving bad grades. Moreover, fear of cane was the most frequently mentioned demotivating factor (58%) (Table 1). Other frequently mentioned demotivating factors were lack of encouragement (54%), and not knowing the usefulness of each of the topics in the syllabus (50%).

Item	Frequency	Percentage
Fear of Cane	140	58
Lack of attention/encouragement from math teachers	130	54
Not knowing usefulness of each of the topics	120	50
Teachers rushing through topics	85	35
Negative attitude of math teachers to the weak students	80	33
Overloaded syllabus and time factor	80	33
Procrastination/not seeking for help from friends	60	25
Laziness/complacency by the students	40	17
Over ambition of parents/guardians/teachers	40	17
Continuous failure/fear of falling behind	40	17
Unfriendly learning environment	30	13

Table 1: Causes of Demotivation of Students in Mathematics

As many as 75% and 50% of the interviewees indicated that tertiary admission and support from teachers respectively are their preferred method of motivation among others, while about, 54% of the interviewees pleaded for the abolition of the use of canes. For example, one of the students begged '...our teachers should stop the use of cane on us so that we can relax to learn mathematics well...' Some of the students' explanations during the interview were opposite to their earlier position on the Likert scale. For example, fifty students from different schools have this or its variation to say:

Learning math brings joy if you succeed but pains if you don't. There is a great joy within me whenever I succeeded in solving a question in math. It's awesome that I could under-

stand how certain formulae were put together to solve questions in word problem. Unfortunately, my teacher wouldn't acknowledge me because I always score less than half of the total marks.

The aforementioned students' responses to the items on intrinsic motivation seemed to be based on relieves from the pain of failure whenever they succeeded rather than the joy of learning mathematics for its inherent satisfaction and the expectations of some of the students for acknowledgement render it instrumental. Thus, this is a call to researchers to be cautious when designing instruments for data collection.

Conclusion

This study serves as a warning to researchers not to rely too heavily on one measure alone when conducting research. Methodological triangulation and use of mixed methods in this study managed to clarify initially confusing survey results. It could be said in a way that the students in this study did not fall into one or the other category of motivation. The Likert survey suggested the students to be both extrinsically and intrinsically motivated but in the interview they were mainly providing extrinsic motivation explanations.

The data also revealed that the importance of the national examination cannot be overemphasized. It is a high stake examination and as such, students were focusing on producing satisfying results. Secondly, many of the students felt they did not get enough feedback from the teachers and the observation of the lessons and the verification of the students' exercises by the researcher confirmed their claims that some teachers did not provide constructive feedback or the feedback was not motivating because it was too harsh.

Thirdly, even though there are restrictions regarding the use of cane in Ghanaian schools, some teachers are terrorizing their students with apparent impunity. The researcher was a witness to many such illegal acts which were immediately addressed by talking with the teachers who promised to change the practice. Fear of cane seems to be the most important factor influencing student motivation. However, corporal punishment seems to create problems in schools as it creates an atmosphere of fear. Agbenyega (2006) suggests that this might lead to truancy and dropping out, and without important skills a person is in danger of social segregation or becoming an outcast. Thus, the use of cane could be a recipe for disaster.

Acknowledgments

Writing of this research report has been greatly supported by PME workshop for early career researchers in Ghana and my workshop mentor Prof. Markku Hannula.

References

- Agbenyega, J. S. (2006). Corporal Punishment in the Schools of Ghana: Does Inclusive Education Suffer? *The Australian Educational Researcher*, 33(3), 107-122.

- Andrew, J. M. (2003). The student motivation scale: Further testing of an instrument that measures school students' motivation. *Australian Journal of Education*, 47(1), 88-106.
- Deci, E. L., & Ryan, R. M. (1985). *Intrinsic motivation and self-determination in human behavior*. New York, NY: Plenum.
- Deci, E. L., & Ryan, R. M. (2000a). The "what" and "why" of goal pursuits: Human needs and the self-determination of behaviour. *Psychological Inquiry*, 11(4), 227-268.
- Gaspard, H., Dicke, A.-L., Flunger, B., Schreier, B., & Hafner, I. (2015). More Value Through Greater Differentiation: Gender differences in Value Beliefs About Math. *Journal of Educational Psychology*, 107(3), 663-677.
doi:<http://dx.doi.org/10.1037/edu0000003>
- Gershoff, E. T., & Font, S. A. (2016). Corporal punishment in U.S. public schools: prevalence, disparities in use, and status in State and Federal Policy *Society for Research in Child Development*, 30(1), 1-26.
- Lepper, M. R., Corpus, J. H., & Iyengar, S. S. (2005). Intrinsic and extrinsic motivational orientations in the classroom: Age differences and academic correlates. *Journal of Educational Psychology*, 97, 184-196. doi:10.1037/0022-0663.97.2.184
- Lim, S. Y., & Chapman, E. (2014). Adapting the academic motivation scale for use in pre-tertiary mathematics classrooms *Mathematics Education Research Journal*, 27, 331-357.
- Morrow, V., & Singh, R. (2014). *Corporal Punishment in Schools in Andhra Pradesh, India: Children's and Parents' Views*. UK: Young lives.
- Mullis, I. V. S., Martin, M. O., Foy, P., & Arora, A. (2012). *The TIMSS 2011 International Results in Mathematics* Chestnut Hill, MA: TIMSS & PIRLS International Study Center, Boston College.
- Reeve, J., Deci, E. L., & Ryan, R. M. (2004). Self-determination theory: A dialectical framework for understanding the socio-cultural influences on student motivation. In D. McInerney & S. V. Etten (Eds.), *Research on socio-cultural influences on motivation and learning: Big theories revisited* (Vol. 4, pp. 31-59). Greenwich, CT: Information Age.
- Ryan, R. M., & Deci, E. L. (2000a). Intrinsic and extrinsic motivations: Classic definitions and new directions. *Contemporary Educational Psychology*, 25(1), 54-67.
- Vallerand, R. J., & Bissonnette, R. (1992). Intrinsic, extrinsic and amotivational styles as predictors of behaviour: A prospective study. *Journal of Personality and Social Psychology*, 60(3), 599-620.
- Wigfield, A., & Cambria, J. (2010). Students' achievement values, goal orientations, and interest: Definitions, development, and relations to achievement outcomes. *Developmental Review*, 30(1), 1-35. doi:<https://doi.org/10.1016/j.dr.2009.12.001>
- Wigfield, A., & Eccles, J. S. (2000). Expectancy-value theory of achievement motivation. *Contemporary Educational Psychology*, 25, 68-81. doi:10.1006/ceps.1999.1015

LEARNING TO ASSESS: EXPLORING CHANGES TO PRE-SERVICE TEACHERS' CRITERIA FOR A QUADRATICS TASK

Michal Ayalon

Karina Wilkie

University of Haifa

Monash University

This study sought insights into the process of pre-service teachers (PSTs) learning to assess student learning through designing and then refining assessment criteria for a quadratics task using student example responses. Sixty PSTs attempted the task individually, then worked in pairs to develop initial assessment criteria, analyse a selection of student responses, and revise their criteria. The data analysis examined variations in the pre-service teachers' own use of and attention to quadratic features and mathematical language as assessment criteria for this task throughout the activity. This paper discusses evidence suggesting that collaborative analysis of example student task responses can make certain, but not all, assessment criteria salient to PSTs.

In recent years, research initiatives and educational policy have advocated a greater focus on teachers using formative assessment (FA) practices, to motivate students with task-specific feedback and to adjust their teaching approaches during the learning process (Andrade & Cicek, 2010; Black & Wiliam, 2009). These practices contrast with summative assessment, such as standardised testing, that occurs at the conclusion of the learning process. The effectiveness of FA relies in part on the development of useful tasks that gauge deeper understanding and of appropriate criteria for assessing them (Danielson & Marquez, 2016). It crucially requires teachers to be able to provide quality constructive feedback on tasks that students can interpret and use to improve their learning (Sadler, 1998). There remains a lack of both in-service and pre-service professional learning opportunities for teachers to develop formative assessment competencies (Stiggins, 2010). There is little in the research literature on professional learning for using formative assessment in mathematics, and particularly at secondary levels (Panadero & Johnsson, 2013; Scheider & Randel, 2010). We found one study on learning to use provided generic scoring rubrics with example student responses that included secondary algebra teachers (Schafer, Swanson, Bené & Mewberry, 2001). They found empirical support for the benefit to teachers' practice when they experience collaborative professional learning on rubric use. This qualitative study with sixty Israeli PSTs sought to explore the process of learning to assess secondary students' responses to an open-response quadratics task. The design of the sample of students' responses for PSTs to assess focussed on two main criteria: quality of quadratics features (analytical vs. visual) and mathematical language (formal vs. informal). The study focussed on if and how the PSTs might improve their knowledge of quadratic features and the mathematical language needed for the task, and apply this in building assessment criteria. This paper addresses the following research question: *What are the variations in pre-service teachers' own use of and attention to quadratic features and*

mathematical language through the process of designing and refining task assessment criteria with student example responses?

BACKGROUND

The assessment task and related quadratic concepts

The task used in this study (Figure 1) focuses on students' ability to notice and interpret the analytic features of graphs of quadratic functions (Ayalon, Watson, & Lerman, 2016). Quadratic functions are typically the first non-linear type introduced to students, and learning concepts include: the variable (non-constant) gradient and parabolic shape, orientation, the turning point as a maximum or minimum, the line of symmetry, the roots (or solutions or x-intercepts) of equations, transformations, different forms of equations, and the relations between equation parameters and the graph's position and shape. In other words, quadratics provide the move from seeing functions as generating ordered pairs to functions being mathematical objects in their own right with certain properties (Watson, 2013). This could be given as one of the reasons for spending a significant amount of time learning about them in school curricula.

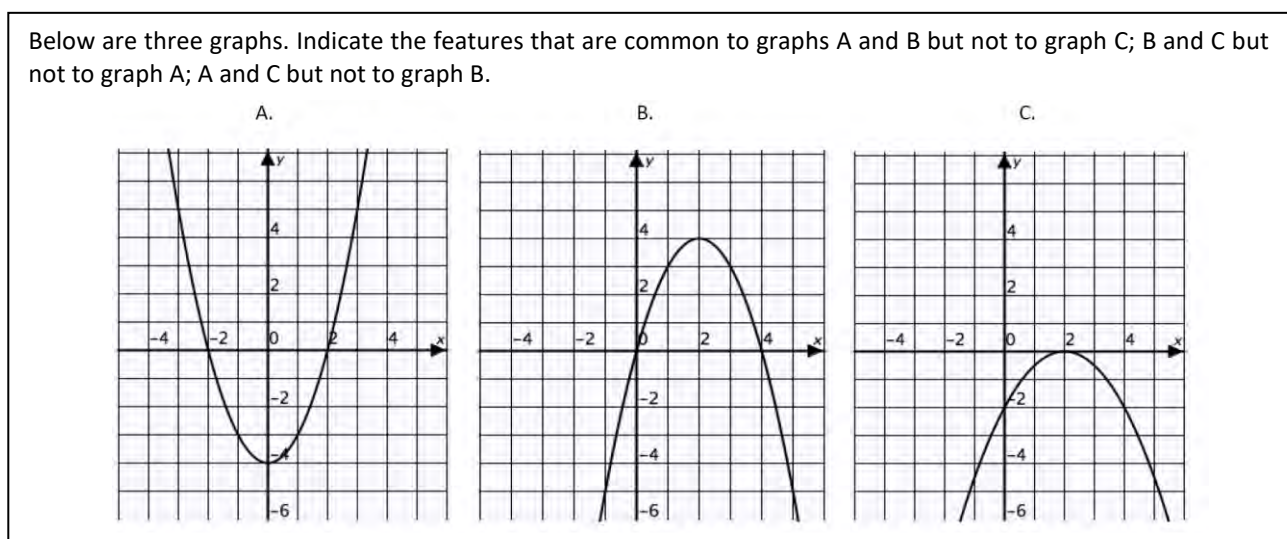


Figure 1: The quadratics assessment task used with the PSTs

Research has shown that students can treat graphs as pictorial or geometric objects (Leinhardt, Zaslavsky, & Stein, 1990), so in this task, students may describe visual (pictorial) rather than analytic (mathematical) features. The level of formality of their mathematical language use and the number of features they attend to in their comparisons may also differentiate them. In most mathematics classrooms both informal and formal language are used and these can be either in written or spoken form. Informal language is the kind that learners use in everyday life to express their mathematical understanding. Formal mathematical language refers to the standard use of terminology (mathematics register), which is usually developed within formal settings like schools. The valued goal in school mathematics is formal, written mathematical competence (Setati & Adler, 2000). Words, phrases, meanings, and different modes of

communication allow discourse about concepts, objects, and processes in mathematics (Temple & Doerr, 2012). Pimm (1991) illuminated the challenges this poses for mathematics teachers:

One difficulty facing all teachers, however, is how to encourage movement in their learners from the predominantly informal spoken language with which they are all pretty fluent, to the formal language that is frequently perceived to be the landmark of mathematical activity (p. 21).

Previous research findings on students' responses to the quadratics task

The quadratics task used in our study of PSTs was previously included in a previous survey from an international comparative research project investigating how functions concepts develop for learners throughout secondary (high) school in different contexts (e.g., Ayalon et al., 2016). Data were collected on student responses to six functions tasks at different year levels from two different curricula systems (Israel and England). Table 1 shows examples of some of the analytical features referred to by Years 9 to 12 Israeli students and how they were expressed, informally or formally.

Analytical feature	Formal/ Informal	Example
Orientation	Informal	B and C look like mountain way up but A is not
Orientation	Formal	Both B and C have maxima
Gradient	Informal	A and B have the same gradient as each other except B is negative
Gradient	Formal	The magnitude of the gradient functions is the same
Turning point	Informal	The top of the arch both contain 2 on the x axis
Turning point	Formal	Both B and C's optimum point is at 2 on the x axis
Zeros	Informal	They both touch the point (2,0)
Zeros	Formal	A and B intercept the x-axis at two points, so have two real solutions

Table 1: Examples of students' references to different analytical features in the quadratics task

The Israeli teachers reported that their students learn to plot quadratic functions from Year 9 onwards and expected them to notice important analytical features. From analysis of the Years 9–12 Israeli students ($n = 80$), five response categories emerged: (1) No reference to features at all; (2) Reference only to visual features (e.g., thin, high, low) that do not express reasoning about variables or functions; (3) Reference to (at least one) analytical feature (e.g., orientation, transformation, turning point, zeroes) informally; (4) Reference to analytical features using mixture of informal and formal language; and (5) Reference to analytical features using formal language. Overall, the students across the secondary years identified analytical features, with orientation and zeroes being the most frequent. Progression in use of formal language was also found, yet with many students still using informal language, or a mixture of informal and formal language at the higher year levels. The current paper follows on from the re-

search on students’ responses to the task, by focusing on the professional learning of pre-service teachers: specifically, how to develop criteria for assessing the task as part of the formative assessment process.

RESEARCH DESIGN

An issue that was raised from the previous research on the students’ quadratic task responses related to teachers’ own ability to use formal language for describing analytical features of quadratic graphs and whether or not they focus on different levels of language use when assessing student responses. This paper attempts to address this question by studying 60 Israeli secondary mathematics PSTs’ (1) use of analytical features and mathematical language in answering the quadratics task, and (2) attention to analytical features and mathematical language as criteria for assessment before and through assessing given examples of (actual) student task responses. We therefore explore the process of the PSTs learning to assess formatively through (1) attempting the quadratics task (individually); (2) creating initial assessment criteria (in pairs); (3) refining them through analysing actual student task response examples (in pairs); and (4) reflecting on the work (individually). Using findings from the literature and the previous student study, we selected the example student responses (presented to the PSTs) to provide a range in the level of quality of quadratic features (visual vs. analytical) and language used to describe them (informal vs. formal). (Examples of responses will be shared at the conference.) The activity lasted for two hours and took place within a unit on learning and teaching algebra as part of the PSTs’ university course. The 60 PSTs live in diverse towns and villages in northern Israel. We discuss variations in changes to the PSTs’ own use of analytical features and mathematical language, and also attention to use of analytical features and mathematical language as criteria for assessing student responses.

FINDINGS AND DISCUSSION

Noticing of features during task attempt

Two PST initial response categories emerged: (1) Reference to analytical features (e.g., orientation, transformation, turning point, zeroes) using mixed *formal* and *informal* language, and (2) Reference to analytical features using *formal* language *only*. Table 2 presents the distribution of these categories.

Code	# PSTs (n = 60)
(1) Analytical features using mixture of formal and informal language	14
(2) Analytical features using formal language only	46

Table 2: Distribution of PSTs’ categories for the quadratics task

Table 2 shows that about a quarter of the PSTs used a mixture of formal and informal language when communicating common features, whereas three quarters used formal language. This finding indicates a much higher proportion of PSTs able to use formal language for quadratics features compared to secondary (high) school students.

Creating initial assessment criteria

As presented in Table 3, the PSTs identified four initial assessment criteria (in pairs after having attempted the task individually):

1. Quality of features of quadratic graphs: In most cases, the PSTs made a list of features they considered important, e.g., zeroes, orientation, positive/negative, and increasing/decreasing domains.
2. Fluency: The PSTs suggested that students who notice a higher number of features (usually more than one or two) deserve a higher result, e.g., “A Student who notices two features or more in the graphs will receive a higher mark than a student who attends to one feature only” (PSTs # 5 & 6).
3. Mathematical (formal) language: Quality of language and terms used to describe features were suggested by the PSTs, e.g.,
We would like the students to use language, which is accepted in mathematics. It is very important that students will know to use the correct terms, for example, functions having maximum and minimum points, and not how many of them like to say ‘smiling’ or ‘crying’ functions (PSTs #53 & 54).
4. Originality: Several PTSs mentioned noticing of unique features, usually referring to this aspect as a bonus, e.g., “If a student sees something unique in the graph, maybe a feature of quadratics not discussed in the classroom, we would give a bonus” (PSTs # 43 & 44).

Criteria for assessment	# PSTs (n = 60)
Quality of features	60
Fluency	25
Mathematical language use	24
Originality	6

Table 3: Distribution of PSTs’ initial assessment criteria

As Table 3 shows, all of the PSTs referred to quality of features as a criterion for assessment of students’ responses. Fluency and mathematical language were suggested by about 40% of the PTSs, with very few referring to originality.

Refining assessment criteria through analysis of student task response examples

We found that the PSTs did not suggest additional types of assessment criteria after their analysis of example student responses, but two salient revisions were noticeable. One related to *quality of features*, where about half of the PSTs not only attended to a list of expected analytical features (as they did initially), but also explicitly differentiated between visual and analytical features, with visual features considered as inappropriate, e.g.,

Students may attend to visual features of graphs, like high and low. These are not functional features. They do not have mathematical meaning, as intersection points and orientation of the graph have. Hence we would like to emphasize that one of the criteria for assessing the students’ work is the quality of features – not visual (PSTS #35 & 36).

The other salient revision was an increase in the number of PSTs suggesting mathematical language as a criterion for assessment: 56 compared to 24 in the initial phase (see Table 4).

Criteria for assessment	# PSTs (n = 60)
Quality of features (with specific attention to analytical vs. visual features)	60 (32)
Fluency	26
Mathematical language use	56
Originality	6

Table 4: Distribution of PSTs' assessment criteria after analysis of student responses

Evidence of PSTs' learning about quadratics and formative assessment

All of the 60 PSTs both attended to analytical features in the quadratic graphs in the first phase of the activity and mentioned particular features that they considered important when identifying initial criteria for assessment. About half went on to demonstrate a noticeable change in making an explicit distinction between analytical and visual features after analyzing the students' example responses, which had included both types. Being aware of this distinction is important for teachers as research shows that students need to progress beyond viewing graphs as pictorial or geometric objects if they are to develop conceptually (Leinhardt et al., 1990). No change was evidenced regarding PSTs specifying fluency and originality as assessment criteria after analysis of the student responses, possibly because these aspects were not salient enough in the examples provided. Fluency and originality are considered two main features of creativity (ref). We were encouraged, however, that at least some PSTs considered them as important for assessing this quadratics task. The most prominent change was associated with language used to communicate quadratics features. Table 5 presents the variations in changes to the PSTs' own use of mathematical language and subsequent attention to students' mathematical language when assessing example student task responses. As seen in Table 5, there were 14 PSTs who initially included informal language when responding to the quadratics task, and also did not suggest mathematical language as one of their assessment criteria in the first phase of the activity. When refining their criteria through the analyzing the example student responses, 10 of them added mathematical language as a criterion. These PSTS described this change in their reflections following the activity, e.g., Only after seeing possible student responses did I realize that I used inaccurate language for talking about the orientation of the graph when solving the problem (I wrote that the parabola is smiling)... The comparison I made among the responses helped me notice that the same feature of the graph can be termed differently, and it made me think about mathematical language as an important but unobvious part of learning and teaching mathematics (PST #2). As also shown in Table 5, 22 out of the 46 PSTs who initially used formal language in their task response did not initially mention language as an assessment criterion, but did refer to it after analyzing the student responses. In their reflections, these participants wrote that at the beginning, language did not occur to them as something that needed particular atten-

tion, and only after they encountered possible student responses did they realize that mathematical language can vary in its level of formality. For example: During the process of assessing the student responses, I found out that mathematical language could be an issue. First I thought that the only things to be evaluated are the kinds of features noticed in the graphs and the number of them. The activity helped me become aware of the fact that students may use informal terms... I think that using correct mathematical language is an important part of learning mathematics and is necessary for developing communication in the classroom. Actually using this task in the classroom, even requiring students to assess the sample responses as we did, might encourage them to pay attention to their language (PST #14). Finally, 24 PSTs used and included mathematical language as a criterion for assessment right from the beginning of the activity.

Noticing of features in the graph: use of language	Initial assessment criteria: Attention to language	Refining assessment criteria through analyzing students' responses: Attention to language	# PSTs (n = 60)
Mixture of formal/informal	No	No	4
Mixture of formal/informal	No	Yes	10
Formal only	No	Yes	22
Formal only	Yes	Yes	24

Table 5: Variations of PSTs' attention to mathematical language during the assessment experience

CONCLUSION

The study focussed on if and how the PSTs might improve their knowledge of quadratics features and the mathematical language needed for the task, and apply this in building assessment criteria. Overall, the findings show that knowledge of quadratics properties – both for themselves and as a pedagogical aim – was evidenced from the beginning, also demonstrated by previous research with secondary students in Ayalon et al.'s (2016) research. Becoming aware, however, of the possibility of students' attention to visual (rather than analytic) features was a consequence of the activity of collaborative student example response analysis. Further research on the choice of example responses is needed to explore possible influences on PSTs attention to creativity as well. The effectiveness of formative assessment relies in part on the development of useful tasks that gauge deeper understanding and of appropriate criteria for assessing them (Danielson & Marquez, 2016). In this study we used an open-response quadratics task and found that analyzing example responses drew PSTs' attention to students' informal mathematical language use. The valued goal in school mathematics classrooms is formal, written mathematical competence (Setati & Adler, 2000) and so these types of tasks seem to have the potential to provide teachers with useful assessment information. FA also crucially requires teachers to be able to provide quality constructive feedback on tasks that students can interpret and use to improve their

learning (Sadler, 1998). There is more to understand about how these types of tasks can be used, particularly at secondary levels, for effective formative assessment.

References

- Andrade, H., & Cizek, G. J. (2010). Preface. In H. Andrade & G. J. Cizek (Eds.), *Handbook of formative assessment* (pp. vii-xii). New York, NY: Routledge.
- Ayalon, M., Watson, A., & Lerman, S. (2016). Reasoning about variables in 11 to 18 Year Olds: Informal, schooled and formal expression in learning about functions. *Mathematics Education Research Journal*, 287, 379-404.
- Black, P. & Wiliam, D. (2009) Developing the theory of formative assessment. *Educational Assessment, Evaluation and Accountability*, 21(1), 5-31.
- Danielson, C., & Marquez, E. (2016). *Performance tasks and rubrics for high school mathematics: Meeting rigorous standards and assessments* (2nd Ed.). New York, NY: Routledge.
- Leinhardt, G., Zaslavsky, O., & Stein, M. (1990). Functions, graphs and graphing: Tasks, learning and teaching. *Review of Educational Research*, 60(1), 37–42.
- Panadero, E., & Jonsson, A. (2013). The use of scoring rubrics for formative assessment purposes revisited: A review. *Educational Research Review*, 9, 129-144.
- Pimm, D. (1991). Communicating mathematically. In K. Durkin and B. Shire (eds.), *Language in Mathematical Education* (pp. 17–23). Open University Press, Milton Keynes,
- Sadler, D. R. (1998). Formative assessment: Revisiting the territory. *Assessment in Education: Principles, Policy & Practice*, 5(1), 77-84.
- Schafer, W. D., Swanson, G., Bene, N., & Newberry, G. (2001). Effects of teacher knowledge of rubrics on student achievement in four content areas. *Applied Measurement in Education*, 14(2), 151-170.
- Schneider, M. C., & Randel, B. (2010). Research on characteristics of effective professional development programs for enhancing educators' skills in formative assessment. In H. Andrade & G. J. Cizek (Eds.), *Handbook of formative assessment* (pp. 251-276). New York, NY: Routledge.
- Setati, M., & Adler, J. (2000). Between languages and discourses: Language practices in primary multilingual classrooms in South Africa. *Educational Studies in Mathematics*, 43, 243-269.
- Stiggins, R. (2010). Essential formative assessment competencies for teachers and school leaders. In H. Andrade & G. J. Cizek (Eds.), *Handbook of formative assessment* (pp. 233-250). New York, NY: Routledge.
- Temple, C., & Doerr, H. (2012). Developing fluency in the mathematical register through conversation in a tenth-grade classroom. *Educational Studies in Mathematics*, 81(3), 287–306.
- Watson, A. (2013). Functional relations between variables. In A. Watson, K. Jones, & D. Pratt (Eds.), *Key ideas in teaching mathematics: Research-based guidance for ages 9–19* (pp. 172–199). Oxford: Oxford University Press.

ELEMENTARY SCHOOL TEACHERS' IMPLEMENTATION OF DYNAMIC GEOMETRY USING MODEL LESSON VIDEOS

Anna Baccaglini-Frank*, Pietro Di Martino*, and Nathalie Sinclair**

* University of Pisa, Italy, ** Simon Fraser University, Canada

In this study, a set of activities on line symmetry in a Web Sketchpad environment, published on the website of a Canadian university, were adapted by two Italian researchers for 1st and 2nd grades with an Interactive White Board in the classroom. The activities were proposed to a 2nd grade during two video-recorded lessons conducted by one of the researchers. The videos were viewed by three teachers who then proposed the same activities in their 1st and 2nd grade classes. The study was carried out over a 6-month period. One of its aims, the main focus of this paper, was the following: to study teachers' implementation of the activities and to identify aspects of the study's teacher instructional improvement cycle that were most influential in their implementations of the technology-based activities.

INTRODUCTION

As Crawford and Adler (1996) underline, there is little evidence that the knowledge generated by researchers in mathematics education is embodied in the teaching practices in school. The gap between educational theory and teachers' practice is a major issue in the field of mathematics education (Jaworski, 1998; Mason, 1998). Surely a critical issue concerns the difficulties related to the dissemination of research results in the public domain: many scholars underline the insufficient dissemination of research results for practitioners (Artigue, 2016) and the need for more two-way modes of communication between researchers and practitioners (Venkat, 2016). But this is only one side of the coin. Beyond the dissemination issue, there is also the issue of teachers' interpretation and use of curriculum materials developed by researchers in mathematics education. Indeed, teachers are decision makers, and their decisions are influenced by factors such as knowledge, but also values, beliefs, emotions and previous experiences (Malara & Zan, 2008). Teachers do not approach their professional learning or curriculum materials as blank slates: they have a wide range of experiences, wants, needs, worries affecting their interpretation and use of professional opportunities (Cuoco, 2001). It has been recognised that teachers act as interpreters and mediators of curriculum materials (Remillard, 2005). This reflects a broader pattern in which the unfolding of innovation in education is shaped by the sense making of the agents involved (Spillane et al., 2002). Teachers typically select from and adapt curriculum materials, incorporating these materials into wider systems of classroom practice. Therefore, it is natural to expect adaptations to curriculum material developed by researchers during a teachers' implementation of that material. In the context of technology integration, Ruthven (2009) has identified five structuring features of

classroom practice that shape the ways in which teachers adapt particular tools to their classroom contexts. Moreover, Ruthven (2016) writes of *interpretative flexibility* to refer to how technology is taken up to aligned with user concerns and adapted to the situations in which use takes place. This opens the way to variation in modalities of use between different user groups and between different settings for use, and to change in these modalities over time. This may lead, in turn, to the product being redesigned, launching a further cycle of adaptation. From a sociocultural perspective, “the conceptualization of instruments [is] an activity distributed between designers and users” (Rabardel & Waern, 2003, p. 643).

In our research, we aim to gain insight into teachers’ adaptation decisions by identifying the particular structuring features come into play and which might be the determining factors. Typically, the aims of specific didactical materials are explicit, while the ways of instantiating such materials in the classroom are not. Therefore, in this study, we wanted to flip this point of view, agreeing with the teachers on the content of a set of activities, and then providing a video of an instantiation of the technology-based activities by a researcher acting as teaching in a classroom. At no point were the teachers asked to imitate the researcher. In other words, we designed a *teacher instructional improvement cycle* in which, once the mathematical topic had been chosen and discussed, and a set of technology-based tasks planned, we decided to video-record a researcher as she instantiated the activities in a 2nd grade class. We saw these videos as boundary objects (Star & Griesemer, 1989) useful to study the nature of the teachers’ adaptations. We were especially interested in studying the impact of this relatively uncommon design feature on the teachers’ implementation decisions. We explicitly chose experienced teachers for two reasons: we conjectured that their consolidated teaching styles and identities would increase the likelihood that adaptations would emerge, and that they would be more aware of their decisions and, therefore, it would be easier for them to express and discuss them.

THE DESIGNED TEACHER INSTRUCTIONAL IMPROVEMENT CYCLE

The mathematical content chosen was line symmetry, a topic with which many primary school teachers do not feel at ease, and that is considered difficult for students, as well; it is typically taught for the first time in 1st and 2nd grade (ages 6-8). We started with a set of activities on line symmetry in Web Sketchpad, published on the Canadian website <http://www.sfu.ca/geometry4yl.html>. The study was carried out over six months; it involved three elementary school teachers and four classes (three 2nd grades and one 1st grade) in three Italian elementary schools. The teachers’ experience was the following: T1 – 20 years of experience and 16 teaching math; T2 – 23 years of experience and 21 teaching math; T3 – 35 years of experience and 25 teaching math. Three researchers (the authors of the papers) were involved in the study: the third author had designed the original digital activities and conducted classroom-based research using them (see Ng & Sinclair, 2015); the other two authors – working at the same Italian university – organized the implementation of the activities and attended the meetings with the teachers. The phases of the study were: (1) Re-design of the lesson plans and

interactive files on the site, a priori considerations on such activities; (2) Presentation to the three teachers of the newly designed materials, explaining the changes made; (3) Instantiation of the activities in a 2nd grade class by the first author; (4) T1, T2, T3 reception of the videos and completion of a questionnaire; (5) T1, T2, T3 carried out the activities in their classes (T1 in a 1st grade, T2 and T3 in 2nd grades); these sessions were video-recorded; (6) A posteriori analysis of the video-recordings by all three researchers; and (7) Final questionnaire on how the lessons went, and meeting with the Italian researchers.

In this paper, we report on the questionnaire data and the final meeting (phases 4 and 7). In particular, we discuss the *role played by the video-recordings* of the implementation of the activities in the teachers' processes of decision-making when adapting and implementing the materials. Indeed, the initial instantiation of the activities by a researcher was a distinctive feature of this study.

The activities on line symmetry

The final version of the activities – after the adaptation developed by the Italian researchers – made use of interactive files projected on an IWB. The files contained sets of colored circles symmetrically arranged on two sides of a line that could be continuously dragged on the screen: when a circle is dragged, its corresponding circle moves so as to preserve symmetry. The line is also draggable.

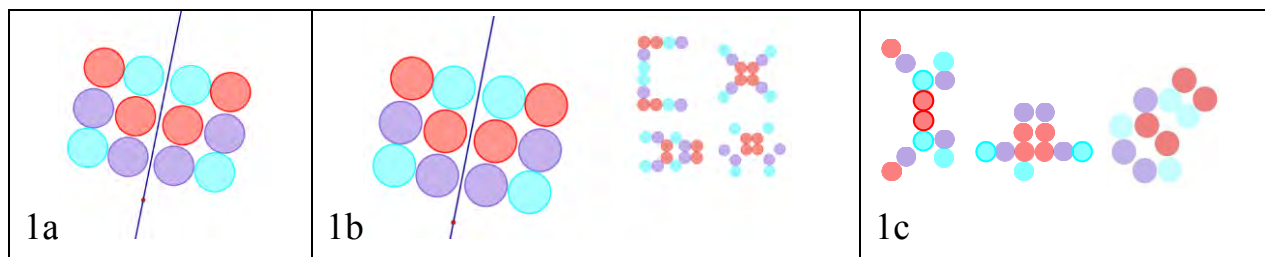


Figure 1a, 1b, 1c: screenshots from the interactive files used for the activities.

The activities consisted of the following five tasks, to be assigned over 3 class periods (about 2.5h): (1) What do you see? What happens when you drag the circles? (Fig. 1a) (2) Make predictions: describe all that will move before dragging to check. (3) In pairs, one student tells the other what to do to reproduce a figure (Fig. 1b). (4) Draw a picture of how the interactive file worked. (5) Which of the pictures can/can't you make with the file? Why? (see Fig. 1c). Tasks 4 and 5 were assigned to all students, at their desks, with paper and pencil. Selected students' answers for Task 5 were discussed by the whole class, using the IWB.

THEORETICAL CONSIDERATIONS

We draw on both Ruthven's (2009) framework to guide our analysis and on the notion of boundary objects, each of which is described below.

Structuring features of classroom practices

In his research analysing the ways in which teachers integrate (or not) digital technologies, Ruthven has identified five structuring features of classroom practices that teachers must often adapt in order to make effective use of the intended affordances of these technologies: working environment (room location, physical layout, use of IWB), resource system (complementing and connecting with existing resources), activity structure (the action and interaction of participants), curriculum script (choosing appropriate tasks, recognising difficulties), and time economy. If any of the structuring features of a teacher's existing classroom practice is challenged by a task and/or tool, the teacher will adapt it, sometimes thereby shifting its intended use (by a researcher or designer).

The videos as boundary objects

Star and Griesemer (1989) describe boundary objects as “scientific objects which both inhabit several intersecting worlds [...] *and* satisfy the informational requirements of each of them” (p. 393). Boundary objects are “both plastic enough to adapt to local needs and the constraints of the several parties employing them, yet robust enough to maintain a common identity across sites” (p. 393). They can also “coordinate academic and practitioner perspectives” (Venkat, 2016, p. 187). In the present study, various objects played the role of boundary objects: the activities present on the website; their modifications designed by the Italian researchers; and the videos of the researcher's lessons, that were passed to the teachers and discussed before teachers' implementation of the lessons. All these objects lie at the intersection of different worlds and communities, and have the potential of serving as vehicles to communicate and convey meaning across different communities, even if different communities can define and interpret them in different ways. The activities on the website were at the intersection of the Canadian practices and the Italian ones; the Italian re-designed activities were at the intersection of the researchers' (ideal envisioned classroom and research influence on practices) and teachers' community (actual classrooms and everyday practices); the videos were at the intersection between the single (real) classroom communities and future, potential classrooms communities in which the activities would be realized again.

TEACHERS' REACTIONS TO THE RESEARCHER'S VIDEOS

Question Q1 (What do you see as the major potentials and pitfalls of the activities?) was assigned after the teachers had received only the activities, questions Q2 (What difficulties do you foresee in implementing the activities in your classroom?), Q3 (Having seen the videos, which aspects will you try to replicate and which will you change when teaching yourself?) and Q4 (Are there didactical choices or mathematical considerations in the videos that you did not find clear?) were assigned after they had also viewed the researcher's videos, but still before the teachers' implementations; Q5 (What similarities and differences did you notice in your implementations with respect to that of the researcher?) was assigned after the teachers' implementations.

After receiving only the materials, the teachers (in their answers to Q1) identified a number of potentials in them. These dealt with the resource system, the activity structure and the curriculum script; specifically, the greater ease in working with oblique lines of symmetry compared to previously used materials, and their beliefs – founded in previous experiences – about children’s excitement and “surprise” in using the IWB and “moving things around”. Interestingly, in their answers to Q2 the main difficulties foreseen by the teachers, after viewing the videos, were only partially related to the pitfalls identified after having viewed only the activities; these had to do with both activity structure and curriculum script. In terms of the former, the teachers were concerned about keeping the children’s attention and silence for long periods of time. All three teachers wrote that this kind of activity requires long periods of attention, and with large classes (23-29 children) it may be difficult to maintain silence and concentration. Moreover, they wrote that all children would want to go to the IWB and it would be hard to call them all. T2 also commented on the fact that maintaining silence and order in the classroom would be even harder for her since she is not an “external expert” (unlike the researcher).

In terms of curriculum script, the teachers were concerned about handling students’ presumed difficulties in responding to the “creative” drawing request. T2 expressed worry in the prediction task, and T1 wrote: “A pitfall might be the absolute abstractness of the material”. They also expressed concern about coordinating discussion about the behaviour of the objects on the screen in a way that would facilitate understanding without putting words in the students’ mouths. This was not an issue in their usual classroom practice, in which they would begin by giving students definitions of objects and then tasks that used these objects. Indeed, speaking about difficulties related to language, T3 wrote: “I think I will have trouble calling ‘the objects’ in particular the line of symmetry and relationships between this and the balls (parallelism, perpendicularity) with the names given by the students.” T2 wrote: “the ‘mental experiments’ will be maybe the most difficult part but also the most interesting. It will not be easy to manage the lesson when they will have to come up with words to describe the movements without me giving hints.” T3 added: “Also with so small and ‘ignorant’ [in its etymological meaning] children with respect to the language of geometry, I could have trouble using an ‘alternative’ language that is easy enough to understand.”

In response to Q3, the teachers also referred to activity structure and curriculum script analysing critical features of the video. In particular, they appreciated the ideas of: using ‘oral descriptions’ and words instead of gestures for Tasks 1 and 2; highlighting the terminology used by the children and agreeing on their meanings; using arms and hands to help indicate ‘parallel’ and ‘perpendicular’ and seeing if the students tilt their heads to better perceive the line of symmetry. On the other hand, T2 and T3 mentioned that in the researcher’s videos they noticed students’ difficulties in making up names other than ‘rows’ and ‘columns’ to describe perpendicular and parallel alignments of the circles with respect to the line of symmetry. This appeared to them as problematic especially when the line was oblique, so they proposed to modify the activities by

asking the students what they meant by ‘rows’ and ‘columns’. We note that this proposed adaptation seems to be in keeping with the curriculum script modeled in the video. In response to Q4, all teachers wrote that they appreciated the didactical choices implemented by the researcher, and they found all mathematical considerations in the videos clear. The only issues that T2 and T3 mentioned related to the resources system in that they preferred to think of these activities as part of a longer sequence that “integrates also the body and manual skills, that is a laboratorial activity”.

Finally, after having implemented the activities, the teachers noticed many similarities between what happened in their classes and in the researcher’s video. The teachers primarily noticed aspects of curriculum script: difficulties in considering the distance from the line of symmetry, usefulness of gestures with arms and hands to indicate parallel and perpendicular alignments with respect to the line of symmetry, use of the word ‘mirror’ to refer to the line of symmetry, difficulties in speaking “with respect to the mirror”, use of arrows to indicate movement in the static drawings, and preference for horizontal or vertical lines of symmetry. However, the teachers also noticed some differences: In T2’s class the children preferred speaking of ‘axis’ when referring to the line of symmetry; moreover, this class had an interesting discussion about whether the line of symmetry was finite or infinite. In T1’s class (the 1st grade) the children seemed to take a longer time to realize they could move the line of symmetry. Finally, T3 remarked, again, how she thought that the children in the researcher’s video were more quiet and attentive than her students, which she believed, depended on the novelty of a different teacher in the classroom.

While many outcomes and comments from the concluding meeting also fit well with Ruthven’s framework, some were more difficult to interpret through such framework. For example, T1 referred to the *content safety* that the videos provided: “It gave me great peace of mind to work next to the researchers, because I knew that what I was going to propose was mathematically sound, and I did not have to worry about appropriateness and depth of the mathematics I was teaching. I knew what properties were important and what to aim for.”

This may be seen as being related to curriculum script, but it emerged because of our novel design, in which the mathematical affordances of the tool use were made explicit for the teachers. T2 also focused on the mathematical dimension of her implementation, saying that the activities helped her engage her students in mathematical discussions: “With dynamic geometry, there was extra support for discussing properties of the line of symmetry, and I could point to the screen and describe properties of a physical object that was coherent with the mathematics.”

DISCUSSION AND CONCLUSIONS

The teachers’ critical analysis of the researcher’s instantiation of the technology-based activities seemed to affect their opinion about potentials and limits of the activity and, in particular, about students’ difficulties. This appears clearly comparing teachers’ answers to Q1 and to Q2. For example, in answering Q1 there was no mention of the

language-related difficulties that, instead, appeared heavily in the answers to Q2. The awareness of the delicate issue of how to speak of new mathematical objects and properties characterized many of the teachers' answers to the other questions, as well. So, in a way, the videos planted a new awareness in the teachers, which lead to them paying particular attention to their own words and gestures, as well as to those of their students. This awareness seemed to elicit a new tension in the teachers: in the sense of complex collection of opposing forces of wants, needs and self-assessments of own capabilities that complicate the decision making processes of teachers (Liljedhal et al., 2015). However, the videos also offered helpful suggestions to solve this tension, which were noticed and appreciated by the teachers: how the researcher picked up on students' words and gestured, and how specific language and gestures were agreed upon and used to facilitate discourse on line symmetry. Indeed, the teachers commented on analogies and differences especially on these aspects of the curriculum script. Overall the teachers seemed to appreciate what they saw in the researcher's videos, and decided to implement the activities, seemingly trying to reproduce the activity structure and curriculum script with a very high degree of fidelity. They decided to do this despite their concerns, for example, about the extreme abstractness of the technology-based tasks (an aspect of the curriculum script), or the risky activity structure. In this sense, the researcher's videos seemed to allay the tensions. This may in part be due to the fact that the researcher video was recorded in their school, with students they were familiar with.

Finally, a comment on Ruthven's structuring features. These have provided an insightful tool for analyzing what teachers decided to adapt in order to make effective use of the intended affordances of these technologies. However, some important issues emerged in this study, which do not seem to be properly captured by this framework. The first issue is that of the use of words and gestures in the classroom; this could be seen as part of the curriculum script, or, possibly of the activity structure. But its nature and the strength with which it emerged in this study suggests it should be considered a new feature altogether. We conjecture that this feature may have emerged this strongly in part because of the grade levels involved in the study (attention to words and gestures may play a more major role in early elementary grades than in high school), and in part because of the researcher's video in which particular attention was paid to these aspects of the activities. Also the reference to what we called content safety may be specific to professional development cycles of primary school teachers.

References

- Artigue, M. (2016). Mathematics Education Research at University Level: Achievements and Challenges. In E. Nardi, C. Winslow & T. Hausberger (Eds.), *Proceedings of the 1st conference of INDRUM* (pp. 11-27). Montpellier, France.
- Crawford, K. & Adler, J. (1996). Teachers as Researchers in Mathematics Education. In: Bishop A.J., Clements K., Keitel C., Kilpatrick J., Laborde C. (Eds.), *International Handbook of Mathematics Education* (pp. 1187-1205). Dordrecht: Springer.

- Cuoco, A. (2001). *Mathematics for Teaching*. *Notices of the AMS*, 48(2), 168–174.
- Jaworski, B. (1998). Mathematics Teacher Research: Process, Practice and the Development of Teaching. *Journal of Mathematics Teacher Education*, 1, 3-31.
- Liljedahl, P., Andrà, C., Di Martino, P. & Rouleau, A. (2015) Teacher tension: important considerations for understanding teachers' actions, intentions, and professional growth needs, In K. Beswick, T. Muir & J. Wells (Eds.), *Proceedings of the XXXIX Congress of the IGPME*, vol. 3, (pp. 193-201).
- Malara, N. & Zan, R. (2008). The complex interplay between theory and practice: reflections and examples. In L. English (Ed.), *Handbook of International Research in Mathematics Education* (pp. 539-564). New York: Routledge.
- Mason, J. (1998). Enabling Teachers to Be Real Teachers: Necessary Levels of Awareness and Structure of Attention. *Journal of Mathematics Teacher Education*, 1, 243-267.
- Ng, O. & Sinclair, N. (2015). Young children reasoning about symmetry in a dynamic geometry environment. *ZDM – The International Journal on Mathematics Education*, 51(3), 421-434.
- Rabardel, P. & Waern, Y (2003). From artefact to instrument. *Interacting with Computers*, 15, 641- 645.
- Remillard, J. (2005). Examining key concepts in research on teachers' use of mathematics curricula. *Review of Educational Research*, 75(2), 211-246.
- Ruthven, K. (2009). Towards a naturalistic conceptualisation of technology integration in classroom practice: The example of school mathematics. *Education & Didactique*, 3(1), 131–149.
- Ruthven, K. (2016). Constructing dynamic geometry: the interpretative flexibility of mathematical software in teaching practice (pp. 1–15). *Invited lecture to the ICME-13, Hamburg*, July 2016.
- Star, S. L., & Griesemer, J. R. (1989). Institutional ecology, 'translations' and boundary objects: amateurs and professionals in Berkley's museum of vertebrate zoology, 1907–1939. *Social Studies of Science*, 19, 387– 420.
- Venkat, H. (2016). Connecting research and mathematics teacher development through the development of boundary objects. In J. Adler & A. Sfard (Eds.), *Research for Educational Change: Transforming researchers' insights into improvement in mathematics teaching and learning* (pp. 182-194). London: Routledge.

AFFORDANCES AND TENSIONS IN TEACHING BOTH COMPUTATIONAL THINKING AND MATHEMATICS

Sandy Bakos and Mathieu Thibault

Simon Fraser University and Université du Québec à Montréal

This study reports on a dynamic geometry approach to the teaching of looping in a grades 2/3 classroom. The study is part of a large initiative to integrate computational thinking in the primary mathematics classroom. Descriptive analysis of interview data suggests that the majority of young learners were capable of interpreting and creating multiple action loops. However, we identify certain difficulties that some students experienced and report on the affordances and tensions involved in trying to combine computer science and mathematical concepts in a mathematics classroom.

INTRODUCTION

With the increasing use of digital technologies in mainstream society, there is an increasing recognition that students require exposure to and instruction related to computational thinking (CT), in order “to function, problem-solve, engage in digital innovation and advance a society already heavily technology-based” (Kotsopoulos *et al.*, 2017, p. 155). Wing (2008) defines CT as “an approach to solving problems, designing systems and understanding human behaviour that draws on concepts fundamental to computing” (p. 3717). Researchers argue that CT concepts such as variables and loops, and CT practices such as abstraction and decomposition are shared across many other disciplines, including mathematics (Lye & Koh, 2014).

Indeed, these connections have been explored at least since the work of Papert (1980). But, as Benton *et al.* (2017) note, “findings from research into the ‘impact’ of computer programming on children’s learning of mathematics had been inconclusive” (p. 118). These authors cite two challenges facing previous research: (1) the syntactical difficulty of programming languages such as *Logo* and (2) the difficulty of transferring from the language of *Logo* to that of the mathematics classroom. The former challenge has been attenuated by the emergence of block-based languages such as *Scratch*, which has been found to help support students’ learning of angle (Benton *et al.*, 2017) and probability (Gadanidis *et al.*, 2017). In part motivated by the second challenge, Sinclair and Patterson (2018) have proposed that dynamic geometry environments (DGE) can be seen as a visual computer programming language whose semantics are those of Euclidean geometry.

This paper reports on a portion of a larger study aimed at exploring the potential of using DGE at the primary school level, in order to support *both* geometric thinking and CT. In this sense, we use the lens of affordances as a way to describe DGE’s potential for developing the concept of looping with young learners.

AFFORDANCES AND TENSIONS AS A CONCEPTUAL FRAMEWORK

Affordances were first defined by Gibson (1986) from an ecological perspective, as “a relation between an organism and an object with the object perceived in a relation to the needs of the organism” (Hammond, 2010, p. 205). Affordances are usually talked about in relation to the potential of a particular technology to help learn a particular concept. Hammond (2010) argued that affordances in the interaction of the tool and the person provide both opportunities and constraints. Further, affordances are neither rigid or fixed. In the original design of dynamic geometry environments, there were few affordances related to the learning of computer science concepts. However, seen in an expansive way, a particular tool such as a DGE, along with the activities in which it is used, may give rise to new affordances, ones that will be different from the affordances of environments such as *Logo* or *Scratch*—both in terms of the opportunities and constraints. In this paper, we are particularly interested in the concept of looping, which is fundamental in computer science, but can also be seen as relevant to mathematics as a way of recognising repeated patterns. However, despite some overlap, there are also differences in the manner in which looping arises in CT and mathematics, which may require the teacher to make choices between which to privilege. Such pedagogical tensions are not uncommon in the face of “competing and worthwhile aims” (Ball, 1993, p. 373).

THE CONCEPT OF LOOPING AT THE CORE FOR LESSON DESIGN

Looping has been recognised as a challenging concept for students. At the middle school level, Grover and Basu (2017) report that students have difficulty distinguishing a loop from its initial and terminal events, counting the number of times the loop will repeat, and repeating each action of a multiple action loop separately. Mladenovic, Boljat and Zanko (in press) found that grades 5-6 students had less difficulty with looping when using *Scratch* than when using *Logo*, though the three errors mentioned above were still in evidence, especially the third one. In an effort to better support the looping concept, Grover, Lundh and Jackiw (2017) devised a more dynamic approach that addressed underlying ideas of computer science. Using a comic strip metaphor, students rearranged panels on a screen to learn about looping by focusing on sequence, pattern and repetition—which connect well to the patterning aspects of the mathematics curriculum. This dynamic approach was the starting point of our own work with much younger students (grades 2/3).

In designing the two 90-minute lessons, we also drew on the computational thinking pedagogical framework proposed by Kotsopoulos *et al.* (2017). The four pedagogical phases of learning that comprise this frame include: unplugged, tinkering, making and remixing, all of which may relate to physical, digital, computer-based or conceptual objects. Foundational in developing CT, *unplugged* experiences provide a means of introducing “preliminary and overlapping concepts related to CT” (p. 159) without the use of technology. *Tinkering* experiences involve taking existing objects apart and making changes and/or modifications for the purposes of exploration and considera-

tion of the implications of these changes. Through the use of existing objects, the cognitive demands of object building are relieved during tinkering, allowing learners to focus their attention on “[a]pplication, simulation and problem solving” (p. 161). When tinkering with computer programs, students can easily “see the connection between changes in the program and the outcome and... know immediately that an error has occurred” (p. 161) as a form of debugging. Tinkering experiences provide a rich “context for conjecturing, problem-solving, generalising and predicting – all which can lead to deeper mathematical understanding” (pp. 161-162). *Making* experiences involve constructing new objects, and require learners to make use of their foundational skills of knowledge and understanding to “problem-solve, make plans, select tools, reflect, communicate, and make connections across concepts” (p. 162). *Remixing* activities “involve the appropriation of objects or components of objects” (p. 154) which are then modified, adapted and/or embedded within another object, and used for substantially different purposes.

PARTICIPANTS AND CONTEXT OF THE RESEARCH

We worked with a group of twenty-one grades 2/3 students (8-9 years old) in a diverse and affluent urban neighbourhood in British Columbia, Canada. Looping was the third of three concepts we worked on, each of which we designed in order to connect strongly with geometric ways of thinking.

We started with the comic strip activity described above. Pairs of students shared an iPad and were asked to sequence the panels to tell a story. Each panel depicted a swimmer engaged in different activities: diving into a pool, swimming towards the right, swimming towards the left and stopped at the pool’s edge. After this tinkering experience, we asked students about how far the swimmer travelled and how the sequence might look if the swimmer travelled twice as far. At this point, a sketch (see Figure 1) was projected onto the SmartBoard and was used to explore these longer sequences as a whole class. The dynamic sketch afforded a visualisation of the concept of looping, showing the repeating parts of the sequence, and also initial and terminal events. The four pictures were displayed on the screen, along with a large coloured “bracket”, which surrounded the repeat block of the pattern and the number of desired repeats could be typed into a small box. The swim generator sketch would then arrange the pattern created in a comic strip format. Students were then asked what it would look like for the swimmer to travel five times as far. If each lap was 25 m, how many repeats would be needed for 200 m, 1 km, 10 km? The children were asked to draw what it would look like to have the swimmer swim 350 m, using the bracket notation to indicate the repeated actions.

The lesson concluded with two unplugged activities. First the children were asked to identify other situations in their everyday lives where they participated in activities that repeat like this. The children mostly offered other sporting situations. One student discussed interacting with her pet hamster, and how she opens up the cage, and keeps patting, patting, and then closes the cage. This response provided an opportunity to

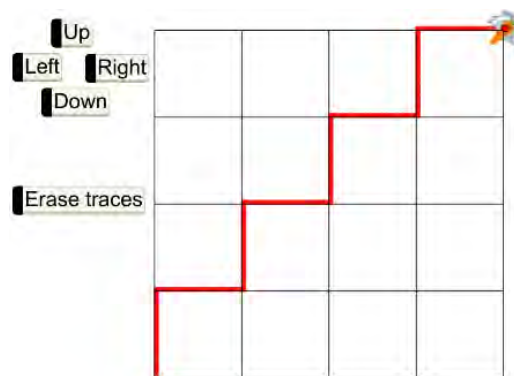


Figure 2: Rabbit and carrot sketch

The making experiences also afforded opportunities for tinkering. After successfully moving the rabbit to the carrot, and reviewing the sequence of instructions written on the chart paper, students were challenged to make or tinker with finding other paths that the rabbit could take to the carrot. Students were then asked to identify a faster way of saying one of the patterns recorded earlier on the chart paper. Using the idea of repeat, a student eventually said, “we could say right up and repeat it four... three.” The hesitation around the counting was discussed (‘repeat 4’ means there are 4 sets of repeated actions in total) and the teacher used the verbal order of the student’s statement to record (*right, up*) *repeat 4* on the chart paper. The class continued to experiment with these ideas through both making and tinkering experiences. Reinforcement was required regarding the looping aspect, and use of the repeat terminology after a student stated the sequence and then “do it again” rather than repeat. Transitioning back to an unplugged experience, students were given various rabbit-carrot grid scenarios in a booklet, and asked to either draw the path on the grid for a given procedure or to provide the procedure for a given drawing.

Throughout the teaching lessons related to this project, extensive written notes and photos were taken, and records of student work were retained. In order to ascertain student understanding of the looping concepts introduced and explored, four interview

questions were devised by the research team. Sixteen individual student interviews were conducted on two separate occasions, the first session was a week after, and the second session occurred two weeks after the final looping lesson. There were four items in the interview that pertained to looping: three unplugged and one on the iPad in a DGE. The first question asked the students to physically demonstrate a given procedure, which included a repeat component. The second question, the students were asked to write a procedure for a particular path shown on a grid (see Figure 3 or 4). For the third question the students were given a written procedure and asked if it would allow the rabbit to successfully reach the carrot. Students could use the dynamic sketch of the rabbit and carrot on the grid to confirm their response. The final question provided students with a written procedure and the accompanying visual trace of the path (see Figure 5), and asked them to rewrite the procedure using fewer words.

Data consists of student written responses, the screen capture data that recorded student actions on the iPad, and audio recordings of the interviews, which were transcribed for data analysis.

DATA ANALYSIS

The first question afforded the majority of students to successfully demonstrate the procedure [stand up, (look right, look left) repeat 3, sit down]. Of the three children who were unsuccessful, one of them could not complete it, and the remaining two had difficulty with the looping component: 1) completion of the sequence once without any acknowledgement of the repeat component; 2) repetition of the procedure three times, demonstrating a misconception regarding the repeat function.

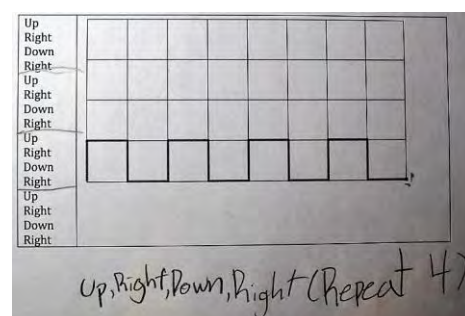
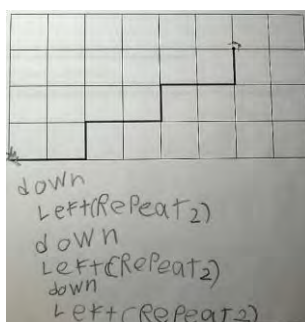
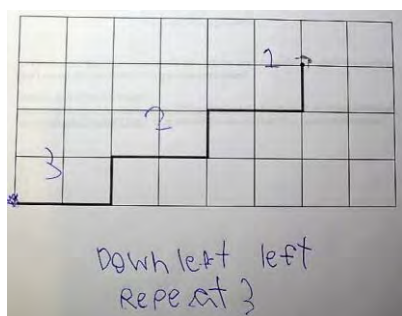


Figure 3: Sequence looping

Figure 4: Unit looping

Figure 5: Question 4

For the second question, there were three students who wrote out the procedure without any use of looping. One explained verbally that, “Instead of doing repeat, I just like writing it.” Two students got the number of repeats wrong. One student verbalised the initial sequence of the path as “down, left, left” and then pointed to the remaining two sections of the sequence and wrote, “repeat two”. The other wrote “repeat once.” The responses of the 13 students who used loops fell into two distinct categories, one of which we term *sequence looping* and the other *unit looping* (see Table 1). *Sequence looping* is considered the ideal response and involves identifying the entire loop and indicating how many times this sequence would repeat (see Figure 3). *Unit looping* involves identification of a smaller unit that repeats within the sequence (see Figure 4).

Use of sequence looping	Use of unit looping	No use of looping
7	6	3

Table 1: Question 2 results for the 16 students who attempted it

For the third question, the following procedure was used: [(up) repeat 4, (right, down) repeat 4, (up) repeat 4]. All students, except one confirmed that the rabbit would reach the carrot. Most children approached this as a making activity (see Table 2), by creating the path on the iPad screen to confirm that the path was, indeed correct. Many children transitioned back and forth between making and tinkering, often due to self-correction of an error. There were two children who did not correctly complete the procedure. When the rabbit did not reach the carrot, one child stated that “No, I don’t think so. It’s not going at the ending”, and moved on to the next question. The second child, after making an error, simply adjusted the procedure to ensure that the rabbit would reach the carrot.

Unplugged	Making	Making & Tinkering
1	6	8

Table 2: Question 3 results for the 15 students who attempted it

For the final question, 11 children used *sequence looping* to write the procedure (see Figure 5 and Table 3): nine students responded correctly, including one who wrote ‘x4’ instead of repeat four, and two got the number of repeats incorrect. Only two students attempted to use *unit looping* when rewriting the procedure, and both did so incorrectly: 1) by counting and ‘collecting’ the number of ups (4), rights (8) and downs (4), then writing a procedure to reflect this without respecting the path provided; 2) by noticing some patterns and attempting to identify the units and looping that would go with them. One student rewrote the entire sequence without any use of looping.

Use of sequence looping	Use of unit looping	No use of looping
11	2	1

Table 3: Question 4 results for the 14 students who attempted it

DISCUSSION

We first highlight some affordances of working across CT and geometry. The activities focused on looping engaged children in identifying patterns that could be repeated to describe a sequence. As the classroom teacher remarked, this practice is similar to pre-algebra patterning work in the mathematics classroom, where students are asked to identify the “unit” of a pattern that repeats. The teacher motivated the shift to the ‘repeat’ notation by challenging the students to describe it in more efficient ways—efficiency being valued both in computer science and in mathematics. In this shift to the grid, the students engaged with looping in a more geometric context in

which they had to use spatial reasoning to predict and describe the movement of the rabbit. The grid is a fundamental object both in computer science and in mathematics.

Despite these affordances, there were three issues that arose in relation to the different expectations and values that arise in computer science and geometry. The first relates to notation. In programming languages, the common looping notation involves a set of actions bracketed by a “repeat n ”, as shown in Figure 1. This formulation “repeat n ” means that the set of actions appear exactly n times; that is, they are not repeated n times after the first set of actions. The latter interpretation would be entirely acceptable in mathematics. Further, the placement of the “repeat n ” doesn’t have a convention in mathematics. In choosing to place it after the articulation of the set of actions, the teacher hoped to make communication about the pattern easier. However, a tension arose between this desire to ease communication in a context where the placement of the ‘repeat n ’ was not important, and a stricter adherence to the disciplinary norms of computer programming.

A second issue relates to the use of the grid. Many students struggled with the directional language, especially with right versus left. Given the focus on looping, the teacher focused more on the identification of the repeated actions than on the correctness of the directions, which meant that several students did not get sufficient opportunities to develop their spatial language. Of course, this subordination of spatial language to looping could also have occurred with a different concept, including a mathematical one. Nonetheless, it highlights tensions that teachers can face when trying to integrate CT in their mathematics teaching practices.

A final issue relates to the use of sequence versus unit loops. In a mathematics context, identifying patterns of repetition is the main goal of learning, so both sequence and unit loops could be equally valued. Seeing global patterns as well as patterns within patterns can be important in many different mathematical contexts. The norms of computer programming instruction, however, privilege the sequence loop over the unit loop as it is more efficient. Given the large number of students who preferred using a unit loop, a tension might arise in relation to whether it is important to insist on sequence looping given the CT aims, or whether unit looping can also be encouraged given the mathematical aims of the teacher.

CONCLUSION

When integrating CT into the mathematics classroom, the geometric context of a dynamic geometry approach was appropriate for facilitating the learning and application of the CT looping concept with grades 2/3 students. Although our analyses identified some tensions related to student difficulties, it suggests that the looping activities afforded the majority of the learners to interpret and create multiple action loops. Furthermore, the students demonstrated an ability to independently choose and move amongst various pedagogical experiences, such as making and tinkering, in order to respond to the CT questions. The findings of this study suggest that although there are affordances in combining computer science and mathematical concepts in the context

of a mathematics classroom, there can also be unexpected tensions that confront teachers in relation to pedagogical decisions that privilege either the CT or the mathematics.

Acknowledgment: We wish to thank Dr. Nathalie Sinclair for her thoughtful feedback on early drafts of this paper.

References

- Ball, D. (1993). With an eye on the mathematical horizon: Dilemmas of teaching elementary school mathematics. *The Elementary School Journal*, (93)4, 373-397.
- Benton, L., Hoyles, C., Kalas, I., & Noss, R. (2017). Bridging primary programming and mathematics: Some findings of design research in England. *Digital Experiences in Mathematics Education*.
- Gadanidis, G., Hughes, J., Minniti, L. & White, B. (2017). Computational thinking, grade 1 students and the binomial theorem. *Digital Experiences in Mathematics Education*, (3)2, 77-96.
- Gibson, J. (1986). *The ecological approach to visual perception*. New York: Lawrence Erlbaum Associates.
- Grover, S. & Basu, S. (2017). Measuring student learning in introductory block-based programming: Examining misconceptions of loops, variables, and Boolean logic. *Proc. 48th ACM SIGCSE Technical Symposium on Computer Science Education* (pp. 267-272). Seattle, USA: ACM.
- Grover, S., Lundh, P., & Jackiw, N. (2017). Thinking outside the box: integrating dynamic mathematics to advance computational thinking for diverse student populations. Poster presentation at the *American Educational Research Association*. San Antonio, TX.
- Hammond, M. (2010). What is an affordance and can it help us understand the use of ICT in education? *Education and Information Technologies*, 15(3), 205-217.
- Kotsopoulos, D., Floyd, L., Khan, S., Namukasa, I.K., Somanath, S., Weber, J., & Yiu, C. (2017). A pedagogical framework for computational thinking. *Digital Experiences in Mathematics Education*, 3(2), 154-171.
- Lye, S. & Koh, J. (2014). Review on teaching and learning of computational thinking through programming: What is next for K-12? *Computers in Human Behavior*, 41, 51-61.
- Mladenovic, M., Boljat, I., & Zanko, Z. (2017). Comparing loops misconceptions in bloc-based and text-based programming languages at the K-12 level. *Educ. Inf. Technol.*
- Papert, S. (1980). *Mindstorms: Children, computers, and powerful ideas*. New York: Basic Books.
- Sinclair, N. & Patterson, M. (2018). The dynamic geometrization of computer programming. *Mathematical Thinking and Learning*.
- Wing, J. (2008). Computational thinking and thinking about computing. *Philosophical Transactions of the Royal Society A*, 366(1881), 3717-3725.

THE TRANSITION FROM HIGH SCHOOL TO UNIVERSITY MATHEMATICS: ENTERING A NEW COMMUNITY OF PRACTICE

Amalia-Christina Bampili¹, Charalampos Sakonidis², and Theodossios Zachariades¹

¹National and Kapodistrian University of Athens, Greece;

²Democritus University of Thrace, Greece

The transition from secondary to tertiary mathematics encompasses a complex interaction of social, institutional and mathematical context changes, including a vast array of emotions, beliefs and issues. The present paper reports on a study of the challenges faced by two first year mathematics undergraduate students during their transition from secondary to tertiary education through the lenses offered by the Communities of Practice framework. Data were gathered over the students' first two semesters of attendance predominately through interviews. The results indicate a powerful interaction between social and institutional issues shaping their initiation into a new practice of mathematics.

INTRODUCTION

The secondary-tertiary transition is itself an exciting and often confusing experience for students. After tough examinations, the successful students have yet to adjust to new learning environments, new modes of study, and above all, higher expectations of the self.

The problems encountered in the transition from high school to university mathematics are common in educational systems worldwide. Several researchers identify a “gap” between school and university mathematics content (Luk, 2005; Kajander & Lovric, 2005; Winsløw, 2013), while others point at important changes that affect students during the secondary-tertiary transition. These include the new academic and social environment as well as the shift required to a way of thinking and studying mathematics that differs from that promoted at school (Cherif & Wideen, 1992; Tall, 1992).

The paper reports on a study of how two first-year students in a Mathematics Department of a Greek University dealt with transition issues through the lens offered by the Communities of Practice framework, focusing on the ways that social and institutional parameters shaped their shift to a new practice of ‘doing mathematics’ required.

LITERATURE REVIEW

Mathematical practices at university level are distinguished from those at secondary level for reasons related to the mathematical content as well as to the participants in each of the two practices (i.e., teachers and students). In the new institutional environment students are adults considered responsible for their choices, including their

choice of studying mathematics, and they are expected to study and learn independently (Biza, Jaworski, & Hemmi, 2014).

The transition from high school to university mathematics can be seen as the result of many interacting transitions: social, institutional, mathematical content transitions as well as others (Alcock & Simpson, 2002). University as an institution and university mathematics are encountered as new worlds, where new communication and participation rules are required to live in, which might make the novice student feel like a foreigner (Gueudet, 2008).

Hernandez-Martinez et al. (2011) considered social aspects as the most important when entering university. Students argue that the beginning of university life can be a quite scary and nerve-racking phase for many. The change from a structured, parent-disciplined life to a self-disciplined university life is difficult. First-semester students claim that the change of the education environment, new expectations and unlimited freedom are the biggest problems (Cherif & Wideen, 1992; Clark & Lovric, 2008).

Concerning institutional issues, students in transition undergo changes requiring an adjustment of learning strategies, time management skills and a shift to more independent studying. The new environment demands a different type of critical thinking, something for which students are not necessarily prepared (Cherif & Wideen, 1992).

As far as the mathematical content is concerned, for first-year university students

the move to more advanced mathematical thinking [which] involves a difficult transition, from a position where concepts have an intuitive basis founded on experience, to one where they are specified by formal definitions and their properties reconstructed through logical deductions (Tall, 1992, p. 1).

First-year undergraduates are confronted with a significant change from a computational to a proof-based learning and teaching approach; they witness an increased emphasis on the precision and rigor of the mathematical language, and this is very new for them (shock of the new) (Clark & Lovric, 2009). In other words, a shift from “instrumental understanding” to more “relational understanding” is on demand.

The above suggest that students studying mathematics at university level enter a new community where the practice of being a student differs from that of the school community. Hence, the need of shifting to new ways of “being” and “belonging” signifies the need to developing a new identity of practicing mathematics.

THEORETICAL FRAMEWORK

We employed the Communities of Practice (CP) framework based on the work of Lave and Wenger (1991) and Wenger (1998) to explore the ways in which the subjects of the study dealt with identity issues through analyzing the changing forms of their participation to the studying university mathematics practice during the transitional phase: from entrance as a newcomer, through becoming an old-timer.

Within this perspective, the person is defined by as well as defines relations, which are in part systems of relations among persons (Lave and Wenger, 1991). Activities and understandings are part of broader systems of relations in which they have meaning. In this sense, identity in practice arises out of an interaction of participation and reification. Lave and Wenger consider learning as increasing participation in CP, which concerns the whole person acting in the world:

...a community of practice is a living context that can give newcomers access to competence and also can invite a personal experience of engagement by which to incorporate that competence into an identity of participation (Wenger, 1998, p. 214)

The transition from newcomer to old-timer involves differing trajectories of identity. According to Wenger (1998), a trajectory can be seen as a continuous motion through time that connects the past, the present and the future. Wenger states: “As we go through a succession of forms of participation, our identities form trajectories, both within and across communities of practice” (p.154), including peripheral (never leading to full participation) and inbound (from the periphery to the centre) trajectories. Furthermore, an individual’s identity is shaped by combination of participation and non-participation in the community of practice. With respect to the interaction of participation and non-participation, Wenger (1998) distinguishes two cases: peripherality (some degree of non-participation enables a less full participation) and marginality (non-participation prevents full participation).

In the following section, we present two cases of first year students entering university mathematics practice, using CP framework to make sense of characteristics and issues of their struggle to become participants in the new practice.

THE STUDY

Situated within the literature reviewed above, the study reported here is part of an ongoing research project aimed to examine the interface between social, institutional and mathematical content aspects of the transition from high school to university mathematics. In particular, the research question pursued in the study is as follows:

How do social and institutional issues shape the initiation to the studying university mathematics practice and thus to the development of a new identity by first-year students?

Greek students go through hard preparation to pass the university entry exams. During their final high school year, most of the students undergo a strictly structured life program, including many hours of daily study almost always under the guidance of school teachers and private teachers in paid courses after school. They are introduced to Calculus, the emphasis of teaching being, however, more on computational than conceptual learning/understanding. At university level first-semester students are introduced to mathematical theory which involves very specific and rigorous rules and processes (such as theorems, definitions and proofs). This constitutes a qualitatively big jump for their thinking. Furthermore, there is hardly any support around provided

either by the academic staff (e.g., in the form of learning advisors), or by later-year students and/or the Students' Association.

In October 2015 we started surveying incoming first-year students (October 2015-June 2016), collecting information. Twelve students volunteered to be interviewed individually to help us look at the issues described above. Four semi-structured interviews (in the beginning of the first semester, before the semester exams, in the middle of the second semester and before the second semester exams) were carried out, each lasting between 25 and 45 minutes; these were audio-recorded and fully transcribed. Students were asked about their conceptions of university mathematics, how their experience of mathematics at school differed from that at the university, how their study habits or ways of working had changed, how they felt being a member of a new institutional environment and how they dealt with the changes in their social-personal life.

Two of these students, Nefeli and Asli (pseudonyms), are the focus of the work presented here: Nefeli's responses during the data collection strongly indicated that she was undergoing changes (from a peripheral participant to an almost full participant) regarding studying mathematics: although she was doing well in mathematics (her grades were good at school and also in the university entry exams), in the beginning of her first university year she felt that perhaps it had not been a good decision to study mathematics. She was negatively affected because of the overwhelming changes imposed in her lifestyle and the new institutional environment that strongly influenced her studies. She even considered quitting. Only after the first semester exams did she started adapting to the new environment, and at the end of the first year she almost felt well adjusted. On the other hand, Asli's responses indicated that she was struggling (at least to be a peripheral participant) regarding studying mathematics: although she was doing well in mathematics (her grades were good at school and almost good in the university entry exams), from the beginning to the end of her first university year she felt that she could not meet the requirements of her studies, although she never doubted that it was a good decision to study mathematics. As Nefeli, she was also negatively affected by the changes imposed in her lifestyle and the new institutional environment. She could not pass the first semester exams and she was facing problems adapting to the new environment. At the end of the first year she almost lost her motivation.

RESULTS

Nefeli's and Asli's representative comments and thoughts related to transition and expressed in four interviews were organized along social, institutional and mathematical content aspects, as presented in Tables 1, 2 and 3. In the following, some central issues emerging along each of these aspects are discussed.

The social issues of the transition were seen by both as among the most important (but also worrying) creating, among other things, time managing problems (Table 1).

Interviews	Social issues
first	SN ₁ : "I am negatively affected because of the long home-university distance". SA ₁ : "I have also a part-time job. I have not enough time to study".
second	SN ₂ : "I manage time better, but I'm still undergoing a total change in my former well-organized life".
third	SN ₃ : "I have the opportunity to manage my time as I want, although not so effectively all the time". SA ₃ : "It is hard for me to manage time better. I am always tired and bored to study".
fourth	SN ₄ : "I also manage the time between lectures more qualitatively.... I go to the library and study". SA ₄ : "Sometimes I think that maybe it's an excuse that because of my part time job I have no time to study".

Table 1: Social aspects through the interviews

Interviews	Institutional issues
first	IN ₁ : "I have great expectations for academic staff support, like in high school" IA ₁ : "Students have to deal alone with their studies more than I expected".
second	IN ₂ : "I believe that professors take it for granted that students understand mathematics. I am afraid to ask the professor, because he may think that I am stupid". IA ₂ : "...professors do not make any effort to help students understand their lectures".
third	IN ₃ : "I am negatively influenced by the absence of help from the Student Association and the absence of a Student Learning Advisor". IA ₃ : "I am negatively affected by the absence of any help. We do not have either a Student Learning Advisor or a Tutor".
fourth	IN ₄ : "...Although I have to admit that some professors guided us well enough, my adjustment was getting better after a long time with great mental and spiritual effort". IA ₄ : "I need some help. I struggle alone to find out which courses to take, how to study effectively...".

Table 2: Institutional aspects through the interviews

Both students experienced big changes in the new institutional environment. A vast array of answers is identified in their interview responses: from great expectations for a creative teacher-student relationship and academic staff support to their statement that some professors do not care at all if students understand their lectures (Table 2).

Regarding studying mathematics, Nefeli lost her self-confidence at the beginning. As time went by, she confronted studying mathematics as a challenge: to turn her disappointment and stress to something powerful and effective. On the other hand, Asli could not find a way to study and learn independently (Table 3).

Interviews	Mathematical content issues
first	<p>MN₁: “In high school, we did not pay much attention to the conceptual understanding. Teachers told us what to study and how. When I started studying university mathematics, I was desperate. I was wondering if I had taken the right decision”.</p> <p>MA₁: “I am negatively affected by the fact that for the university entrance exams I could not achieve the grades that I expected”.</p>
second	<p>MN₂: “If I could say only one thing that I still struggle with, this is the difficulty of the subject. ... I felt I turned my love to mathematics to something sick....”.</p> <p>MA₂: “I think it was a good decision to study mathematics, but the subject is more difficult than I expected”.</p>
third	<p>MN₃: “...I realized that to do well on the first semester exams, I had to use my “simple” knowledge inductively to solve a problem, rather than knowing many things”.</p> <p>MA₃: “I could not pass the 1st semester exams. I think that even if I study hard I will again fail the exams”.</p>
fourth	<p>MN₄: “I feel more confident. I passed the exams with good grades”.</p> <p>MA₄: “I am struggling a lot. If I get my degree with a low score, how will I find a job afterwards?”</p>

Table 3: Mathematical content aspects through the interviews

The results indicate the dynamics and the connections identified within all aspects. Nefeli, as a newcomer, had to deal with her expectations concerning her social and academic life (SN₂, IN₁) and “move away” from her former way of living and studying (SN₁, MN₁). The lack of studying support, which is a feature of the new institutional environment, affected her almost until the end of the first year. She struggled a lot to achieve necessary changes (to come closer to full participation), something that also affected her self-confidence as a mathematics student (IN₂, MN₁, MN₂). Her great mental and emotional effort as well as the influence of some inspiring professors

(MN₄, IN₄) helped her to take the next step. After the first semester exams and more clearly near the end of the first year, it looks like she had also managed to find the needed way of studying (MN₃). Overall it seems that she was close to finding her place within her new community, which is a feature of an old-timer (SN₃, SN₄, MN₄). Her success in Calculus I and II exams (8/10 and 10/10) can be seen as a positive outcome of her efforts.

With regard to Asli, the analysis of the data show that she could not deal with the changes in her social and academic life (SA₁, SA₃). The lack of studying support affected her until the end of the first year (IA₂, IA₃, IA₄). Although she had no doubts about her decision to study mathematics (MA₂), the fact that she could not achieve the grades that she expected for the university entrance exams (MA₁), affected her self-confidence as a math student (MA₃). It seems that it was difficult for her to find her place within the new community (forming an identity of non-participation which is close to marginality). She failed the first semester exams and almost felt losing her motivation (MA₄).

DISCUSSION AND CONCLUSIONS

The results of our analysis reveal that the students' shift to unknown social and institutional communities has a powerful effect on their studying of university mathematics. When Nefeli and Asli were high-school students, the social (family, friends-classmates) and the institutional (school and private lessons) communities were aligned: all supported them with the aim of passing university entrance exams, an achievement highly valued in Greek society. Hence, their life was well organized by others (family, school and teachers) for this goal to be fulfilled. As first year university mathematics students, they are expected to deal, mainly on themselves, with social and institutional issues never encountered before: to participate in these new communities practices in ways beneficial to their studies, with hardly any support provided either by the academic staff, or by later-year students and/or the Students' Association and even by their new friends and fellow students.

In the context of the research reported here, studying university mathematics is seen as a social activity, and specifically as participation in interacting communities of practice shaping the development of a new identity of practicing mathematics. The ways students reflect on being school students and novice university mathematics students, and how they deal and conceptualize social, institutional and mathematics studying issues may evolve. In CP terms, developing an identity as a student of mathematics is about negotiating what counts as legitimate "being" within various communities, in university and school, comprising shifting conceptions of what mathematics studying is or should be. We thus assume that possible changes in mathematics students' accounts constitute evidence of a developing identity, in which the student's changing images of the self are embedded.

Acknowledgment: The third author thanks Special Account for Research Grants and National and Kapodistrian University of Athens for funding to attend the conference.

References

- Alcock, L. & Simpson, A. (2002). Definitions: Dealing with Categories Mathematically. *For the learning of Mathematics*, 22(2), 28–34.
- Biza, I., Jaworski, B., & Hemmi, K. (2014). Communities in university mathematics. *Research in Mathematics Education*, 16(2), 161–176.
- Cherif, A. & Wideen, M. (1992). The problems of the transition from high school to university science. *Catalyst* 36(1), 10–18.
- Clark, M. & Lovric, M. (2008). Suggestion for a theoretical model for secondary-tertiary transition in mathematics. *Mathematics Education Research Journal*, 20(2), 25–37.
- Clark, M. & Lovric, M. (2009). Understanding secondary-tertiary transition in mathematics. *International Journal of Mathematical Education in Science and Technology*, 40(6), 755–776.
- Gueudet, G. (2008). Investigating the secondary-tertiary transition. *Educational Studies in Mathematics*, 67, 237–254.
- Hernandez-Martinez, P., Williams, J., Black, L., Davis, P., Pampaka, M., & Wake, G. (2011). Students' views on their transition from school to college mathematics: rethinking 'transition' as an issue of identity. *Research in Mathematics Education*, 13(2), 119–130.
- Kajander, A. & Lovric, M. (2005). Transition from secondary to tertiary mathematics: McMaster University experience. *International Journal of Mathematical Education in Science and Technology*, 36(2–3), 149–160.
- Lave, J., & Wenger, E. (1991). *Situated learning. Legitimate peripheral participation*. New York, NY: Cambridge University Press.
- Luk, H.S. (2005). The gap between secondary school and university mathematics. *International Journal of Mathematical Education in Science and Technology*, 36(2–3), 161–174.
- Tall, D. (1992). The Transition to Advanced Mathematical Thinking: Functions, Limits, Infinity and Proof. In D. A. Grouws (Ed.), *Handbook of Research on Mathematics Teaching and Learning* (pp. 495–511). New York: Macmillan.
- Wenger, E. (1998). *Communities of practice: Learning, meaning and identity*. Cambridge: Cambridge University Press.
- Winsløw, C. (2013). The transition from university to high school and the case of exponential functions. In B. Ubuz, C. Haser & M. A. Mariotti (Eds.), *Proceedings of the Eighth Congress of the European Society for Research in Mathematics Education* (pp. 2476–2485). Ankara: ERME.

SITUATIONAL AND DISTAL SOURCES OF MEANING IN A MULTILINGUAL MATHEMATICS CLASSROOM

Richard Barwell

University of Ottawa, Canada

In research on mathematics learning in contexts of language diversity, much work has focused on students' meaning-making, using the notion of language as a resource. In this report, I use an alternative though related perspective that sees language in terms of sources of meaning. This perspective is based on a dialogic, Bakhtinian theory of language. Sources of meaning are examined in terms of discourses, voices and languages. I present preliminary analysis of the sources of meaning used in one multilingual Canadian mathematics classroom. I show how the sources of meaning drawn on by students and the teacher in mathematical meaning-making can be usefully distinguished as situational or distal in nature (or both). Distal and situational sources of meaning are implicated in the stratification of mathematics classroom interaction.

INTRODUCTION

There are many situations in which language diversity can arise in mathematics classrooms. Many school students around the world learn mathematics in second- or additional-language mathematics classrooms, multilingual mathematics classrooms, content-integrated learning mathematics classrooms, or language immersion mathematics classrooms. Despite persistent assumptions that such students face barriers or must struggle to learning mathematics, in all these contexts, students do learn mathematics, and sometimes outperform other students (e.g. Clarkson, 2007). How do they do this? For many years, researchers have been developing a more complex understanding of how multilingual students learn mathematics in different contexts (for a review, see Barwell, Moschkovich & Setati Phakeng, 2017).

A key strand of this work has examined students' meaning-making processes in mathematics. This work has identified many language "resources" used by students. The most widely examined resource is the language that students use in addition to the language of instruction (Clarkson, 2007; Planas, 2014; Planas & Setati, 2009; Setati, 2005). Setati (2005) showed how students in South Africa used both English (the language of instruction) and Setswana (their home language) in their elementary school mathematics class. Clarkson (2007), showed how students from immigrant backgrounds in Australia used their home language when problem solving. In both contexts, their teachers were not always aware that students used their home languages to think about mathematics. Another strand of work has looked at discursive resources used by students, including gestures and features of diagrams (e.g., Moschkovich, 2008), as well as generic and narrative forms in the context of word problems, and

features of written language (e.g., Barwell, 2005). In a detailed analysis of the work of two Spanish-speaking students in the United States, Moschkovich (2008) showed how they drew on various resources, including markings on a graph, which were indicated through the use of gestures, as well the multiple meanings made available through discussing the mathematics problem and hearing each other's interpretations (of a graph, in this case). In my own work involving students from minority language backgrounds in the UK, I have shown how students made sense of word problems by drawing on narrative accounts of their experiences of the situations in the word problems (e.g. shopping), their familiarity with the generic form of word problems, as well as their emerging familiarity with features of written English, such as spelling and verb tenses (e.g., Barwell, 2005).

This work has been important in challenging deficit orientations towards language diversity in mathematics classrooms, by highlighting the many features of language available to students for making mathematical meaning. These features are often thought of in terms of resources. Recent work has started to explore and critique the notion of resource (e.g., Barwell, 2015; Planas, 2017). For example, a resource perspective can suggest a view of language that fails to capture the fluid, mutually shared nature of language in use. In this report, I draw on an alternative approach derived from a dialogic view of mathematics classroom language.

THEORETICAL FRAMEWORK: SOURCES OF MEANING

Rather than language as a resource, I see language in terms of sources of meaning. To explain this perspective, I first need briefly to set out some general ideas about language and its use in classrooms. I see language as dialogic and organised around two opposing forces first described by Bakhtin (1981). Centripetal language forces work to impose standard forms of language at every level, from pronunciation and accent, to written grammars and standardised style sheets. Centrifugal forces work in the opposite sense, in a constant diversification that is apparent in every utterance. These forces are apparent in mathematics. Centripetal forces arise whenever we refer to formal mathematical language, or invoke the notion of a single mathematical discourse or mathematics register. Centrifugal forces are apparent in the variation that exists whenever people talk or write about mathematics.

The inherent diversity of language, sometimes known as heteroglossia, can be organised into three broad *sources of meaning* (see Busch, 2014). *Multiple discourses* relate to different forms of social organisation, such as the language of teaching or the law, the language of families, or language of activities like mathematics (Bakhtin, 1981, pp. 271-272). In mathematics, multiple discourses arise, for example, in different mathematical domains. *Multiple voices* arise in every utterance, which necessarily reflect not just the speaker's voice, but also the voice of those who have previously used similar words. In learning a new mathematical term, for example, students voice their own ideas, but the teacher's voice is also present. Finally, *multiple languages* are increasingly encountered in mathematics classrooms around the world, in which students

have some proficiency in more than one language, even if, in many classrooms, these languages are rarely heard. These broad sources of meaning are seen less as fixed resources waiting to be used, and more as a flow (like a spring or source of water) in which students are immersed and which they shape to their intentions.

These sources of meaning reflect centripetal and centrifugal forces, which are in turn related to the stratification of language and of society (Bakhtin, 1981). The standard forms of language imposed by centripetal forces tend to reflect the language of the powerful. In mathematics classrooms, for example, teachers will often impose the dominant language of schooling, and will determine what counts as ‘correct’ mathematical language. Heteroglossia, driven by centrifugal forces, reflects the wide variety of forms of expression, and so includes marginal, informal and unorthodox discourse, voices and languages (Duranti, 1998). Students who do not learn to use ‘standard’ forms of mathematical language are likely to be seen as less proficient.

Based on this theoretical perspective, for the analysis reported in this paper, I addressed the question: What sources of meaning are apparent in the mathematical meaning making in one multilingual mathematics classroom in Canada?

DATA COLLECTION AND ANALYSIS: A CASE STUDY OF ONE CLASS

My analysis focused on data collected in one Grades 5-6 class (ages 10-12 years) in Quebec, Canada. The majority language in Quebec is French and the class is in a school in the French language school system of that province. The 18 students were all recently arrived immigrants to Quebec and came from various backgrounds, including South America, West Africa and the Middle East. Students’ home languages included Spanish, Arabic and Swahili. At the start of the school year, none of the students could speak French. The main objective of the class, according to the teacher, was to teach students enough French to be able to enter mainstream classes, and to prepare them for Quebec school life.

Data were collected as part of a larger ethnographic study of four second language mathematics classrooms of different sorts. For this class, I collected the following data over several weeks in May and June 2010: fieldnotes from classroom observations; audio-recordings of whole-class and small-group work; images of classroom artefacts, worksheets, textbooks and students’ written work; and interviews with students and the teacher. After each visit to the class, I wrote up a summary ‘visit report’ of my observations. Altogether, nine full mathematics classes were observed by myself or a research assistant. For the analysis reported here, I reviewed the visit reports to identify moments in which multiple discourses, multiple voices or multiple languages featured in mathematical meaning-making. These dimensions are not mutually exclusive and occur in the same moments. To understand how these sources of meaning played out in the interaction, I also consulted other data, such as transcripts or images of students’ work.

RESULTS

The classes I observed were mostly devoted to work on geometry. The teacher placed a strong emphasis on vocabulary, such as the names of shapes and angles, and terms used for the properties of shapes. My findings are organised around the broad three sources of meaning. As I worked through the data, I noticed that for each of set of sources of meaning, the specific features of language could be organised into two groups. Some discourses, voices and languages arose within and relied on the immediate situation. Meanwhile, some had broader societal origins, such as ideas about the nature of formal mathematical language or ‘good’ French. For the former I refer to *situational* discourses, voices and languages, and for the latter, I refer to *distal* discourses, voices and languages.

Multiple discourses

Discourse features apparent in the class included aspects of both oral and written language. Orally, students participated in genres of explanations of vocabulary items, discussion of tasks and problems, whole class discussion, posing questions, and the use of deictic language and gestures. Deictic language ‘points’ to locally present referents, often using words like ‘this one’ or ‘up there’. Written discourse features included formal definitions, vocabulary in print or written on the blackboard, worksheet genres containing simple and complex tasks, and the teacher’s blackboard writing.

Throughout my observations, the teacher emphasised pronunciation and repeated key terms many times. She also spoke slowly and enunciated clearly, using many gestures, diagrams and physically present objects. For example, one class focused on types of angle:

First, the teacher introduced the notion of a right angle. This concept seemed familiar to the students, since several of them gave examples of angles by identifying objects in the classroom. Most of them pronounced this word without difficulty, even if some seemed to say ‘dwa’ instead of ‘droit’. Next, she introduced the notion of acute (aigu) angle. Even if most pronounced it egou, all the students seemed to understand the concept, since they all made gestures of acute angles with their hands. For a question about obtuse angles, Luis replied ‘angle ouvert’ (open angle). (Visit report, 19 May, 2010)

This account illustrates several of the discourse features I have mentioned. The teacher introduces vocabulary items relating to angles, explains them orally using examples and asks students to use these new terms. She poses closed questions to the students (an oral classroom genre) in which they must identify if an angle made with her hands or feet is obtuse, acute or right-angled. Luis seems to understand but uses a non-standard term ‘open angle’ rather than the term the teacher has introduced ‘obtuse angle’.

The next day, I observed another student working on a problem about an image of a boat presented on a worksheet:

I watch [a student] trying to draw a sail that has 4 right angles and 4 equal sides. He tries several times, erasing each time. It seems that he is attending to 4 sides, but not the specific criteria. A couple of times he draws three sides with obtuse angles before erasing. After a

while I read out one of the criteria – ‘quatre angles droits’ [‘four right angles’]. He then starts drawing rectangles, suggesting he hadn’t attended to the ‘droit’ before. He tries a couple of rectangles and erases them. I read out the other criterion – ‘quatre angles égaux’ [‘four equal angles’]. He keeps drawing rectangles. I read it again and repeat égaux a couple of times. He still has a rectangle so I ask him how long one of the sides is and then another and I say they should be the same – 4 and 4 or 3 and 3 or 2 ½ and 2 ½. Now he works on drawing a square. (Visit report, 20 May 2010)

This episode illustrates students’ encounters with formal discourses of school mathematics, including the written genre of the worksheet, the relatively formal formulation of the parameters for the shape of the sail and the use of standard mathematical vocabulary. My observations suggest that the student has interpreted the instructions in terms of ‘four’, ‘angles’ and possibly ‘sides’ (the word ‘droit’ could be interpreted to mean ‘line’) but has not initially interpreted ‘equal’ or ‘right angle’ in a conventional way.

These examples serve to show the presence of situational and distal discourse features. Situational features include the use of gestures (the teacher’s creation of angles with her hands), Luis’s informal expressions of mathematical ideas and use of non-standard vocabulary (e.g. ‘open angle’) and students’ informal or initial interpretations of key terms and text genres. These features are situational because they make sense within the immediate situation and rely on that immediate context to make sense. For example, if Luis talks about open angles elsewhere, the term is unlikely to be understood, since it is not a conventional mathematical way of describing angles. Distal features include the teacher’s introduction and reinforcement of formal vocabulary, her regular enunciation and repetition of correct pronunciation, the use of formal written definitions, and written problem and mathematical text genres.

Multiple voices

Both whole class and small group interaction involved an interweaving of students’ informal expressions of mathematics, revoiced and reworked by the teacher in more formal terms, so that students’ voices intermingled with the teacher’s. For example, on one occasion, a student answered that a particular shape is “convex”, but with a Spanish accent. The teacher then rehearses the pronunciation with him:

Student 1: conbexe [*conbex*]

Teacher N: non: (.) convexe ok redis-le (.) non-convexe [*non (.) convex ok say it again (.) non-convex*]

Student 1: non (.) con (.) bexe non-converse? [*non (.) con (.) bex non-converse?*]

Student 2: non (.) con-vexe [*non (.) con-vex*]

Teacher N: non-convexe con (.) v (.) v (.) vexe [*non-convex con (.) v (.) v (.) vex*]

Student 1: non-convexe? [*non-convex?*]

Teacher N: ouais c’est pas un B ah tu comprends? [*yes it’s not a B ah you understand?*]

(Transcription from recording, 10 May 2010)

This process of rehearsal results in the student using words that came from his teacher's mouth. When he correctly utters the word 'non-convexe' we hear both the student's voice and the teacher's. This multi-voicedness extends to non-verbal features of interaction:

[The teacher] developed the students' explanation to introduce words like straight and curved and convex. I noticed that she used lots of gestures – for example for *ligne droite* she gestured a vertical sweep directly in front of her. The students also gestured frequently and the gestures appeared to echo hers. (15 May, 2010)

Similarly, to when students revoiced the teacher's words, they also used gestures introduced by the teacher, combining their own meanings with those of the teacher. Voices from outside the classroom were also sometimes apparent, such as when students were asked to group a set of shapes:

Luis's group explained their grouping based on what he had learned about angles at his previous school. Jeanne explained the same thing and talked about corners rather than angles. (15 May, 2010)

One group of students revoices ideas encountered at a previous school, while another revoices these ideas, this time using non-standard terms to refer to angles.

These examples illustrate situational features of students' voices, such as, in particular, when they take up words or gestures introduced by their teacher to talk about their classification of a set of shapes. Distal features are more aligned with the teacher's voice: she represents standard ways of talking about mathematics and using the French language, as apparent in her rehearsal of one student's pronunciation. Over time, her preferred (i.e. more formal) ways of talking about mathematics prevail.

Multiple languages

Students brought many languages to the class. One student, Luis, for example, spoke accented non-standard French, as well as Spanish words and pronunciation. He also made reference to Spanish and English vocabulary as he made connections between French vocabulary and words with which he was already familiar. In one class, Luis contributed to a discussion about the properties of shapes:

Luis added that "a diamond is not a square that turns, I already know it in English. The angles of a diamond are bigger (*grandé*) on two sides". (Visit report, 11 May, 2010)

This example illustrates Luis's repertoire of languages combining as he uses newly acquired French, sometimes with a Spanish pronunciation (the French word 'grande' is pronounced like the Spanish equivalent) and with reference to English.

The influence of students' status as learners of French clearly shaped their participation in the class:

I heard one student say 'angles' (i.e. /a^hngelz/) while speaking in French. The teacher then asked groups to go to the blackboard and stick their groups of shapes into the two circles and explain their reasoning. There were plenty of examples of students struggling to find words. Sometimes they would ask how to say something (*comment dire*) or would repeat a

word. I also noticed variants of pronunciation of certain words. One student asked if he could explain en Espagnol but the teacher said no. (Visit report, 11 May, 2010)

In this class, then, multiple languages were present and often invoked by different students as they sought to express mathematical thinking. Moreover, the imposition of French (the main goal of the class) clearly shaped how students were able to talk about mathematics (or not).

Students' home languages appear to act as situational aspects of multiple languages in this class. These languages are sometimes used but are not available to all members of the class or in wider society outside the classroom. Nevertheless, they form part of students' mathematical meaning-making. The officially enforced use of French occupies a distal position, both in terms of its official status, as well as in the emphasis on its formal features. And French is widely used in the school and in the community of which the school is a part.

DISCUSSION AND CONCLUSIONS

My analysis has shown the role of multiple language features in students' mathematical meaning making, organised around three broad sources of meaning. The multiple discourses, multiple voices and multiple languages I have identified echo various disparate research findings from the literature (e.g., Moschkovich, 2008; Setati, 2005; Planas & Setati, 2009). The sources of meaning approach allows, however, for a more systematic examination of students' meaning-making and associated language practices. It is clear from the examples I have provided that discourses, voices and languages are all implicated simultaneously. For example, in the teacher's rehearsal with a student of the term non-convexe, discourses (a formal mathematical term), voices (the student's and the teacher's) and languages (French and Spanish) are all involved in this brief meaning-making moment.

My analysis also reveals how students draw on both situational and distal discourses, voices and languages, thus drawing on locally generated or locally meaningful features of language, combined with more conventional or widely recognisable features and, again, they overlap. The non-convexe moment, for example, illustrates situational and distal discourses (non-convexe vs. non-convexe), voices (student vs. teacher), and languages (Spanish vs. French). The situational and distal dimensions should not be seen as a dichotomy; they are poles of an axis that reflects the underlying centripetal and centrifugal forces inherent in language. Moreover, situational and distal language features reflect and reproduce the stratified nature of mathematics classroom interaction. In general (although not always), distal discourses, voices and languages appear to reflect institutional and ultimately political preferences, in this case for standard or conventional forms of mathematical discourse and of the French language. Similarly, situational discourses, voices and languages appear to be largely aligned with the unconventional forms of these second-language learners' mathematical meaning-making. These findings add to the accumulating work that suggests that mathe-

mathematics teachers need to pay careful attention to students' multiple ways of making meaning in mathematics.

Acknowledgements

The data referred to in this report come from the project Mathematics in a Multilingual World (2008–2012), funded by the Social Science and Humanities Research Council of Canada, grant 410-2008-0544. I am grateful to the school, students and teacher for their generous participation, and to Elysée Cadet and several other research assistants.

References

- Bakhtin, M. M. (1981). *The dialogic imagination: Four essays*. (ed., M. Holquist; trans, C. Emerson and M. Holquist). Austin, TX: University of Texas Press.
- Barwell, R. (2005). Integrating language and content: Issues from the mathematics classroom. *Linguistics and Education*, 16(2), 205-218.
- Barwell, R. (2015). Language as a resource: Multiple languages, discourses and voices in mathematics classrooms. In K. Beswick, T. Muir, & J. Wells (Eds.), *Proceedings of the 39th conference of the International Group for the Psychology of Mathematics Education* (vol. 2, pp. 97-104). Hobart, Australia: University of Tasmania.
- Barwell, R., Moschkovich, J. & Setati Phakeng, M. (2017). Language diversity and mathematics: second language, bilingual, and multilingual learners. In J. Cai (Ed.), *Compendium for Research in Mathematics Education* (pp. 583-606). Reston, VA: National Council of Teachers of Mathematics.
- Busch, B. (2014). Building on heteroglossia and heterogeneity: The experience of a multilingual classroom. In A. Blackledge, & A. Creese (Eds.), *Heteroglossia as practice and pedagogy* (pp. 21-40). Dordrecht, The Netherlands: Springer.
- Clarkson, P. C. (2007). Australian Vietnamese students learning mathematics: High ability bilinguals and their use of their languages. *Educational Studies in Mathematics*, 64(2), 191-215.
- Duranti, A. (1998). *Linguistic anthropology*. Cambridge, UK: Cambridge University Press.
- Moschkovich, J. N. (2008). I went by twos, he went by one: multiple interpretations of inscriptions as resources for mathematical discussions. *The Journal of the Learning Sciences*, 17(4), 551–87.
- Planas, N. (2014). One speaker, two languages: Learning opportunities in the mathematics classroom. *Educational Studies in Mathematics*, 87(1), 51-66.
- Planas, N. (2017). Multilingual mathematics teaching and learning: Language differences and different languages. In Kaur, B., Ho, W. K., Toh, T. L. & Choy, B. H. (Eds.), *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education* (vol. 4, pp. 65-72). Singapore: PME.
- Planas, N., & Setati, M. (2009). Bilingual students using their languages in the learning of mathematics. *Mathematics Education Research Journal*, 21(3), 36-59.
- Setati, M. (2005). Teaching mathematics in a primary multilingual classroom. *Journal for Research in Mathematics Education*, 36(5), 447-466.

KNOWLEDGE OF STATISTICAL TESTS BY PROSPECTIVE HIGH SCHOOL TEACHERS

Carmen Batanero, M^a del Mar López-Martín, M^a Magdalena Gea, and Pedro Arteaga
University of Granada, Spain

This research was aimed to evaluate prospective high school teachers' common knowledge of hypothesis tests. The responses given by 73 Spanish prospective teachers to an open problem similar to those included in the previous years at the entrance to university tests are analysed. Although the majority of participants set correct hypotheses and select a test consistent with the same, only part of them correctly complete the procedure, make the correct decision and contextualize the results. The implication is the need for a better preparation of teachers in this topic.

INTRODUCTION

Statistical inference plays a prominent role in human science research and is a content in the high school Spanish curriculum (MECD, 2015), as well as in the entrance to university tests, which have included an inference problem in the past 15 years (López-Martín, Batanero, Díaz-Batanero, & Gea, 2016). This topic is not simple, as shown by common criticisms of their incorrect use (e.g., Nickerson, 2000) and the students' errors (Castro Sotos, Vanhoof, den Nororgate, & Onghena, 2007; Makar & Rubin, 2018, Vera, Díaz, & Batanero, 2011). Improving the understanding and application of the topic implies the adequate preparation of the teachers responsible for its teaching, which has hardly been taken into account in the previous research (Haradine, Batanero, & Rossman, 2011, Makar & Rubin, 2018). This work tries to complement these works of research.

PREVIOUS RESEARCH

The most commonly misinterpreted concept in inference is the level of significance α , defined as the probability of rejecting the null hypothesis, when it is true. Students change the two terms of the conditional probability in this definition, and interpret α as the probability that the null hypothesis is true, once the decision to reject it has been taken (Birnbbaum, 1982). Other students think that α is the probability of being wrong when rejecting the null hypothesis (Haller & Krauss, 2002; Vallecillos, 1999). Similar errors appear when interpreting the p -value or probability of finding a value of the sample statistic equal or more extreme than that observed, when assuming the null hypothesis is true (Castro- Sotos et al., 2007). Other outstanding errors are related to the role played by the null and alternative hypotheses since when carrying out a hypothesis test, the null hypothesis is proposed to be rejected, while the alternative is the complement (Batanero, 2000). The null hypothesis is assumed to be true and the sampling distribution of the test statistic is determined by this assumption. In addition,

students confuse unilateral and bilateral tests, define hypotheses that do not cover the parametric space and use the sample statistic instead of the population parameter to define the hypotheses (Vallecillos, 1999; Vera et al., 2011).

METHOD

Although there is a recent tendency to change the teaching of inference towards informal approaches (Makar & Rubin, 2018), in countries like Spain formal inference is still a part of curriculum and we have to prepare teachers to teach this topic. In this paper we analyse the existence of the above errors in prospective teachers. The sample was made up of 73 students in a Master's Degree compulsory for those who intend to become high school mathematics teachers in Spain. 56% of them were graduates in Mathematics or Statistics and the remaining students have completed undergraduate studies in Engineering, Architecture or Science. All of them took some statistics courses in their undergraduate studies and 57% had some teaching experience. The assessment was part of an activity aimed at developing the participants' content knowledge of inference. In this paper we analyse participants' solutions to the following problem.

Problem: The average life expectancy in a study developed by the United Nations is 69.2 years with a standard deviation 10. In a random sample of 16 European countries the average life expectancy was 78 years. Assuming that the life expectancy follows the normal distribution, propose a hypothesis test, with a level of significance of 5%, to analyse if the average life expectancy in Europe is higher than that obtained in the whole set of countries.

The solution involves a unilateral hypothesis test on the average of a normal population with known variance, a content included in the compulsory examination to become high school mathematics teachers, as well as in the curricular guidelines for high school and the entrance to university tests in Spain. Through the analysis of participants' answers we examine the prospective teachers' knowledge of the following points: a) The way they establish the tests hypotheses; b) If the procedure followed to solve the problem is consistent with the hypotheses proposed; c) Possible errors in the procedure and d) Interpretation of the results.

RESULTS AND DISCUSION

We performed a qualitative analysis of the written solutions to the problem to classify the responses in different categories, starting from previous research and refining the categories through a cyclic and inductive process typical of qualitative research.

Setting the hypotheses

In the first place, the hypotheses proposed by the participants were analysed and classified in the following way:

C1. Correct hypotheses. The student differentiates the theoretical value of the population mean (μ_0) from the observed average in the sample (\bar{x}); next, correctly identifies the null and alternative hypotheses and takes into account that it deals with a unilateral

test. The null hypothesis is that we wish to reject (the average life expectancy in European countries is lower or equal to the general life expectancy in all the countries) and the alternative hypothesis is the opposite. As we see in the following example (ACG's answer), the two hypotheses are complementary and cover the parametric space:

$H_0: \mu_E \leq 69.2$; $H_1: \mu_E > 69.2$.

C2. Correct hypotheses expressed verbally with no symbolization.

C3. Correct hypotheses with incorrect symbolization.

C4. The hypotheses do not cover the parametric space and are not complementary, as in the following example (LUG's answer); this error was described by Vallecillos (1999):

$H_0: \mu_E < \mu$; $\mu_E < 69.2$; $H_1: \mu_E > \mu$; $\mu_E > 69.2$.

C5. Confusing the null and alternative hypotheses. The participant BRR takes as a null hypothesis $\mu > 69.2$, which is, in fact, what the researcher wants to prove. The confusion was described by Vallecillos (1999) and Vera et al. (2011).

We propose the hypothesis test: $H_0: \mu > 69.2$; $H_1: \mu \leq 69.2$.

C6. Confusing the sample and population mean, an error reported by Harradine et al. (2011). As a consequence, the student ALM uses the value of the sample mean to set the hypotheses.

$\bar{x} = 69.2$; $\sigma = 10$; $n = 16$ (the sample, European countries); $\mu_0 = 78$ years is the average life expectancy of European countries [...]. Then, it is a left-tailed test: $H_0: \mu \geq 78$; $H_1: \mu < 78$

C7. Only sets the null hypothesis. Following example (VRM's answer) shows an answer of this category:

The hypothesis is: $H_0: \mu_E \leq 69.2$.

C8. Setting only the null hypothesis and confusing population and sample mean. For example the DGM's answer:

Our variable is the life expectancy and our null hypothesis that the average is 78.

Other errors include (C9) Setting the hypotheses in terms of proportions, instead of using the mean; (C10) including the population mean in the alternative hypothesis. And (C11) including the population mean in both hypotheses that are not complementary. Although most participants correctly set the hypotheses (Table 1), we found some errors described in the previous research (Harradine et al., 2011; Vallecillos, 1999, Vera et al., 2011) such as: hypotheses were not exclusive, do not cover the parametric space or the null or alternative hypothesis are confused; we also found new errors, such as including the hypothetical value in the alternative hypothesis or expressing hypotheses in terms of proportions.

Code	Setting the hypotheses	Frequency	%
C1	Correct hypotheses and symbolization	49	67.1
C2	Correct hypotheses with no symbolization	4	5.5
C3	Correct hypotheses with incorrect symbolization	2	2.7
C4	Hypotheses do not cover the parametric space	2	2.7
C5	Confusing null and alternative hypotheses	3	4.1
C6	Confusing hypotheses and population and sample mean	3	4.1
C7	Only setting the null hypothesis	2	2.7
C8	C7 and confusing population and sample mean	1	1.4
C9	Setting the hypotheses in terms of proportions	1	1.4
C10	Including the population mean in the alternative hypothesis	4	5.5
C11	Including the population mean in both hypotheses	2	2.7
Total		73	100

Table 1: Frequency and percentage of participants according to setting of hypotheses

Selecting the test

Once the hypotheses are set, the next step is deciding whether it deals with a unilateral or bilateral test, since all the computations depend on this decision. Then, we analysed whether the participant specified the type of test and this decision was consistent with the established hypotheses. We have not found this type of analysis in previous research, so this section is an original contribution. The following categories were found:

C1. Correct answer. We expect the student to specify that we deal with a right-tailed test (since the alternative hypothesis is that the mean is greater than 69.2).

C2. Specifying a left-tailed test (critical region to the left side), but building a right side critical region.

C3. Specifying a unilateral test (right-tailed), with no indication of the critical region position, and developing a bilateral test.

C4. Specifying a unilateral test, with no indication of the critical region position, and developing a right-tailed test.

C5. Not specifying the type of test, and developing a right-tailed test.

C6. Not specifying the type of test, and developing a bilateral test or not developing the test.

In Table 2 we present the results, where the majority of students develop a right-tailed test, according to the established hypotheses but did not indicate what type of test they

were performing. We note that some students specified the test, with no indication of whether it was a right tailed test while others confuse the right tailed and left tailed tests, a small percentage applied a bilateral test or failed to develop the test.

Code	Selected test	Frequency	%
C1	Right-tailed test	3	4.1
C2	Left-tailed test	1	1.4
C3	Right-tailed test and apply bilateral	1	1.4
C4	Unilateral	18	24.7
C5	No specification; Right-tailed test	40	54.8
C6	No specification; Not developing the test or bilateral test	10	13.7
Total		73	100.0

Table 2: Frequency and percentage of participants according to the selected test

Developing the procedure

The majority of the students (69.9%) followed the methodology proposed by Fisher (1971/1935) to perform a significance test, in which the probability of obtaining the sample statistic observed or another more extreme (p -value) is computed, and the null hypothesis is rejected if the p -value is smaller than the significance level α . The remaining students (30.1%) followed other procedures (all of them analysed in Rivadulla, 1991), such as computing the critical and acceptance regions (16.4%), translating the problem into a confidence interval problem (6.8%), comparing the likelihood of the data in the assumptions of the null or the alternative hypothesis being true (4.1%) and some failed to solve the problem (2.7%). Below the application of the chosen method is described, including the errors committed.

C1. Correct procedure. When the p -value and the sample distribution are correctly defined. In addition, the participant correctly performs the computations using the normal distribution.

C2. Correct calculation for a bilateral test, and therefore, the critical value, confidence interval or p -value (depending on the methodology applied) is incorrect.

C3. Correct calculation, but using the *Student's t* instead of the normal distribution.

C4. Expressing the p -value as a non-conditional probability.

C5. Not explicitly including the observed value of the statistic in the computation of p -value.

C6 Defining the p -value as the probability that the continuous random variable takes a constant value (that theoretically is equal to zero).

C7. Describing the computation of p -value and being unable to compute it.

C8. Approximating the p -value by the rule of the central intervals in a normal distribution that contain about 68.2%, 95.4% and 99.6% of the distribution.

C9. Approximating the p -value by 0.

C10. Trying to solve the test with an incorrect confidence interval; this is based on the sample mean and not on the population mean.

C11. Exchanging the acceptance and critical regions when using the maximum likelihood method.

Table 3 displays the results in this variable, where the percentage of correct answers is small, which is a cause of concern, considering that participants are prospective teachers. Since the majority of them base their procedure on the p -value, most errors refer to this concept, which is interpreted as simple probability or as a point probability. In other cases, it is not computed and is approximated sometimes incorrectly. Errors in categories C4 to C10 were not described in previous research and account for 71.2% of the study participants.

Code	Computations	Frequency	%
C1	Correct	12	15.1
C2	Correct, bilateral test	9	12.3
C3	Exchanging critical and acceptance regions	1	1.4
C4	Expressing p -value as a simple probability	8	11.0
C5	Not including the observed value in the computation	8	11.0
C6	Interpreting p -value as a punctual probability	23	31.5
C7	Describing the p -value and being unable to compute it	6	8.2
C8	Using the 68%, 95% and 99% normal rule	1	1.4
C9	Approximating p -value by zero	3	4.1
C10	Exchanging regions in maximum likelihood method	3	4.1
Total		73	100.0

Table 3: Frequency and percentage of participants according to computations made

Interpreting the results

Once the calculations are completed, we expect that participants conclude that the result is statistically significant and the null hypothesis should be rejected. When contextualising this result, the conclusion is that we should reject the hypothesis that the average life expectancy in Europe is the same (or smaller) than that in the group of countries. This final step is important, since it involves a deep understanding of the problem and understanding the interest of hypotheses tests. In Table 4 we present the interpretation made by participants, less than 36% of which made the right decision

and contextualised the problem. The remaining participants either did not conclude or made an incorrect decision. Some showed misunderstanding of hypothesis tests, such as assuming that the p -value is the probability of the null hypothesis being true or holding a deterministic view of results.

Code	Interpreting the results	Frequency	%
C1	Correct and contextualises	26	35.6
C2	Correct, interpret p as the probability of the hypothesis	3	4.1
C3	Correct, suggesting that the null hypothesis is false	2	2.7
C4	Correct, does not contextualise	24	32.9
C5	Describing how to make the decision and contextualising	2	2.7
C6	Describing how to make the decision and not contextualising	7	9.6
C7	Incorrect, contextualising	5	6.8
C8	Making no decision	4	5.5
Total		73	100.0

Table 4: Frequency and percentage of participants according to interpretation of results

FINAL IMPLICATIONS

The results suggest the need to prepare prospective Spanish high school teachers in statistical inference, and more specifically in hypothesis tests. In addition to being included in the curricular guidelines and entrance to university tests in Spain. The critical reading of mathematics education research by these professionals requires a basic knowledge of the topic, since this research often includes hypothesis tests results. The errors detected reproduce those identified in students in previous research (e.g., Vallecillos, 1999; Vera et al., 2011), and raise the question of the extent to which such errors could be transmitted in teaching. Moreover, although many participants made a correct decision in the problem, few interpreted the result in the context of the problem, which is worrisome as it is expected that teachers develop the ability to interpret statistical studies in their students. For example, one learning goal in the curriculum is (MECD, 2015, pp. 398 and 407): "Interpret a statistical study based on situations close to the student." Therefore, it is necessary that prospective teachers are able to contextualize the information resulting from statistical tests to help the student acquire this interpretation capacity.

Of course mathematical knowledge is only insufficient for the success of teaching and it is necessary to continue this research with the analysis of pedagogical knowledge on hypothesis tests. But such didactic knowledge cannot be constructed without a solid mathematical knowledge. We hope this problematic will interest teacher educators and other researchers in statistical education and we can ensure a good mathematical and didactic training of high school teachers to teach statistical inference.

Acknowledgement

Project EDU2016-74848-P (AEI, FEDER) and group FQM126 (Junta de Andalucía).

References

- Batanero, C. (2000). Controversies around the role of statistical tests in experimental research. *Mathematical Thinking and Learning*, 2(1-2), 75-98.
- Birnbaum, I. (1982). Interpreting statistical significance. *Teaching Statistics*, 4, 24-27.
- Castro Sotos, A. E., Vanhoof, S., Van den Nororgate, W., & Onghena, P. (2007). Student's misconceptions of statistical inference: A review of the empirical evidence from research on statistical education. *Educational Research Review*, 2(2), 98-113.
- Fisher, R. A. (1971). *The design of experiments*. Edinburgh: Oliver y Boyd (original work published in 1935).
- Haller, H., & Krauss, S. (2002). Misinterpretations of significance: A problem students share with their teachers?. *Methods of Psychological Research*, 7(1), 1-20.
- Harradine, A., Batanero, C., & Rossman, A. (2011). Students' and teachers' knowledge of sampling and inference. In C. Batanero, G. Burrill y C. Reading (Eds.). *Teaching statistics in school mathematics- Challenges for teaching and teacher education. A Joint ICMI/IASE Study* (pp. 235-246). New York: Springer.
- López-Martín, M. M., Batanero, C., Díaz-Batanero, C., & Gea, M. (2016). La inferencia estadística en las pruebas de acceso a la universidad en Andalucía, *Revista Paranaense de Educação Matemática*, 5(8), 33-59.
- Makar, K., & Rubin, A. (2018) Learning about statistical inference. In D. Ben-Zvi, K. Makar, & J. Garfield(Eds), *International handbook of research in statistics education*. New York; Springer.
- MECD (2015). *Real Decreto 1105/2014, de 26 de diciembre, por el que se establece el currículo básico de la Educación Secundaria Obligatoria y del Bachillerato*. Madrid: Author.
- Nickerson, R. S. (2000). Null hypothesis significance testing: a review of an old and continuing controversy. *Psychological methods*, 5(2), 241.
- Rivadulla, A. (1991). *Probabilidad e inferencia científica*. Barcelona: Anthropos.
- Vallecillos, A. (1999). Some empirical evidence on learning difficulties about testing hypotheses. *Bulletin of the International Statistical Institute*, 58, 201-204.
- Vera, O., Díaz, C., & Batanero, C. (2011). Dificultades en la formulación de hipótesis estadísticas por estudiantes de psicología. *Unión*, 27, 41- 61.

THE EFFECTIVENESS OF INTEGRATING GEOGEBRA HOTS ACTIVITIES ON THE DEVELOPMENT OF CREATIVE MATHEMATICAL THINKING

Nimer Baya'a¹, Wajeeh Daher^{1,2}, Dania Abo-Mokh¹, and Ahlam Anabousy¹

¹Al-Qasemi Academic College of Education, Baqa-El-Gharbia, Israel;

²An-Najah National University, Nablus, Palestine

Educating programs are suggested for encouraging creativity among school students. We examined the influence of a mathematics teaching unit based on higher order thinking skills activities on students' creativity. The participants were 64 grade 9 high achievers who were divided randomly into two groups: 31 students in the experimental group, and 33 students in the control group. The research results indicated that, using different pre and post tests involving multiple solution tasks, the educating program affected positively and significantly the fluency and flexibility of the students in the experimental group, but not their originality. In addition, using pre and post tests with same types of multiple solution tasks, the educating program affected positively and significantly the experimental group's flexibility and originality, but not its fluency.

INTRODUCTION

Two areas are at the core of the current study: (1) mathematical creativity and (2) higher order thinking skills (HOTS). These two areas are described below.

Mathematical creativity

Creativity has recently come to be considered a major component of mathematics education (Van Harpen & Sriraman, 2013) and an essential skill that teachers should enhance in all students (Kattou, Kontoyianni, Pitta-Pantazi & Christou, 2013). There is no single perspective or definition of creativity (Leikin & Kloss, 2011; Mann, 2006). Mann (2006) claims that there are more than 100 definitions of creativity in the literature. Eryvynck (1991) defines mathematical creativity as the ability to solve problems or develop structured thinking, as well as make connections in the mathematical content. He emphasises that creative activity is not related to algorithms, but to a novel concept, definition, argument or proof. In the present study, we draw on a definition of creativity as including three components: fluency, flexibility and originality (Guilford, 1950; Torrance, 1966). Fluency is associated with the number of correct solutions that a student provides to a problem. Flexibility is associated with the number of solution types suggested for a problem, or with the number of problem-solving strategies that have been implemented. Originality is associated with the number of solutions offered that very few or no other persons proposed (Torrance, 1966). This is also true for the present study, specifically, when we followed Leikin (2009) to evaluate the creativity components' scores.

Studies on mathematics creativity report its positive effect on students' learning (Mann, 2006), suggesting that creativity tasks are highly effective in the mathematics classroom. This is one reason why we wanted to encourage grade 9 students' creative thinking based on a mathematical teaching unit rich in higher order thinking tasks. In addition, our interest in nourishing grade 9 students' creative thinking in mathematics meets the call of educational institutions and researchers to encourage school students to use higher order thinking skills, including creative thinking skills, because doing that prepares them to be 21st century citizens by possessing the appropriate skills (e.g. National Council of Teachers of Mathematics, 2000).

Researchers have attempted to encourage students' creativity through tasks, tools and educating programs. Some of these researchers suggest open-ended tasks (e.g. Mihajlović & Dejić, 2015) or multiple solution tasks (e.g. Levav-Waynberg & Leikin, 2012) for cultivating students' mathematical creativity. In addition, Daher and Anabousy (2016) reported the positive influence of what-if-not strategy and technology on mathematics pre-service teachers' flexibility. The present research attempts to follow these studies and intends to examine the effect of an educating program on grade 9 students' three components of creativity: fluency, flexibility and originality. This educating program is based on higher order thinking tasks.

Higher order thinking skills

Educating students' for higher order thinking in mathematics prepares them for the 21st century citizenship by possessing the appropriate skills (e.g. National Council of Teachers of Mathematics, 2000). The call for nourishing students' high order and creative thinking is due to its ability to support their problem solving through encouraging diverse solutions for a problem (Imai, 2000). Moreover, educating students for higher order thinking skills provides them with tools that turn them into more critical thinkers. This supports them in overcoming life problems that they encounter, as well as becoming an integral part of the society. In addition, high order thinking is associated with different thinking types. King, Goodson and Rohani (1998) say that high order thinking includes critical, logical, reflective, metacognitive, and creative thinking. From the other hand, high order thinking takes place in the higher levels of the hierarchy of cognitive behavior (Ramos et al., 2013). In the present research, we want to examine the influence of an education program based on higher order thinking skills on grade 9 students' creativity.

Research rationale and goals:

Piggott (2011) argues that creativity in the mathematics classroom is not only related to what students do but also to what teachers do, where the mathematical experiences that teachers offer in this classroom can open up opportunities for students to be creative. Torrance (1972) describes several ways to teach students for creative thinking. We adopted in the present study one of these ways, namely a teaching unit that encourages higher order thinking skills. Little research has been done on the influence of education programs based on higher order thinking skills on students' mathematical creativity,

which is one reason why we are interested to examine this influence in the present study. This study would enable educators to plan better their mathematics teaching, especially regarding ways that encourage students' three components of creativity that are essential for supporting the twenty first century skills (Pásztor, Molnár & Csapó, 2015).

Research question:

Would an educating program based on higher order thinking skills influence grad 9 students' fluency, flexibility and originality?

METHODOLOGY

Research context and participants

The experiment was conducted in a secondary school (9th grade-12th grade) that includes three grade 9 classes. The participants were 64 high achievers (according to the students' mathematics grades in previous exams) who volunteered to participate in the study from these classes. These students were divided randomly into two groups: 31 students constituted the experimental group, and 33 students were in the control group.

The decision on which group is the experimental one was based on the creativity pre-test that yielded significant difference between the averages of the two groups scores in flexibility. The group with the lower average in flexibility was taken as the experimental group.

Research procedure and teaching unit

The teaching unit included six lessons; three algebra lessons about the solution of a system of two equations (two linear equations, two quadratic equations, and one linear equation and one quadratic equation), and three geometry lessons about circles. Each lesson in the teaching unit included activities using GeoGebra that stimulate students' higher order thinking skills, as conjecturing, raising different points of view, pupils asking questions, identifying components and relations, categorizing, comparing, concluding, combining, using different representations, claiming, reasoning and evaluating (Daher and Baya'a, 2015). The teaching unit was implemented in a computer lab in the school for a period of ten lessons. The students in the experimental group carried out the activities using GeoGebra to investigate the mathematical concepts or relations in each activity. The control group learned the same unit as in the formal mathematics book.

Data collection:

All the participants performed a pre-test at the beginning of the experiment, which included seven multiple solution tasks. After the experiment, both groups performed a post-test, which included also seven multiple solution tasks different than the pre-test, except for four tasks that were of the same type as in the pre-test. For example, one

repeated task was the solution of a system of two linear equations with two variables. In the repeated task the coefficients were changed.

Data processing

The scoring of the three components of creativity; fluency, flexibility and originality for the experimental and control groups, before and after the experiment, were calculated according to Liken (2009). These scores were analyzed using t-tests to determine if there were significant differences between the students' scores in the two groups and in one group before and after the experiment.

FINDINGS AND DISCUSSION

Differences in creativity components between the two research groups

In the pre-test of creativity that the two groups took, the students in the two groups had only significant differences in the flexibility component of creativity, which was in favour of the students in the control group, as can be seen in Table 1.

	Experimental		Control		t
	M	SD	M	SD	
Fluency	3.54	1.14	3.32	1.20	0.73
Flexibility	18.51	4.48	21.38	6.60	2.02*
Originality	12.00	6.60	13.76	7.30	1.00

* $p < .05$

Table 1: Means, standard deviations and t-test for creativity components scores before the experiment (N = 31 for the experimental group and N = 33 for the control group)

These results were appropriate for the start of the experiment. In the post-test of creativity that the two groups took, the students in the experimental group had significantly higher scores in fluency and flexibility, but not in originality. This is shown in Table 2.

	Experimental		Control		t
	M	SD	M	SD	
Fluency	3.74	1.31	2.82	0.84	3.12**
Flexibility	24.67	9.20	17.76	4.54	3.58**
Originality	9.56	5.28	8.58	4.47	0.73

** $p < .01$

Table 2: Means, standard deviations and t-test for creativity components scores after the experiment (N = 24 for the experimental group and N = 29 for the control group)

The previous findings agree with Leikin, Levav and Guberman (2011) who found that educating programs, based on employment of multiple solution tasks, resulted in significant differences in the participants' fluency and flexibility in favour of the group who participated in the educating program, and, at the same time, did not result in significant differences in the participants' originality. These results could be explained using the claims of Leikin, Levav and Guberman (2011) that "when students become more fluent they have less chance to be original" or that fluency and flexibility are of a dynamic nature, whereas originality is of a "gift" nature. In addition, we claim that high flexibility scores indicate that the majority of the participants in the experimental group gave various strategies of solutions for the tasks after the experiment, which means that few non-traditional solutions were given by less than 15% of them.

Differences in creativity components in each research group before and after the experiment

We wanted to examine the scores of the three creativity components for each research group to find if there is significant difference in these scores before and after the experiment. Table 3 shows the results for the experimental group. These results are related for the four tasks that were of the same types before and after the experiment.

	Before		After		t
	M	SD	M	SD	
Fluency	4.57	1.43	5.42	2.61	1.59
Flexibility	24.78	6.76	30.86	13.57	2.23*
Originality	11.03	5.31	20.36	16.02	3.07**

*p<.05, **p<.01

Table 3: Means, standard deviations and t-test for creativity components scores of the experimental group before and after the experiment (N = 31 before and N = 24 after)

Table 3 shows that there were significant differences in the scores of creativity components of the students in the experimental group before and after the experiment, except for the fluency score. Previous studies reported this trend but others did not. For example Tooranposhti and Gholamzadeh (2011) found that technology resulted in significant differences in flexibility, originality and elaboration, but not in fluency of 16-17 years old students according to the figurative creativity test of Torrance (form A). On the other hand, Daher, Tabaja-Kadan and Gierdien (2017) found that an educating program based on higher order thinking improved significantly six grade students' scores in fluency, flexibility and originality. The results of the present research is similar more to those of Tooranposhti and Gholamzadeh (2011), probably

because the participants in the two studies are almost of the same age, so education is expected to influence them similarly.

In addition to the said above, the current results were obtained because the students in the experimental group were interested, in the post test, to use different strategies, for the tasks of the same type that they have solved before, more than to give different solutions, which increased significantly their flexibility and originality, but not their fluency. This happened because the educating program helped the students think of other strategies of solutions due to the various higher order thinking strategies and processes in which they engaged.

As to the difference between the participants' scores in creativity components in the control group, Table 4 shows the scores for the control group before and after the experiment.

	Before		After		t
	M	SD	M	SD	
Fluency	4.37	1.36	3.80	2.19	1.28
Flexibility	29.13	6.60	22.44	12.78	2.67*
Originality	15.93	8.18	14.90	12.43	0.39

* $p < .05$

Table 4: Means, standard deviations and t-test for creativity components scores of the control group before and after the experiment (N = 33 before and N = 29 after)

Table 4 shows that there were no significant differences in the creativity components scores of the students in the control group before and after the experiment, except for the flexibility scores which were higher before the experiment. The significant difference of the flexibility scores in favour of the pre-test could have resulted from the reluctance of the participants in the control group to solve tasks of the same type that they have solved before in various strategies. This could be especially true for high achievers as those who participated in the present research.

CONCLUSIONS

Educating programs are suggested for encouraging creativity among school students (Daher et al., 2017; Levav-Waynberg & Leikin, 2012; Torrance, 1972). In the present research, we examined the influence of a mathematics teaching unit based on higher order thinking skills on students' creativity. The research results indicated mixed trends. On one hand, the educating program affected positively and significantly the participants' fluency and flexibility, but, on the other hand, it did not do so for their originality. This indicates that fluency and flexibility are of a dynamic nature, whereas

originality is less so (Leikin et al., 2011), at least for the school level of the students who participated in the present study. At the same time, when considering the scores of the participants only in the repeated types of tasks, the results indicated that the educating program affected positively and significantly the flexibility and originality of the students in the experimental group, but not their fluency. This could indicate that the high achievers in the experimental group were not interested to give more solutions, but more types of solutions. The research results indicate that educating programs could be an effective method for developing students' creativity components, especially their flexibility.

References

- Daher, W. & Anabousy, A. (2016). Flexibility of pre-service teachers in problem posing in different environments. *Paper presented at the 13th international congress in mathematics education*. July 24-31, Hamburg, Germany.
- Daher, W. & Baya'a, N. (2015). Integrating HOTS Activities with GeoGebra in Pre-Service Teachers' Preparation. *World Academy of Science, Engineering and Technology, International Journal of Social, Behavioral, Educational, Economic, Business and Industrial Engineering*, 9(7), 2441-2444.
- Daher, W., Tabaja-Kidan, A., & Gierdien, F. (2017). Educating Grade 6 students for higher-order thinking and its influence on creativity. *Pythagoras*, 38(1), a350.
- Ervynck, G. (1991). Mathematical creativity. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 42–53). Dordrecht: Kluwer.
- Guilford, J.P. (1950). Creativity. *American Psychologist*, 5(9), 444–454.
- Imai, T. (2000). The influence of overcoming fixation in mathematics towards divergent thinking in open-ended mathematics problems on Japanese junior high school students. *International Journal of Mathematical Education in Science and Technology*, 31, 187–193.
- Kattou, M., Kontoyianni, K., Pitta-Pantazi, D., & Christou, C. (2013). Connecting mathematical creativity to mathematical ability. *ZDM*, 45(2), 167-181.
- King, F.J., Goodson, L., & Rohani, F. (1998). *Higher order thinking skills: Definitions, strategies, assessment*. Tallahassee, FL: Center for Advancement of Learning and Assessment.
- Leikin, R. (2009). Exploring mathematical creativity using multiple solution tasks. In R. Leikin, A. Berman, & B. Koichu (Eds.), *Creativity in mathematics and the education of gifted students* (pp. 129–145). Rotterdam: Sense Publishers.
- Leikin, R. & Kloss, Y. (2011). Mathematical creativity of 8th and 10th grade students. In M. Pytlak, T. Rowland, & E. Swoboda (eds.), *Proceedings of the Seventh Conference of the European Society for Research in Mathematics Education - CERME7* (pp. 1084-1093). Rzeszów, Poland: ERME.

- Leikin, R., Levav-Waynberg, A., & Guberman, R. (2011). Employing multiple solution tasks for the development of mathematical creativity: Two comparative studies. In M. Pytlak, T. Rowland, & E. Swoboda (Eds.), *Proceedings of the 7th Congress of European Research in Mathematics Education* (pp. 1094-1103). Rzeszów, Poland: University of Rzeszów.
- Levav-Waynberg, A., & Leikin, R. (2012). The role of multiple solution tasks in developing knowledge and creativity in geometry. *Journal of Mathematical Behavior*, 31(1), 73–90.
- Mann, E. L. (2006). Creativity: The essence of mathematics. *Journal for the Education of the Gifted*, 30(2), 236–260.
- Mihajlović, A., & Dejić, M. (2015). Using open-ended problems and problem posing activities in elementary mathematics classroom. In F.M. Singer, F. Toader, & C. Voica (Eds.), *Proceedings of the Ninth Mathematical Creativity and Giftedness International Conference* (pp. 34–39). Sinaia, Romania: The International Group for Mathematical Creativity and Giftedness.
- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.
- Pásztor, A., Molnár, G., & Csapó, B. (2015). Technology-based assessment of creativity in educational context: the case of divergent thinking and its relation to mathematical achievement. *Thinking Skills and Creativity*, 18, 32-42.
- Piggott, J. (2011). *Cultivating creativity*. Available from <https://nrich.maths.org/5784>
- Ramos, J. L. S., Dolipas, B. B., & Villamor, B. B. (2013). Higher order thinking skills and academic performance in physics of college students: A regression analysis. *International Journal of Innovative Interdisciplinary Research*, 1(4), 48-60.
- Tooranposhti, M. G., & Gholamzadeh, M. (2011). The effect of education of computer on creativity. IPEDR, 20. <http://www.ipedr.com/vol20/34-ICHSC2011-M10014.pdf>
- Torrance, E.P. (1966). *The Torrance tests of creative thinking-norms-technical manual research edition-verbal tests, Forms A and B – Figural tests, Forms A and B*. Princeton, NJ: Personnel Press.
- Torrance, E. P. (1972). Can we teach children to think creatively? *Journal of Creative Behavior*, 6(1), 114-143.
- Van Harpen, X.Y., & Sriraman, B. (2013). Creativity and mathematical problem posing: an analysis of high school students' mathematical problem posing in China and the USA. *Educational Studies in Mathematics*, 82(2), 201-221.

INDIVIDUAL DIFFERENCES IN FRACTIONS' CONCEPTUAL AND PROCEDURAL KNOWLEDGE: WHAT ABOUT OLDER STUDENTS?

Maria Bempeni^{*}, Stavroula Pouloupoulou^{**}, Ioanna Tsiplaki^{*}, and
Xenia Vamvakoussi^{*}

^{*}University of Ioannina, ^{**}Athens University of Economics and Business

We constructed and calibrated an instrument targeting conceptual and procedural fraction knowledge. We used this instrument in a quantitative study with 126 secondary students (7th and 9th graders), testing the hypothesis that there are individual differences in the way students combine the two types of knowledge. Cluster analysis revealed four distinct student profiles: Students who were either stronger or weaker than expected with respect to both types of knowledge; students who were stronger with respect to conceptual knowledge; and students who were stronger with respect to procedural knowledge. These findings support the individual differences hypothesis.

THEORETICAL BACKGROUND

Procedural knowledge typically refers to the ability to execute action sequences to solve problems and is usually tied to specific problem types, whereas conceptual knowledge is defined as the knowledge of concepts and principles pertaining to a domain (Rittle-Johnson, Siegler, & Alibali, 2001; but see also Star (2005) for a plea to reconsider how procedural knowledge is conceptualized). Research in the area highlights the fact that conceptual and procedural mathematical knowledge (hereafter, CKn and PKn) are equally important for mathematical competence (Rittle-Johnson & Schneider, 2015), yet mathematics education wavers between giving precedence to one or the other type of knowledge (Moss & Case, 1999; Star, 2005). As a consequence, the issue of procedural and conceptual knowledge in mathematics learning is not only theoretically interesting, but also educationally relevant.

There has been a lot of discussion regarding which type of knowledge develops first. Procedures-first theories assume that children first learn procedures for solving problems in a domain and then derive domain CKn from their experience solving problems. Concept-first theories support that students initially acquire CKn and then build PKn through repeated practice (Rittle-Johnson et al., 2001). Noting that, regardless of which type of knowledge comes first, the relation between CKn and PKn is typically bi-directional, Rittle-Johnson and colleagues (2001) argued for an iterative model, according to which the two types of knowledge develop in a hand-over-hand process and gains in one type of knowledge lead to improvements in the other, which in turn increases the first type of knowledge.

However, the relations between CKn and PKn remain a complex issue: For example, increases in one type of knowledge do not always result in equal amount of increase in the other; moreover, it appears that there are individual differences in the way students combine the two types of knowledge (Rittle-Johnson & Schneider, 2015). Hallett and colleagues (Hallett, Bryant, & Nunes, 2010; Hallett, Nunes, Bryant and Thrope, 2012) investigated such individual differences in students' fraction knowledge. They assessed CKn and PKn of students at Grade 4 and 5 (2010) as well as at Grade 6 and 8 (2012) and identified groups of students who had strong (or weak) CKn as well as PKn. However, they also consistently traced two substantial groups of students who demonstrated relative strength with one form of knowledge and weakness with the other, with differences between the two types of knowledge becoming less salient with age.

In an in-depth qualitative study, we found, similarly to Hallett and colleagues (2010, 2012), individual differences in the extent to which 9th graders rely on CKn and PKn in the area of fractions (Bempeni & Vamvakoussi, 2015). More specifically, we found students who displayed flawless procedural performance but failed even in the simplest of tasks that required conceptual understanding of fractions, and vice versa. Such findings are theoretically interesting because they challenge the view that all children follow a uniform sequence in gaining the two types of knowledge (see also Canobi, 2004) and illustrate the possibility that CKn and PKn may not develop in a hand-over-hand manner, putting a challenge to the Rittle-Johnson et al.'s iterative model (2001). Moreover, tracing salient individual differences at grade 9 could indicate that individual differences may persist, despite the general tendency to diminish overtime (Hallett et al., 2010; 2012).

An issue that needs to be addressed in this research area is the fact that very little attention has been given to measurement validity. As Rittle-Johnson & Schneider (2015, p. 1128) noted:

However, before more progress can be made in understanding the relations between conceptual and procedural knowledge, we must pay more attention to the *validity* of measures of conceptual and procedural knowledge. Currently, no standardized approaches for assessing conceptual and procedural knowledge with proven validity, reliability, and objectivity have been developed.

To further investigate this issue, we constructed and calibrated a new instrument measuring CKn and PKn of fractions. We administered this instrument to secondary students testing the hypothesis that there are individual differences in the way students combine CKn and PKn that remain salient at the secondary level.

METHOD

Research instrument

The research instrument—in its final form after the evaluation (see Instrument Evaluation below)—comprised 26 fraction tasks grouped in two categories, procedural

(12) and conceptual (14) tasks. The procedural tasks were paper-and-pencil tasks requiring knowledge of procedures taught at school (e.g., to execute fractions operations, to find an equivalent fraction, to cross-multiply, to simplify a complex fraction, and to compare dissimilar fractions).

The conceptual tasks were based on our materials from previous studies (Bempeni & Vamvakoussi, 2015) as well as on other instruments assessing conceptual understanding of fractions (e.g., Van Hoof, Verschaffel, & Van Dooren, 2015) and targeted many important aspects of fraction CKn. For example, students were asked to interpret and evaluate fraction representations with area models as well as the number line; to mentally compare fractions; to estimate the outcome of fraction operations; and to select appropriate fraction operations to solve problems.

The conceptual tasks were posed in the form of multiple choice questions in order to ensure that students would not use paper-and-pencil. This is because some conceptual tasks could be tackled with procedural strategies (e.g., the comparison of fractions), in which case students' PKn rather than their CKn would be assessed (see also Rittle-Johnson & Schneider, 2015).

Participants

The participants of the study were 126 Greek students: 66 seventh graders and 60 ninth graders.

Procedure

The same questionnaire was issued in two versions (A and B), varying the order of presentation of the conceptual tasks so as to prevent students from cheating. The students had fifty minutes to solve the fractions tasks.

DATA ANALYSIS AND RESULTS

The responses of the tasks were coded as correct or wrong. For the data analysis we used the Statistical Package for the Social Sciences (SPSS) and the R Project for Statistical Computing.

Instrument Evaluation

We conducted a clinical pilot with 61 students in order to evaluate the reliability and the validity of our instrument. The instrument, in its initial form, included 39 fraction tasks (12 procedural and 27 conceptual tasks).

The instrument showed strong *face validity* given that all tasks were assessed as clear, reasonable and accurate by 6 mathematics education experts. The experts were also asked to rate the relevance of each item to the aim of the instrument, on a 4-point scale. The calculation of the *content validity* index ($CVI=1 > 0.83$) for each item confirmed the high consistency between experts (Polit, Beck, & Owen, 2007). Moreover, multi-trait analysis illustrated that all the items of the PKn scale showed *convergent validity* and *divergent validity* by demonstrating high correlation with the procedural scale and low correlation with the conceptual scale respectively. More specifically, the value of

correlation for all procedural items was above 0.7. Eight items of the CKn scale showed low correlation with conceptual tasks or higher than expected correlation with PKn tasks, possibly due to their great diversity (see also Hallett et al., 2012). We decided to exclude these items from subsequent analyses.

To establish whether the items on this questionnaire all reliably measure the same construct we calculated Cronbach's alpha. The instrument showed a high degree of *internal consistency* (0.921 and 0.731 for the PKn and CKn measure, respectively). We also calculated Cronbach's alpha for CKn and PKn scales in case we removed a particular task. In all cases the value of Cronbach's alpha was less than 0.921 and 0.731 for the PKn and CKn scales respectively and as a result there was no reason to delete any of the items (Cronbach, 1951). We also assessed the *external consistency* of the instrument over a period of three weeks with a test-retest method. The value of intra-class correlation coefficient was high for all the PKn tasks ($r > 0.8$). However, this was not the case for all the CKn tasks. Five of them presented intra-class correlation coefficient below 0.5. We thus decided to remove them from the final form of our instrument (Ware & Gandek, 1998).

Main study

Our data were analyzed using cluster analysis. Following Hallett et al. (2010), we used the residualized scores of PKn and CKn in our analysis, the raw scores being the percentages of correct answers out of the total of answered questions. This is because the two scales (CKn and PKn) are expected to be correlated, and the use of residualized scores provides a way of measuring CKn and PKn that excludes this common part of variation from both scales (Cohen, Cohen, West, & Aiken, 2003). The residualized scores were obtained using linear regression. More specifically, residuals for CKn were obtained by regressing conceptual knowledge against procedural knowledge, while the residuals for PKn were obtained by regressing PKn against CKn. The residualized scores represent relative, rather than absolute, strength with respect to PKn and CKn, in the sense that a positive residual, for instance in PKn, means that a person's PKn is stronger than expected given their CKn.

In order to identify different profiles of individual differences a cluster analysis was performed based on the two residualized scales using the k-means method and Euclidean distance as a distance measure. In the literature, a wide variety of indices have been proposed to find the optimal number of clusters in a partitioning of a data set during the clustering process. In order to determine the optimal number of clusters at our dataset we used the R package NbClust which provides 26 indices (Brock, Pihur, Datta, & Datta, 2008). The majority of these indices suggested a four cluster solution. The mean (and standard deviation) of the conceptual and procedural scores for the students of the four clusters are presented in Table 1. Table 1 also presents the mean and standard deviation for raw and residualized scores by cluster. The first cluster *Stronger than expected in CKn* (N=21, 16.7%) is characterized by positive CKn re-

siduals and negative PKn residuals which means that students in this cluster performed better than expected in CKn tasks, given their performance in PKn tasks.

	Cluster 1		Cluster 2		Cluster 3		Cluster 4		
	<i>Stronger than expected in CKn</i>		<i>Stronger than expected in PKn and CKn</i>		<i>Stronger than expected in PKn</i>		<i>Weaker than expected in PKn and CKn</i>		
	Mean	SD	Mean	SD	Mean	SD	Mean	SD	
Procedural Residual	-1,4	0,4	0,4	0,2	0,8	0,4	-0,7	0,4	<.0001
Conceptual Residual	0,8	0,5	1,3	0,8	-0,7	0,4	-0,3	0,3	<.0001
Procedural Score	10,3	12,8	85,9	10,4	70,1	16,6	19,6	14,8	<.0001
Conceptual Score	30,9	8,5	54,5	14,2	17,5	8,3	13,3	6,3	<.0001

Table 1: Mean and standard deviation for raw and residualized scores by cluster

The second cluster *Stronger than expected in PKn and CKn* (N=22, 17.5%) and fourth cluster *Weaker than expected in PKn and CKn* (N=31, 24.6%) cluster comprised students with either good or poor performance, respectively, in both measures. The third cluster *Stronger than expected in PKn* (N=52, 41.3%) is characterized by negative CKn residuals and positive PKn residuals which means that students in this cluster performed better than expected in PKn tasks, given their performance in CKn tasks.

We present the cases of two students illustrating extreme individual differences regarding PKn and CKn. The student of the *Stronger than expected in PKn* profile achieved 100% score in procedural tasks but only 14.29% in conceptual tasks, failing even in the simplest ones (e.g.:area model, Figure 2).

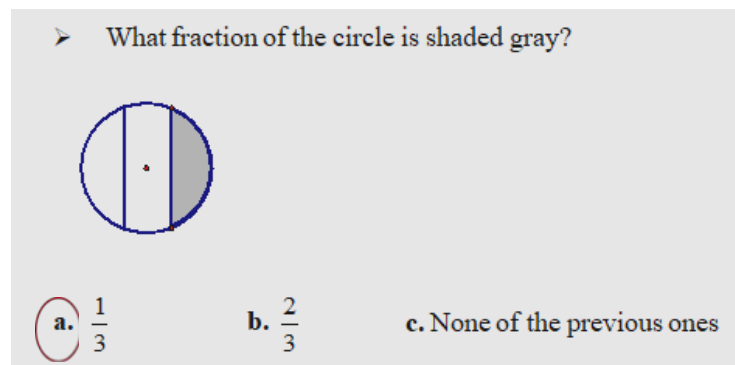


Figure 1: An extreme case in the *Stronger in PKn* cluster

On the other hand, the student of the CKn profile failed in all PKn tasks and responded correctly in 50% of the CKn tasks. Figure 2 illustrates the fact that this particular student was able to estimate a sum of fractions, despite the fact that she had failed to perform fraction addition.

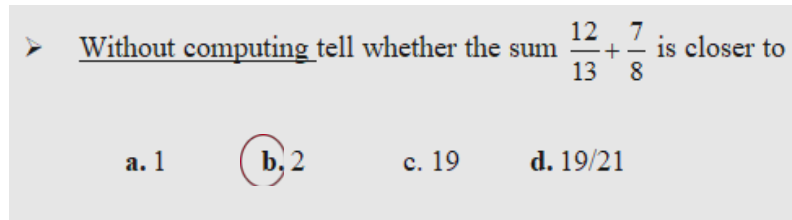


Figure 2: An extreme case in the Stronger in CKn cluster

The chi-square test of independence showed that there were no significant differences in the distribution of the two age groups among the clusters ($\chi^2 = 0.31$, $df = 3$, $p\text{-value} = 0.96$). The distribution of cluster membership across grade is presented in Table 2. The mean raw scores of the two age groups in CKn & PKn tasks are presented in Figure 3.

	Cluster 1	Cluster 2	Cluster 3	Cluster 4
Grade	<i>Stronger than expected in CKn</i>	<i>Stronger than expected in PKn and CKn</i>	<i>Stronger than expected in PKn</i>	<i>Weaker than expected in PKn and CKn</i>
Seven	10	12	27	17
Nine	11	10	25	14
Total	21	22	52	31

Table 2: Distribution of clusters across grade

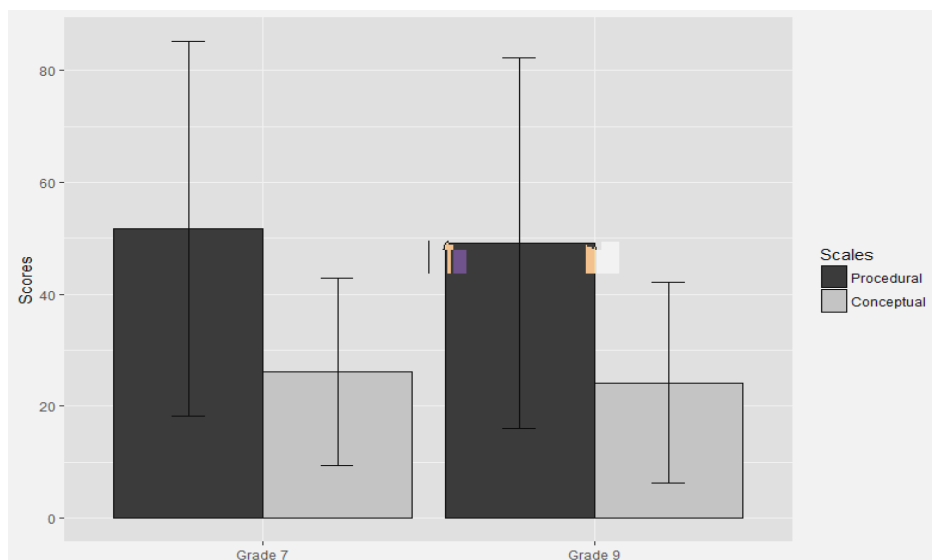


Figure 3: Mean raw scores of the two age groups

CONCLUSIONS-DISCUSSION

In response to the need for valid and reliable measures of conceptual and procedural mathematical knowledge, we constructed and calibrated an instrument targeting conceptual and procedural fraction knowledge. The instrument we developed demonstrated good indicators of validity and reliability and thus can be characterized as an efficient instrument for fraction CKn and PKn.

This instrument was used in a study testing the hypothesis that there are individual differences in the way that students combine CKn and PKn of fractions that remain salient at the secondary level. The results supported this hypothesis. We point out that 43.2% of our sample belonged to the *Stronger than expected in CKn* and *Stronger than expected in PKn* clusters. This hypothesis is further corroborated by the fact that the student profiles obtained in our study were very similar to Hallett and colleagues' (2010, 2012), despite the fact that a different instrument was used, in a different population (Greek students), and for older students (9th graders). In contrast to Hallett et al. (2012), our findings showed that individual differences in CKn and PKn do not necessarily diminish over time and may even be extreme, as indicated by the examples presented above.

Our findings also indicated that only few students adequately combine CKn and PKn. Given that developing both CKn and PKn is critical for mathematical development, it is important that teaching strategies and techniques that support both types of knowledge are used in instruction (see Rittle-Johnson & Schneider, 2015, for a review). We note that the third cluster *Stronger than expected in PKn* comprised the greater part of our sample, suggesting that more attention should be paid to conceptual knowledge during instruction. Moreover, despite the fact that we expected school experience to lead to improvements in students' fraction knowledge, we did not find any significant differences in the distribution of the two age groups across the clusters.

Further, the validation of such individual differences showcases the importance of differential instruction, based on the students' needs. Students who belong to different profiles could gain from different instructional approaches that make good use of the CKn and PKn that students have (Gilmore & Bryant, 2006).

Finally, the instrument we constructed and calibrated can be used for the assessment of students' understanding of fractions and of the lack in CKn and PKn so as to differentiate appropriately mathematics instruction.

Acknowledgements

This project is implemented through the Operational Program "Human Resources Development, Education and Lifelong Learning" and is co-financed by the European Union (European Social Fund) and Greek national funds.



Ευρωπαϊκή Ένωση
European Union

Επιχειρησιακό Πρόγραμμα
Ανάπτυξη Ανθρώπινου Δυναμικού,
Εκπαίδευση και Διά Βίου Μάθηση
Ειδική Υπηρεσία Διαχείρισης
Με τη συγχρηματοδότηση της Ελλάδας και της Ευρωπαϊκής Ένωσης



References

- Bempeni, M. & Vamvakoussi, X. (2015). Individual differences in students' knowing and learning about fractions: Evidence from an in-depth qualitative study. *Frontline Learning Research*, 3, 17-34.
- Brock, G., Pihur, V., Datta, S., & Datta, S. (2008). clValid: An R package for cluster validation. *Journal of Statistical Software*, 25(4) (p.1-22).
- Canobi, K. H. (2004). Individual differences in children's addition and subtraction knowledge. *Cognitive Development*, 19, 81-93.
- Cohen, J., Cohen, P., West, S. G., & Aiken, L. S. (2003). Applied multiple regression/correlation analysis for the behavioral sciences (3rd ed.). Mahwah, NJ: Erlbaum.
- Cronbach L. (1951). Coefficient alpha and the internal structure of tests. *Psychometrika*, 6, 297-334.
- Gilmore, C. K., & Bryant, P. (2006). Individual differences in children's understanding of inversion and arithmetical skill. *British Journal of Educational Psychology*, 76, 309-331.
- Hallett, D., Nunes, T., & Bryant, P. (2010). Individual differences in conceptual and procedural knowledge when learning fractions. *Journal of Educational Psychology*, 102, 395-406.
- Hallett, D., Nunes, T., Bryant, P., & Thorpe, C. M. (2012). Individual differences in conceptual and procedural fraction understanding: The role of abilities and school experience. *Journal of Experimental Child Psychology*, 113, 469-486.
- Moss, J., & Case, R. (1999). Developing children's understanding of the rational numbers: A new model and an experimental curriculum, *Journal for Research in Mathematics Education*, 30, 122-147.
- Polit, D., Beck, C., & Owen, S. (2007). Is the CVI an acceptable indicator of content validity? Appraisal and recommendations. *Res Nurs Health*, 30, 459-467.
- Rittle-Johnson, B., Siegler, R. S., & Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology*, 93, 346-362.
- Rittle-Johnson, & B., Schneider, M. (2015). Developing conceptual and procedural knowledge of mathematics. In R. Kadosh & A. Dowker (Eds.), *Oxford Handbook of Numerical Cognition* (pp.1118-1134). Oxford: Oxford University Press.
- Star, J.R. (2005). Reconceptualizing procedural knowledge. *Journal for research in mathematics education*, 36, 404-411.
- Ware J. & Gandek B. (1998). Methods for testing data quality, scaling assumptions and reliability: The IQOLA project approach. *J Clin Epidemiol.*, 51, 945-952.
- Van Hoof, J., Verschaffel, L., & Van Dooren, W. (2015). Inappropriately applying natural number properties in rational number tasks: Characterizing the development of the natural number bias through primary and secondary education. *Educational Studies in Mathematics*, 89, 39-56.

PRIMARY SCHOOL CHILDREN'S (9 YEAR OLDS') UNDERSTANDING OF QUADRILATERALS

Melania Bernabeu, Mar Moreno, and Salvador Llinares

University of Alicante, Spain

Our goal was to identify what factors trigger or inhibit the capacity to recognize different properties of quadrilaterals and how these properties were related to classify quadrilaterals. A total of 29 primary school children (9 years old) participated in a teaching experiment focused on representing and distinguishing quadrilaterals to emphasize the transition from description to analytical perspective. Findings suggest that the identification and use of different attributes to represent and classify quadrilaterals is gradual and depends on the attributes used. This result supports the idea that the coordination between the discursive registers and the differentiation of the relevant attributes are key factors in the transition from the perceptual to the analytical perspective in quadrilaterals' conceptual understanding.

INTRODUCTION AND THEORETICAL FRAMEWORK

Children's developing concepts of geometrical shapes indicates that older children rely more on rule-based definitions and less on perceptual similarity than younger children (Satlow, & Newcombe, 1998). However, the transition from the overall domain of geometric shapes to relying in the definition of the shape features is conditioned by the figure's shape and by prototypical examples, as children need a synthesis of verbal declarative and imagistic knowledge, each interacting with the other (Clements, Swaminathan, Hannibal, & Sarama, 1999). This occurs when children identify shapes according to their appearance and do not attend to geometric properties characteristic of the class of figures represented (Clements, et al., 1999). For example, Yesil-Dagli, & Halat (2016) reported that 5-6 years old children could identify prototyped triangles but experienced difficulties in identifying triangles of different sizes, types and orientations (Halat, & Yesil-Dagli, 2016; Yesil-Dagli, & Halat, 2016). Furthermore, the way in which toddlers learn the names of geometric forms (canonical or prototypical as equilateral triangles and non-canonical ones as scalene triangles) points to the existence of a fragmentary knowledge in which defining properties are not well understood (Verdine, Lucca, Golinkoff, Hirsh-Pasek, & Newcombe, 2016). In this development, the prototypical examples of shapes are accepted by children without requiring any justification. It is recognized that they constitute both a help and a hindrance to the formation of concepts.

Since the conceptual meaning of a geometrical figure, such as a triangle or a quadrilateral, enables children to decide whether a shape with three or four sides is a triangle or quadrilateral, the non-intuitive non-examples that bear a significant similarity to

valid examples are sometimes mistakenly identified as examples (Tsmair, Tirosh, & Levenson, 2008). Therefore, in the concept acquisition of geometrical shapes, reasoning about properties allows children to move away from a descriptive perspective for example, and to name the similarities and differences between a parallelogram and a rectangle (Walcott, Mohr, & Kasberg, 2009).

A way to theoretically explain the development from perceptual description to recognition of similarities and differences and classification of figures, was provided by Duval (1999), based on the coordination between the registers of representation (for example, verbal descriptions of a figure and its drawing) and the coordination between perceptual, discursive and operative apprehensions. With perceptual apprehension, children perceive figures in a global way; with operative apprehension they can move and modify the figure (e.g. its orientation) to solve the task (e.g. move the shape to identify it as a quadrilateral). Finally, in discursive apprehension, children associate geometrical properties to shapes involving an anchorage change (from visual to discursive and from discursive to visual). Therefore, in discursive apprehension, several figures can correspond to the same geometrical object, e.g. the quadrilateral concept (Duval, 1999, 1995).

The existence of several registers of representation (verbal and written description and drawings) provides specific ways to process each register in the learning of the conceptual meaning of the quadrilaterals. During their conceptual learning of the quadrilateral concept, children should generate specific visual operations that are particular to each register allowing to change any initial figure into another one, while keeping the properties of the initial figure (Duval, 1999). Therefore, to gain insight into the conceptual learning of quadrilaterals, it is necessary to identify what factors trigger or inhibit the development of the recognition of different properties of the quadrilaterals and how these properties are related to classify different types of quadrilaterals. Students' understanding of inclusion relations of quadrilaterals and the role played by prototypical examples has been studied for secondary students (Fujita, & Jones, 2006; Fujita, 2011). On the other hand, research has showed that preschool children form schemas based on feature analysis of visual forms, while continuing to rely primarily on visual matching to distinguish shapes, even though they are able of recognizing components and simple properties of familiar shapes (Clements, et al. 1999; Levenson, Tirosh, & Tsamir, 2011). Less is known, however, on the role played by the coordination of different registers in the conceptual learning of quadrilaterals in primary school children.

Our research question is: how could several registers of representation and the coordination between different attributes influence 9-year-old (3rd grade) children's capacity to recognize, represent and classify quadrilaterals?

METHOD

Participants and context

The participants in this study were 29 primary school children (9-year-olds, 3rd grade) attending a state school. They were organized into eight groups. All groups included children with high and low levels in mathematics, as well as some children with more verbal communication skills. In the Spanish curriculum in primary education, 9-year-old children are expected to recognize polygons according to their number of sides; recognize concave and convex figures; circumference, the circle and its elements; and identify regularities and symmetries in geometric figures. The learning of quadrilaterals is not specifically included.

Design of an instructional sequence

We designed an instructional sequence of eight teaching sessions and associated tasks corresponding to the characteristics of children's understanding of shapes found in prior studies (Fisher, Hirsh-Pasek, Newcombe, & Golinkoff, 2013; Halat, & Yesil-Dagli, 2016; Satlow, & Newcombe, 1998; Yesil-Dagli, & Halat, 2016) and curricular documents in our country. In each teaching session, children engaged in dyad, small group and large group discussions to solve the tasks (recognize attributes, draw figures according to several attributes and classify a set of figures, define its properties and how different shapes were related). In the last two sessions, five tasks on quadrilaterals were performed (Table 1). In previous sessions, the tasks focused on the polygons, some of their attributes (angles, sides, diagonals, vertices, concavity and convexity, and symmetry axes), and types of triangles.

Sessions	Characteristics	Type of task
S. 7	T.12. Recognize and assign attributes (e.g. sides, parallelism, diagonal, angle, symmetry axes) of quadrilaterals. Prototypical and non-prototypical figures.	Recognize
	T. 13. Identify some attribute and classify a set of quadrilaterals. Prototypical and non-prototypical figures.	Classify
S. 8	T. 14. Justify the attribute of a quadrilateral. Non-examples figures.	Recognize
	T. 15. With the given attributes, draw two quadrilaterals (e.g. draw two quadrilaterals with diagonal perpendicular). Prototypical and non-prototypical figures.	Represent
	T. 16. Classify a set of shapes according to a given attribute. Prototypical and non-prototypical figures.	Classify

Table 1: Characteristics of the tasks on quadrilaterals.

The figures in the tasks were given on individual cards. The different tasks included examples and non-examples, and prototypical and non-prototypical examples of fig-

ures. These card figures were presented in different orientations, specifically flipped and rotated (Martin, Lukong, & Reaves, 2007; Tsamir, et al., 2008). The different quadrilaterals used were parallelogram (square, rectangle, rhombus, rhomboid), and non-parallelogram (kite, different types of trapeze – rectangle, isosceles). Each group of children was given a set of cards with different quadrilaterals and a set of tags to name different attributes (sides, parallelism, diagonal, equal diagonals, perpendicular diagonals, vertex, convex, concave, equal sides, symmetry axes, and so on). The tasks' features and their implementation by teachers aimed at highlighting the nuances of visualization and figure processing indicated by Duval, considering that the existence of several registers of representation provided specific ways to process each register (Duval, 1999). To solve each task, operative and discursive apprehensions had to be made since children had to recognize the figure, relate their attributes and in some cases, modify it mentally to assign properties or classify them. The teaching sessions were video recorded. The data analysed in this research corresponded to the teaching session's transcriptions and children group's notebooks relating to the five tasks on quadrilaterals.

Analysis

We analysed the children's notebooks from each session on quadrilateral tasks. To understand children's answers, we analysed group reasoning for each task. Based on an analysis of the session transcriptions, we identified when they used an operative apprehension (for example when they turned a card around to visualize the figure from a different orientation and linked their discourse to the new position of the card), and how they used different attributes to organize their discursive apprehension to solve the different tasks. We identified the attributes and properties causing most difficulties and for which task (recognize quadrilateral, represent and classify). This procedure allows us identify any factors inhibiting the capacity to recognize different properties of the quadrilaterals and how these properties were related to classify them. With this procedure, we tried to determine patterns and trends in how children recognized, represented and classified quadrilaterals.

RESULTS

The results from the analysis of the five tasks relating to the quadrilaterals show that students did not use the different attributes of the quadrilaterals (for example, diagonals, symmetry axes, parallelism of the sides, length of the sides) in the same way. This fact indicates that the attribute used influences the way children recognize, represent and classify quadrilaterals. We organize these results taking into account the influence of different attributes on the performance of mental activities (recognize, represent and classify) that indicated how coordination between perceptual and discursive apprehensions occurred.

Recognize and Represent

The recognition tasks revealed how the attributes were used by children to justify the difference between quadrilaterals. So, the way in which the different figures were recognized provide us insight about how the figures were understood. We identify three features. First, how the children use the meanings of attribute to justify what a figure is. For example, in the tasks on parallelogram recognition, one of the children justified that the isosceles trapeze was not a parallelogram since the parallel sides were not equal (this justification seems to assume that the parallelograms have the equal sides)

- 1 A2G6: It is not a parallelogram (Figure 1) because the one above and the one below do not coincide.
- 2 I: What do you mean, they do not coincide?
- 3 A2G6: They are not equal.

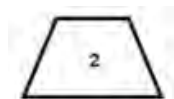


Figure 1: Isosceles trapezoid.

Meanwhile, another child justifies that the isosceles trapeze was not a parallelogram because it did not have parallel sides two to two.

- 4 A1G1: It is not a parallelogram because at some point the lateral sides will cross each other.

A second characteristic is that the attributes were not recognized in the same way for the different figures. For example (Figure 2), in the task where they had to recognize the attributes to different quadrilaterals by means of labels, in the kite the attribute of parallelism was wrongly labelled ("parallel sides two by two") while for the right trapeze, they assigned all the labels correctly.

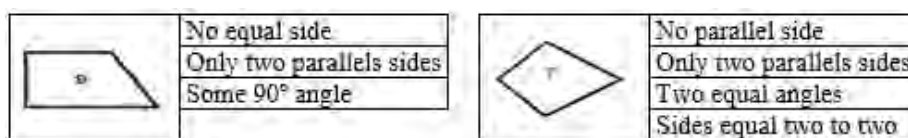


Figure 2: Solution to activity 12.

This reveals that the development of the ability to make register changes (from verbal to figurative, and vice versa) and the coordination of operative and discursive apprehensions depends on the considered attribute and figure. This result indicates the progressive development of discursive apprehensions in the conceptual learning of quadrilaterals.

A third characteristic is that the attributes were not used systematically when they were less common, such as the diagonal perpendicular. In the representation task shown in Figure 3, all the groups drew at least one of the figures (mostly the square), however, they had difficulties in drawing a second quadrilateral with the same attribute. Four

groups (out of eight) drew rectangles confusing the diagonals with the axes of symmetry. This seems to indicate that the use of certain attributes to represent a quadrilateral is not exhaustive and depends on the attribute.

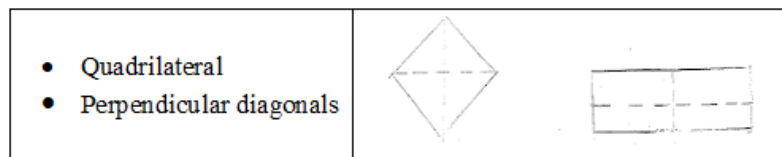


Figure 3: Drawings created by group 3.

Classification

In the task asking to classify quadrilaterals into two groups, seven out of eight groups grouped the quadrilaterals according to the parallelism of the sides. There were two ways of classifying the figures: one parallel side and no parallel side (Figure 4); and parallelograms and non-parallelograms.

Four groups performed the classification correctly and three included a quadrilateral in the group that it did not belong to. The fact that the majority used parallel sides versus other attributes (perpendicular diagonals) in the classification tasks reflects the gradual progression with which children develop their conceptual understanding of quadrilaterals. It reveals a variety of patterns of development relating to the understanding of quadrilaterals defined by the different considered attributes. For example, considering that diagonals are cut at midpoint, as an attribute of parallelograms, is later associated with the conceptual meaning of parallelogram.

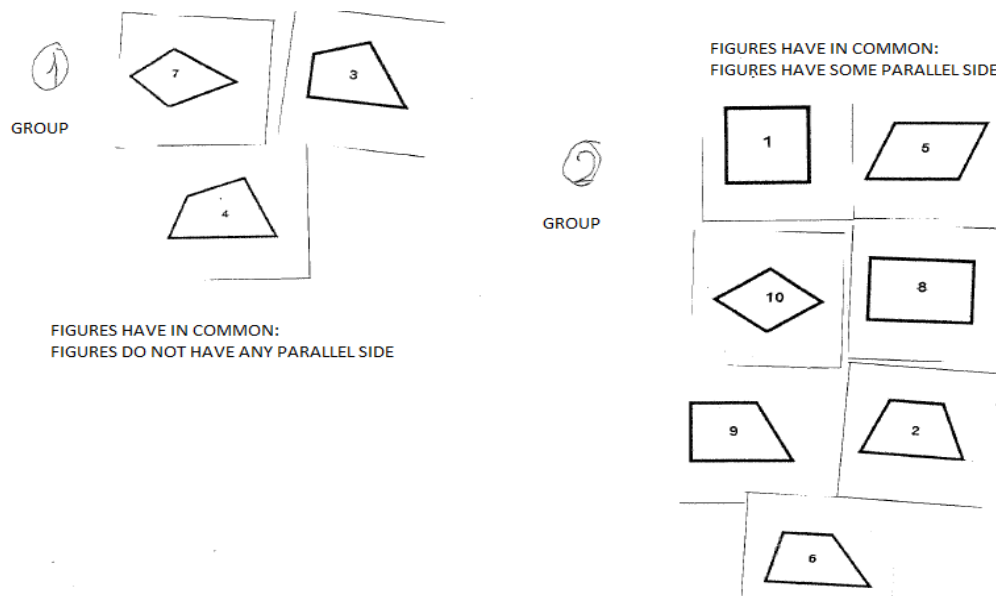


Figure 4: Classification of quadrilaterals.

DISCUSSION

The recognition of different attributes and their use to represent and classify quadrilaterals shows that the conceptual understanding of the different types of quadrilaterals is gradual and depends on the attributes used. The first attributes used to differentiate quadrilaterals were parallelism and the length of the sides, while the use of perpendicular diagonals and axes of symmetry appeared gradually but not exhaustively.

These results indicate the gradual manner in which different attributes are used analytically by children in tasks related to open classification (when no criterion is given) and in tasks related to representing and recognizing quadrilaterals reflecting some attributes. Evidence of this gradual understanding of the concept of quadrilaterals are the manner in which some children justified that a figure had parallel sides or not and the difficulties in using the attribute "perpendicular diagonals" to draw different quadrilaterals. This finding highlights the role of discursive register in the justify given, and use of attributes in the conceptual learning of quadrilaterals. That is, the coordination between the discursive register and the differentiation of attributes is key in the transition from the perceptual to the analytical perspective (Duval, 1999). It is precisely this coordination and differentiation that are revealed in discursive apprehensions.

The way in which children seem to progress when they reason about figures can be explained by the development of the relationship between perceptual, discursive and operative apprehensions (Duval, 1995). In this sense, Duval (1999) emphasizes the importance of generating a new representation from a given representation. This fact becomes evident in the tasks in which children must represent a quadrilateral that fulfils some condition or conditions (for example, quadrilateral with perpendicular diagonals) that implies a change of register, according to Duval (1999). This fact is relevant since the conceptual understanding of quadrilaterals is based on the development of register changes and the progressive coordination between registers (drawing and verbal expression). In this case, our results support the finding that coordination between registers of drawing representation and verbal expression is progressive and depends on the attribute.

Acknowledgements

This research has been financed by Prometeo/2017/135 from Generalitat Valenciana (Spain).

References

- Clements, D. H., Swaminathan, S., Hannibal, M. A. Z., & Sarama, J. (1999). Young children's concepts of shape. *Journal for Research in Mathematics Education*, 30, 192–212.
- Duval, R. (1995). Geometrical Pictures: Kinds of representation and specific processes. In R. Sutherland & J. Mason (Eds.), *Exploiting mental imagery with computers in mathematical education*. Berlin, Springer, pp. 142- 157.

- Duval, R. (1999). Representation, Vision and Visualization: Cognitive Functions in Mathematical Thinking. In F. Hitt and M. Santos (Eds.), *Proceedings of the 21st Annual Meeting North American Chapter of the International Group of PME*. Cuernavaca, México. Columbus, Ohio, USA: ERIC/CSMEE, pp. 3-26.
- Fisher, K. R., Hirsh-Pasek, K., Newcombe, N., & Golinkoff, R. M. (2013). Taking shape: Supporting preschoolers' acquisition of geometric knowledge through guided play. *Child development*, 84(6), 1872-1878.
- Fujita, T., & Jones, K. (2007). Learners' understanding of the definition and hierarchical classification of Quadrilaterals: Towards a Theoretical framing. *Research in Mathematics Education*, 9(1), 3-20.
- Fujita, T. (2012). Learners' level of understanding of the inclusion of quadrilaterals and prototype phenomenon. *Journal of Mathematical Behavior*, 31(1), 60-72
- Halat, E., & Yesil-Dagli, U., (2016). Preschool Students' Understanding of a Geometric Shape, the Square. *BOLEMA*, 30(55), 830-848.
- Levenson, E., Tirosh, D., & Tsamir, P. (2011). *Preschool Geometry. Theory, Research and Practical Perspectives*. Rotterdam: Sense Publishers.
- Martin, T., Lukong, A., & Reaves, R. (2007). The role of manipulatives in arithmetic and geometry tasks. *Journal of Education and Human Development*, 1(1), 1-14.
- Satlow, E., & Newcombe, N. (1998). When is a Triangle not a Triangle? Young Children's Conceptions of Geometric Shapes. *Cognitive Development*, 13, 547-559.
- Tsamir, P., Tirosh, D., & Levenson, E. (2008). Intuitive nonexamples: The case of triangles. *Educational Studies in Mathematics*, 69(2), 81-95.
- Verdine, B., Lucca, K., Golinkoff, R., Hirsh-Pasek, & Newcombe, N. (2016). The Shape of Things: The origin of young children's knowledge of the names and properties of geometric forms. *Journal of Cognition and Development*, 17(1), 142-161.
- Walcott, C., Mohr, D., & Kastberg, S. E. (2009). Making sense of shape: An analysis of children's written responses. *The Journal of Mathematical Behavior*, 28(1), 30-40.
- Yesil-Dagli, U., & Halat, E. (2016). Young Children's conceptual Understanding of Triangle. *Eurasia Journal of Mathematics, Science & Technology Education*, 12(2), 189-202.

VALUE OF PICTURES IN MODELLING PROBLEMS FROM THE STUDENTS' PERSPECTIVE

Matthias Böckmann and Stanislaw Schukajlow

University of Münster

Pictures are an important part of human life, and they often accompany modelling problems. In the present study, we investigated whether the extent to which students believe pictures are valuable for understanding modelling problems differs for decorative, representational, and essential pictures. 217 ninth and tenth graders from nine German middle-track classes were randomly assigned to three groups. One group reported the picture-specific utility value of decorative pictures, whereas two other groups reported the utility value of representational pictures and essential pictures, respectively. Students' picture-specific utility value ratings were higher for essential pictures and representational pictures than for decorative pictures, and their utility value ratings were higher for essential pictures than for representational pictures.

INTRODUCTION

One important goal of mathematics education is to ensure that students are able to solve problems in the real world using mathematics; thus, modelling competence is part of curricula all over the world (Niss et al., 2007). Similar to the problems in the real world, modelling problems that are presented in the classroom should (and often do) include text and pictures. In order to solve modelling problems, students have to use information that can be found in both the text and the pictures.

The Cognitive Theory of Multimedia Learning explains how humans deal with information that is presented in text and pictures. There are two channels in working memory, one that processes information from words and one from pictures (Mayer, 2005). For an efficient processing of information, the usefulness of the pictures has to be recognized. However, recent studies showed inconsistent results with respect to whether students saw pictures as useful while they solved mathematical problems (Dewolf et al., 2015; Elia & Philippou, 2004). More precisely, it is not clear whether students recognize that pictures can be useful for solving problems with a connection to reality. Moreover, although information in the real world is often presented pictorially, we could not find any studies that investigated students' perceptions of the usefulness of pictures for solving modelling problems in the classroom. The current study focuses on students' perceptions of the extent to which pictures that accompany modelling problems can be useful (or the perceived utility value of pictures) for solving these problems.

THEORETICAL BACKGROUND

Utility Value

Values are an important part of affect and have been investigated from social, psychological, and sociological perspectives (Bishop et al., 2003). The expectancy-value theory links expectancies and personal values and proposes that expectancies and values influence performance, task choice, and motivation (Eccles, 1983). A positive relation between values and students' performance was recently confirmed for problems with and without a connection to reality (Schukajlow, 2017).

In line with the expectancy-value theory researchers underline the importance of the utility (or extrinsic) value for motivation and achievement. A task's utility value describes the learner's perceived usefulness of a task (Eccles & Wigfield, 2002) and refers to the importance of a task or its parts (e.g. text or pictures) for career, grades, an accurate solution, or other indicators of success. In this study, we analyzed the utility value of pictures for understanding modelling problems and thus for the solution process. As the pictures that accompany modelling problems can have different functions in the solution process, we were interested in determining whether students would assign different utility value ratings to different types of pictures.

Pictures and their functions in problem solving

The term "pictures" describes static visual illustrations such as photos, paintings, or vector graphics. In this study, we used photos as pictures because photos are closely connected to reality and reflect reality more precisely than other types of pictures.

In combination with text, pictures can serve different functions. In the present study, we adapted a taxonomy of pictures for modelling problems that was developed by Elia and Philippou (2004), who specified different functions of pictures in mathematical problem solving. *Decorative pictures* "do not give any actual information concerning the solution of the problem" (Elia & Philippou, 2004, p. 328). *Representational pictures* show the parts of the situation described in the text. *Essential pictures* (called informational pictures by Elia & Philippou, 2004) present information that is essential for solving the modelling problems.

Utility value and pictures in modelling

A cognitively demanding transfer between real-world problems and mathematics is at the core of mathematical modelling. Activities that are needed to solve modelling problems are understanding the situation and constructing a situation model (Blum & Leiß, 2007). As understanding the problem is the first activity in the modelling process, it can be expected to influence other modelling activities such as constructing a mathematical model and is thus very important for solving a problem. This expectation was confirmed empirically, as the quality of the situation model was found to be closely related to modelling competence (Krawitz et al., 2017; Leiß et al., 2010). Because it is important for students to understand modelling problems, we decided that our study would focus on this modelling activity.

One example of a modelling problem used in our study is the Kite problem (Figure 1). The process of understanding while solving the Kite problem results in the construction of a model of the situation that includes two people, a piece of string, a kite, and the positions of the people and the kite. The height of the kite is unknown, and it can be calculated, for example, by using Pythagoras' theorem and adding an estimate of Lucas' height.

Lucas got a new kite as a birthday present. The kite is 1 m in length and 50 cm in width. Lucas flies the kite with his friend Susan (see picture). They are standing at a distance of 80 m from each other. The kite's string has a length of 100 m. Susan is right under the kite.

How high is the kite flying at this moment?



Figure 1: The Kite modelling problem with a decorative picture

Pictures can support the modelling process by facilitating students' understanding. The decorative picture in Figure 1 does not facilitate an understanding of the problem because it presents only the kite but not the positions of the people flying the kite. Thus, this picture does not enhance a deeper understanding of the problem, which is crucial for constructing a deep situation model and thus also for solving a modelling problem (Krawitz et al., 2017). However, a modelling problem can also include a representational or an essential picture instead of a decorative picture. A representational picture of the Kite problem might show where Susan, Lucas, and the kite are positioned and can help the problem solver structure the information and construct a model of the situation (Figure 2). An essential picture should include important information such as the distance of 80 m between Susan and Lucas in the Kite problem. Without taking this information into account, it is not possible to solve the problem (Figure 3).

A picture's utility value determines whether the picture will be used in the solution process. If students perceive the picture as useful, they might integrate the information from the picture into the solution. In this case, the picture can support the modelling process. This is why we asked students to rate the utility value of different pictures.

Prior findings on the utility value of pictures for solving problems have been contradictory to some extent. Whereas most students recognized that decorative pictures did not help them solve mathematical problems (Elia & Philippou, 2004), students did not always identify the high utility value of pictures with other functions. For example, in the study by Elia and Philippou (2004), students realized the supporting role of representational pictures for solving arithmetic word problems. In the study on realistic problems, however, students did not include information from the picture in their solution, and thus, they did not identify the usefulness of representational pictures for a solution (Dewolf et al., 2015). Further, only a few students recognized the importance of essential pictures for solving arithmetic word problems (Elia & Philippou, 2004).

One explanation for these results is that students could not identify what kinds of information in representational or essential pictures might be helpful for solving problems. The ability to recognize picture-specific utility value might depend, among other things, on the type of problem and might be different for real-world problems.

HYPOTHESIS

Pictures can help people understand real-world problems. Representational and essential pictures enhance the construction of a situation model because these pictures can act as structural aids and can facilitate the step of understanding in the modelling process. These pictures are supposed to have high utility value in helping people understand the problems, whereas decorative pictures do not support the modelling process and therefore have a low perceived utility value in helping people understand the given problems. In addition, essential pictures are necessary for the solution process and might have a higher utility value than representational pictures. Our considerations led to the following hypotheses: (1) Students will assign higher utility value to representational and essential pictures than to decorative pictures in the extent to which these pictures help them understand modelling problems. (2) Students will assign higher utility value to essential pictures than to representational pictures.

METHOD

Sample and design

217 students from nine middle-track classes (lower secondary schools) in grades 9 and 10 (mean age=15.06 years, SD=.79; 49.9% female) participated in the study. The students in each class were randomly assigned to one of the three experimental groups. Each group read modelling problems accompanied by pictures and then rated each picture's specific utility value for understanding the problem. Students in group 1 documented their perceived utility value of pictures with a decorative function. Group 2 reported on the utility value of representational pictures, and group 3 on the utility value of essential pictures. The instructions were: "Read each problem carefully and then answer some questions. **You do not have to solve the problems!**" (cf. Schukajlow, 2017). The participants did not solve the problems because it was not necessary to solve the problems in order to rate a picture's utility value for *understanding* the problem. After reading these instructions, students read each problem and answered the question about utility value.

Sample problems

In the present study, we used six modelling problems on the topic Pythagoras' theorem, that were developed and tested in prior studies (e.g. Schukajlow, 2017; Blum, 2011). In the present study, we reworked the pictures that accompanied the problems and offered students problems with a decorative, representational, or essential picture. A sample problem (i.e. the Kite problem) with a decorative picture was presented above (Figure 1). Figure 2 presents the same modelling problem with a representa-

tional picture. This picture shows the situation described in the task and can support the problem solver's understanding of the problem.

Lucas got a new kite as a birthday present. The kite is 1 m in length and 50 cm in width. Lucas flies the kite with his friend Susan. They are standing at a distance of 80 m from each other (see picture). The kite's string has a length of 100 m. Susan is right under the kite.

How high is the kite flying at this moment?



Figure 2: The Kite modelling problem with a representational picture

In the experimental condition with an essential picture, numerical information about the distance of 80 m between Lucas and Susan is missing from the text but is presented in the picture (Figure 3). This information is important for solving the problem.

Lucas got a new kite as a birthday present. The kite is 1 m in length and 50 cm in width. Lucas flies the kite with his friend Susan. They are standing far away from each other (see picture). The kite's string has a length of 100 m. Susan is right under the kite.

How high is the kite flying at this moment?



Figure 3: The Kite modelling problem with an essential picture

Utility value

A statement about the picture-specific utility value followed each modelling problem. The item that was used to measure the utility value of decorative, representational, and essential pictures for understanding the problem was “The picture helps me understand the problem.” A 5-point Likert scale was used (1=not at all true; 5=completely true) to record the students' answers. The picture-specific utility value was measured by calculating mean values of the answers for all six problems for each type of picture. The Cronbach's alpha reliabilities for the utility value scale for decorative, representational, and essential pictures were .73, .82, and .84, respectively.

In order to analyze group differences in the perceived utility values, a one-way ANOVA was computed. For a post hoc analysis, we used Bonferroni comparisons.

RESULTS

Perceived utility value

To test Hypotheses 1 and 2 and compare the utility values of pictures for understanding modelling problems, we compared the utility value means for the students who rated the pictures with decorative, representational, and essential functions. Table 1 shows the group means for perceived utility value for the three different types of pictures.

function of picture		
decorative	representational	essential
2.14 (.71)	3.26 (.89)	3.74 (.84)

Table 1: Means (SDs) for perceived utility value

Students assigned the lowest utility value to the decorative and the highest to the essential pictures. As expected, there were significant differences in perceived utility value between the three types of pictures ($F(2, 214)=74.40, p<0.01, \eta^2=.41$).

(I) Type	(J) type	Mean Difference (I-J)	Std. Error (SE)	<i>p</i>	Cohen's d
representational	decorative	1.12	.13	<.01	1.39
Essential	decorative	1.60	.14	<.01	2.06
Essential	representational	.48	.14	<.01	0.55

Table 2: Values from the Bonferroni post hoc analysis of differences in utility value

A post hoc analysis employing t-tests with a Bonferroni correction revealed significant differences between representational and decorative pictures ($t(145)=8.46$) and between essential and decorative pictures ($t(135)=12.36$) (Table 2). Thus, this result confirmed our first hypothesis: Students gave higher utility value ratings to representational and essential pictures than to decorative pictures. Furthermore, we found a significant difference between the utility value ratings of the essential and representational pictures ($t(140)=3.33$) and confirmed our second hypothesis: Students gave higher utility value ratings to essential pictures than to representational pictures.

DISCUSSION

In the present study, we analyzed students' utility value ratings of pictures used in modelling problems. The results showed that students' picture-specific utility value ratings differed according to the pictures' functions (Elia & Philippou, 2004). Decorative pictures did not support the modelling process, as they do not include infor-

mation that can help solve the problem and do not facilitate the construction of a situation model. Representational and essential pictures facilitated the construction of a situation model due to their supporting role in the modelling process. As expected on the basis of theoretical considerations and prior findings, students reported a significantly lower utility value for understanding the problems with decorative pictures than for representational or essential pictures. This result is in agreement with Elia and Philippou's (2004) study in which all students recognized that decorative pictures did not help solve a problem and gave decorative pictures lower utility value ratings than representational pictures. However, it is not in line with the results of Dewolf et al. (2015) who did not find indications of the perceived importance of representational pictures for solving realistic problems. One explanation of this finding is a difference in the process of solving realistic problems and the modelling problems used in our study. Whereas students neglect reality when solving realistic problems, they consider reality when solving modelling problems (Galbraith & Stillman, 2006).

The comparison of the utility value ratings of essential and representational pictures confirmed that students gave higher utility value ratings to essential pictures. This finding indicates that students identified information in the picture that was essential for solving the problem. This result was different from Elia and Philippou's (2004) findings. One explanation for the clear differences between the utility value ratings of essential and representational pictures in our study is that the pictures and text were presented simultaneously. Thus, it was easier for the students in our study to identify the usefulness of essential pictures than in Elia and Philippou's (2004) study, in which the students first worked with the text and were given the picture later.

One limitation of our study involves the design of the essential pictures. The numerical information in the essential pictures can attract readers' attention and can thus foster utility value. We tried to counter this limitation by adding "*see picture*" to the text in all conditions. Moreover, students might estimate pictures' utility value superficially, as students might assign higher utility value to such pictures without understanding whether this information is important for solving the problem. This open question should be clarified in future studies. Another important future question is how the utility value of different pictures affects students' modelling performance.

CONCLUSION

Modelling problems often include pictures. However, to the best of our knowledge, the importance of pictures in modelling problems had not been investigated until now. In the present study, we expanded prior findings to include perceptions of the role of pictures in modelling problems. In our study, students recognized that pictures with different functions had different levels of usefulness, and we encourage teachers and researchers to pay attention to the pictures they use in the classroom.

References

- Bishop, A., Seah, W. T., & Chin, C. (2003). Values in Mathematics Teaching — The Hidden Persuaders? In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Second International Handbook of Mathematics Education* (pp. 717-765). Dordrecht: Springer Netherlands.
- Blum, W., & Leiß, D. (2007). How do Students and Teachers Deal with Modelling Problems? In C. Haines, P. Galbraith, W. Blum, & S. Khan (Eds.), *Mathematical modelling. ICTMA12 - Education, engineering and economics* (pp. 222–231). Chichester: Horwood.
- Blum, W. (2011). Can Modelling Be Taught and Learnt? Some Answers from Empirical Research. In G. Kaiser, W. Blum, R. Borromeo Ferri, & G. Stillman (Eds.), *International Perspectives on the Teaching and Learning of Mathematical Modelling. Trends in Teaching and Learning of Mathematical Modelling* (Vol. 1, pp. 15–30). Dordrecht: Springer Netherlands.
- Dewolf, T., van Dooren, W., Hermens, F., & Verschaffel, L. (2015). Do students attend to representational illustrations of non-standard mathematical word problems, and, if so, how helpful are they? *Instructional Science*, 43(1), 147–171.
- Eccles, J. S. (1983). Expectancies, Values, and Academic Behaviors. In J. T. Spence (Ed.), *Achievement and achievement motives. Psychological and sociological approaches* (pp. 75–146). San Francisco, CA: Freeman.
- Eccles, J. S., & Wigfield, A. (2002). Motivational beliefs, values, and goals. *Annual Review of Psychology*, 53, 109–132.
- Elia, I., & Philippou, G. (2004). The function of pictures in problem solving. In M. J. Høines & A. B. Fuglestad (Eds.), *Proc. 28th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 2, pp. 327–334). Bergen, Norway: PME.
- Galbraith, P. L., & Stillman, G. (2006). A Framework for Identifying Student Blockages during Transitions in the Modelling Process. *ZDM*, 38(2), 143-162.
- Krawitz, J., Schukajlow, S., Chang, Y.-P., & Yang, K.-L. (2017). Reading comprehension, enjoyment, and performance: how important is a deeper situation model? In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), *Proc. 41th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 97-104). Singapore: PME.
- Leiß, D., Schukajlow, S., Blum, W., Messner, R., & Pekrun, R. (2010). The role of the situation model in mathematical modelling – task analyses, student competencies, and teacher interventions. *Journal für Mathematikdidaktik*, 31(1), 119-141.
- Mayer, R. E. (2005). Cognitive Theory of Multimedia Learning. In R. E. Mayer (Ed.), *The Cambridge Handbook of Multimedia Learning* (pp. 31–48). Cambridge: Cambridge University Press.
- Niss, M., Blum, W., & Galbraith, P. L. (2007). Introduction. In W. Blum, P. L. Galbraith, H.-W. Henn, & M. Niss (Eds.), *Modelling and Applications in Mathematics Education: the 14th ICMI Study* (pp. 1-32). New York: Springer.
- Schukajlow, S. (2017). Are values related to students' performance? In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), *Proc. 41th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 4, pp. 161-168). Singapore: PME.

CALCULUS STUDENTS' USE OF VISUALIZATIONS WHEN SOLVING VOLUME PROBLEMS

Krista Kay Bresock and Vicki Sealey

West Virginia University

Calculus volume problems are unique in that they involve two areas important in calculus: integration and visualization. This research aims to investigate student understanding of integration when solving volume problems and how students use their drawings to aid in the problem-solving process. Participants were recruited from a large, public research university and interviews consisted of students working through routine and novel volume problems while discussing their thought processes aloud. All students in this study used pictures in the process of solving the volume problems, but the extent to which students could use their sketches meaningfully varied greatly. We recommend greater emphasis on non-traditional, non-revolution volume problems in Calculus 2 classrooms.

INTRODUCTION AND LITERATURE REVIEW

Students' first exposure to integral application problems generally comes in the form of area problems. These problems tend to be straight-forward for students because the 2-dimensional region can be visualized and sketched on paper and the problem solved with a straightforward integral formula. From there, students can move on to volume problems where the cognitive load is increased in a few areas: students must now visualize or imagine a 3-dimensional solid, and the corresponding integral formulas increase in complexity.

In one of the first studies on student understanding of integration, Orton (1983) asked students (age 16-22) to discuss their solutions to various calculus problems, including those on the topic of convergence, Riemann sums, areas, and volumes of solids. Students encountered difficulties on four questions in particular, each of which required their understanding of integration as a limit of a sum.

The results suggested that most students had little idea of the procedure of dissecting an area or volume into narrow sections, summing the areas or volumes of the sections, and obtaining an exact answer for the area or volume by narrowing the sections and increasing their number, making use of a limit process. (p. 7)

From that point forward, research on integration has continued to confirm this observation and has worked to expand our understanding of students' conceptions of integration. Thompson's (1994) study on students' view of integration as accumulation and their understanding of the Fundamental Theorem of Calculus found that students' "images of a Riemann sum [seemed] not to have entailed a sense of motion" which resulted in an insufficient foundation on which to build proper reasoning about a sum's

rate of change. Other studies on the integral as accumulation have been done with similar results (Bezuidenhout & Olivier, 2000; Rösken & Rolka, 2007; Mahir, 2009; Huang, 2010).

More recent research by Sealey (2006, 2014) and Jones (2013, 2015a, 2015b) has added depth to our understanding of students' conceptions of the definite integral. In particular, Sealey (2014) developed a framework for characterizing student understanding of Riemann sums and definite integrals that breaks students' conceptions down into layers. She discovered that students have particular trouble with the "product layer" – $f(x) \Delta x$ – and suggests that we allow students to engage in more in-depth activities that focus on this aspect of the definite integral. Jones (2013) found that students hold certain symbolic forms for the definite integral, and the most productive symbolic form is that of adding-up-pieces, also known as "multiplicatively-based summation" or MBS (Jones, 2015b).

Outside of mathematics education research, many studies have been done on students' conceptions of the integral in applied settings. In particular, physics education research has been particularly robust in this area (e.g., Yeatts et al, 1992; Cui et al, 2006; Meredith & Marrongelle, 2008; Von Korff & Rebello, 2012; Sealey & Thompson, 2016).

One key aspect of an integral volume problem is that of visualization. According to Arcavi (2003),

Visualization is the ability, the process and the product of creation, interpretation, use of and reflection upon pictures, images, diagrams, in our minds, on paper, or with technological tools, with the purpose of depicting and communicating information, thinking about and developing previously unknown ideas and advancing understandings. (p. 217)

In the past, attempts to categorize students by "learning styles" (e.g., Krutetskii, 1976) implied that visualization was an innate characteristic and not necessarily learnable. Research on the relationship between visualization and mathematical performance found that students who were classified as having a preference for visual methods ("visualizers") tended to perform more poorly on mathematical tasks (Lean & Clements, 1981; Battista, 1990). As time went on, we moved away from this idea of visualization as innate learning style and moved toward the more productive "visualization as one of many tools available in the problem-solving process."

Duval (1999) talked of understanding as translation between semiotic registers, with visualization being one of these registers. Even though pictures can be an important tool, Stylianou and Silver (2004) found that it is the quality of the picture that is most important. In their exploration of expert and novice visualization practices, they state that novices do produce pictures and visualizations, but they generally "lack the necessary procedural knowledge that would allow them to use visual representations functionally and efficiently" (p. 380). Bremigan's (2005) results were similar, stating that the presence of a constructed or modified diagram was not a sufficient condition for problem-solving success.

RESEARCH AIM

The aim of this study is to investigate how students use and understand their visualizations and sketches in the process of solving calculus volume problems. In particular, we are interested in examining if student understanding of the underlying Riemann sum structure of the definite integral is related to or evident in their use of visualizations when solving these types of problems. Our research has the goal of mixing methods and ideas in the areas of student content knowledge and visualization.

THEORETICAL FRAMEWORK

The combination of Sealey's (2014) Riemann Integral Framework and Zazkis, Dubinsky, and Dautermann's (1996) Visualization/Analysis (VA) Framework was used in this research to inform both the data collection and analysis of student understanding of the definite integral and their use of visualizations when solving volume problems. The Riemann Integral Framework breaks the constituent parts of the Riemann integral down into pieces – product, summation, limit, and function – and it allows us to examine in which layer students have the most trouble when solving volume problems.

For the VA Framework, there is a first visualization associated with the situation, V_1 (for example, a sketch of a 2-dimensional region), which is then acted on by an analysis event, A_1 . There is a subsequent visualization, V_2 , in which the student's first visualization is reinterpreted as a result of A_1 . Next comes another analysis, A_2 , on V_2 , which leads to another visualization, and so on. The process goes on like this, from visualization to analysis to visualization, optimally resulting in a more complete understanding of the physical situation. We aim to use this framework in the analysis of students' use of pictures and diagrams when solving integral volume problems.

RESEARCH METHODOLOGY

Interviews with students were conducted during summer 2016 (Study 1) and summer 2017 (Study 2). The participants were recruited from summer classes at a large, public research university in the north-eastern United States. There were a total of six interviews in the two studies. The interviews were one-on-one and videotaped, and the students were asked to write their math work on paper or a white board and discuss their thoughts aloud.

During the interviews, students were asked to complete three, second-semester calculus volume problems. In Study 1, the problems were three routine solid of revolution problems (e.g., "Find the volume of the solid obtained by rotating the region bounded by the curves $y = x^2$ and $y = 3x$ about the line $x = -1$.") and the students drew their sketches and wrote their math work on the same paper. In Study 2, the problems were two routine solid of revolution problem and one geometric solid problem ("Find the volume of the pyramid whose base is a square with side length L and whose height is h .") In order to more clearly observe when students were referring to their drawings during the problem-solving process, the method in Study 1 was adjusted so that students completed their math work on a separate piece of paper from their drawings

(which were done on either another piece of paper or a white board). In the interviews, students were probed about their responses and were asked to explain their work and thought processes. Some typical questions that were asked during the interview process were: “How do you know this integral gives you a volume?”, “What does the dx mean?”, and “Can you show on your picture where the different parts of the integral formula come from?”

The video data was transcribed and analysis is in the beginning stages. We use thematic analysis (Braun & Clark, 2006) to identify themes and patterns in the data. In particular, we are using theoretical thematic analysis, which “driven by the researcher’s theoretical or analytic interest in the area, and is thus more explicitly analyst-driven (p. 84). Since we have pre-determined themes that we will be looking for (e.g., working in a specific layer for Sealey’s framework or being in one of the visualization/analysis phases of Zazkis’ framework), we believe this more focused analysis works best for our methods.

PRELIMINARY RESULTS

In both studies, students very heavily relied on self-produced pictures and diagrams, but with varying results. The quality and characteristics of their drawings varied as well, from static, two-dimensional drawings, to dynamic, 3-dimensional drawings (Figure 1); from minimally-detailed drawings, to in-depth drawings. (Figure 2).

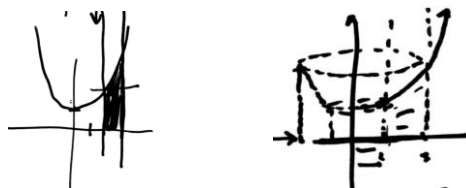


Figure 1: Student 1 and Student 2 drawings with 2D and 3D characteristics

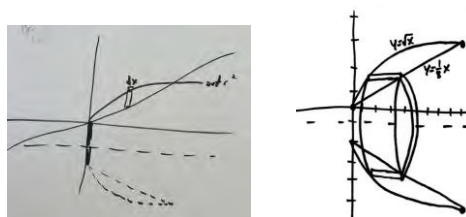


Figure 2: Student 6 and Student 5 drawings with minimal and in-depth detail

Overwhelmingly, students tended to work through the traditional solid of revolution problems in a very formulaic and methodical way. When probed about their reasoning and understanding about their methods for these types of problems, students’ responses exposed reliance on formulas and memorization.

Interviewer: Can you explain how that integral gives you the volume of a solid?

Student 2: I tend to get bogged down in the proofs. Especially with integration stuff, I treat that just as a formula. Physics is the class where I think about and understand.

Student 3: I know the formulas, but sometimes I don't know where to apply them.

As with Sealey's (2014) research, in the solid of revolution problems, students had the most issues in the product layer, but that seemed to be because it was the only layer they really needed to attend to, since they had the area formulas memorized (ie, πr^2 for slicing and $2\pi rh$ for shells).

Only two students (Student 5 and Student 6) worked through the geometric volume problem ("the pyramid problem") in Study 2, but some very interesting preliminary results have come out of this with respect to their methods and visualizations. Student 5 had much success with the solid of revolution problems and responded with some understanding that the formulas come from the limit of a sum of products that approximate the volumes on a small scale.

Interviewer: How do you know that [integral] actually gives you a volume?

Student 5: I know that because that's the basic equation for the volume of a cylinder.

Interviewer: Can you tell me more about that? How that equation in particular gives you the volume of a cylinder?

Student 5: Since $2\pi r$ is the area of a circle, you're basically multiplying that by the height of the cylinder. And you're basically doing the approximation-type thing where you're adding up all the disks.

Although her initial formula for the surface area of the cylindrical shells is not correct (she corrects it later with some guidance), she exhibits understanding of the underlying structure of the definite integral. When asked to work the problem using "the other method" (slicing), she was asked again about the formula and was able to respond more clearly.

Interviewer: How do you know that these particular equations [integrals] give you volumes?

Student 5: I do know that is the area of a circle. You just have to take the area of the bigger one, and if you subtract it from the area of the smaller one, then you get the space that's left over for the washer. And then when you integrate it, with respect to y , you're taking and stacking washers and that gets your third dimension.

Student 5 goes on to attempt to work the pyramid problem and has many issues and is not able to make significant progress. Her pictures that she produces in the process of working through the pyramid problem lack the detail and confidence of her pictures produced while solving the solid of revolution problems (see Figures 2 and 3).

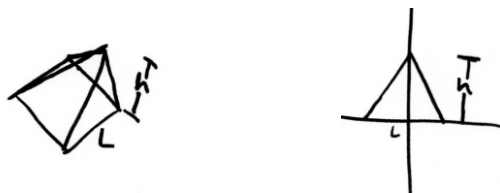


Figure 3: Student 5's drawings for the pyramid problem

The lack of explicit formulas given in the pyramid problem were a major hindrance for Student 5 making meaningful progress. As a side note, Student 5 and Student 6 made it quite clear that they never did any problem like the pyramid problem in their Calculus 2 classes or in their homework.

Student 6, who was the second student to have the pyramid problem, also had interesting, varying results, in almost the opposite direction as those of Student 5. Student 6 was very unsure and weak on his solutions of the traditional solid of revolution problems, and got continually hung up on “the formula”.

Student 6: OK, so shell, 2 pi. Pretty sure it's um, I want to say height times radius but, yeah. I'm not sure about the formula. I know it's 2 pi times the integral, or 2 pi x . It's either height times radius or height times something else.

Student 6: I remember looking at the formula, and just remembering that it was 2 pi, the way I remembered it was the integral of 2 pi x times the height times the radius. And the height and radius and dx I seemed to conceptualize. The x , I'm not really sure, and I just like, just remember that it's there.

Even though Student 6 had difficulties with the revolution volume problems and he did not technically “solve” the pyramid problem, he had more success conceptualizing the pyramid problem than Student 5. Analysis of the pyramid problem responses is ongoing, but one very obvious difference between Student 5 and Student 6 is the quality of their pictures used in solving the pyramid problem (Figure 4).



Figure 4: Student 6's drawings for the pyramid problem

TEACHING IMPLICATIONS AND FUTURE RESEARCH

It is clear from these preliminary results that volume problems can serve as a valuable tool for student understanding of integration. It is our belief that more non-traditional, non-revolution volume problems should be integrated into the Calculus 2 curriculum, which is in line with Sealey's (2014) recommendation that students be exposed to more varied and in-depth integral problems. Volume problems are relatively unique in that they are integral application problems where the physical components can be observed and sketched (as opposed to work integral problems, etc.). This characteristic of volume problems should be exploited and worked with extensively, as visualizing is a valuable skill for students in STEM fields and getting practice with problems like this will be very beneficial.

This research is in the preliminary analysis phase and more interviews on this subject are being planned for 2018; in particular, we will be conducting more interviews with students working through volume problems like the pyramid problem. We would like

to investigate how students use their sketches to build the pieces of the corresponding volume integral and how they interact with the drawing in the problem-solving process. Furthermore, we would like to develop ways in which students can more meaningfully engage in constructing and understanding the product layer of the integral. In the future, we would like to study other aspects of visualization that students use when solving volume problem, such as gesture.

References

- Arcavi, A. (2003). The role of visual representations in the learning of mathematics. *Educational Studies in Mathematics*, 52(3), 215-241.
- Battista, M. (1990). Spatial visualization and gender differences in high school geometry. *Journal for Research in Mathematics Education*, 21(1), 47-60.
- Bezuidenhout, J., & Olivier, A. (2000). Students' conceptions of the integral. In *Proceedings of the 24th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 73-80). Athens, GA: PME.
- Braun, V., & Clarke, V. (2006). Using thematic analysis in psychology. *Qualitative Research in Psychology*, 3(2), 77-101.
- Bremigan, E. (2005). An analysis of diagram modification and construction in students' solutions to applied calculus problems. *Journal for Research in Mathematics Education*, 36(3), 248-277.
- Cui, L., Rebello, N., Fletcher, P., & Bennett, A. (2006). Transfer of learning from college calculus to physics courses. In *Proceedings of the National Association for Research in Science Teaching 2006 Annual Meeting* (pp. 1-7). San Francisco, CA: unpublished.
- Duval, R. (1999). Representation, vision, and visualization: Cognitive functions in mathematical thinking. Basic issues for learning. In *Proceedings of the 21st Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 3-26). Cuernavaca, Morelos, Mexico: PME.
- Huang, C. (2010). Conceptual and procedural abilities of engineering students in integration. In *Proceedings of the Conference of the Joint International IGIP-SEFI Annual Conference*. Trnava, Slovakia.
- Jones, S. (2013). Understanding the integral: Students' symbolic forms. *Journal of Mathematical Behavior*, 32(2), 122-141.
- Jones, S. (2015a). Areas, antiderivatives, and adding up pieces: Definite integrals in pure mathematics and applied science contexts. *Journal of Mathematical Behavior*, 38, 9-28.
- Jones, S. (2015b). The prevalence of area-under-a-curve and anti-derivative conceptions over Riemann sum-based conceptions in students' explanations of definite integrals. *International Journal of Mathematical Education in Science and Technology*, 46(5), 721-736.
- Krutetskii, V.A. (1976). *The psychology of mathematical abilities in school children*. Chicago, IL: University of Chicago Press.

- Lean, G. & Clements, M.A. (1981). Spatial ability, visual imagery, and mathematical performance. *Educational Studies in Mathematics*, 12(3), 267–299.
- Mahir, N. (2009). Conceptual and procedural performance of undergraduate students in integration. *International Journal of Mathematical Education in Science and Technology*, 40(2), 201–211.
- Meredith, D., & Marrongelle, K. (2008). How students use mathematical resources in an electrostatics context. *American Journal of Physics*, 76(6), 570–578.
- Orton, A. (1983). Students' understanding of integration. *Educational Studies in Mathematics*, 14(1), 1–18.
- Rösken, B. and Rolka, K. (2007). Integrating intuition: The role of concept image and concept definition for students' learning of integral calculus. *The Montana Mathematics Enthusiast*, Monograph, 3, 181–204.
- Sealey, V. (2006). Definite integrals, Riemann sums, and area under a curve: What is necessary and sufficient? In *Proceedings of the 28th Annual Meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education*, (Merida, Yucatan, Mexico, Nov. 9–12, 2006), 2, 46–53.
- Sealey, V. (2014). A framework for characterizing student understanding of Riemann sums and definite integrals. *Journal of Mathematical Behavior*, 33, 230–245.
- Sealey, V., & Thompson, J. (2016). Students' interpretation and justification of “backward” definite integrals. In *Proceedings of the 19th Annual Conference on Research in Undergraduate Mathematics* (pp. 1275–1281). Pittsburgh, PA: MAA.
- Stylianou, D., & Silver, E. (2004). The role of visual representations in advanced mathematical problem solving: An examination of expert-novice similarities and differences. *Mathematical Thinking and Learning*, 6(4), 353–387.
- Thompson, P. (1994). Images of rate and operational understanding of the fundamental theorem of calculus. *Educational Studies in Mathematics*, 26(2), 229–274.
- Von Korff, J., & Rebello, N. (2012). Teaching integration with layers and representations: A case study. *Physics Review Special Topics–Physics Education Research*, 8(1), 010125.
- Yeatts, F., and Hundhausen, J. (1992). Calculus and physics: Challenges at the interface. *American Journal of Physics*, 60(8), 716–721.
- Zazkis, R., Dubinsky, E., & Dautermann, J. (1996). Coordinating visual and analytical strategies: A study of students' understanding of the group D_4 . *Journal for Research in Mathematics Education*, 27(4), 435–457.

A COMPARISON OF APPROACHES TO STIMULATED RECALL INTERVIEWS WITH MATHEMATICS TEACHERS IN ORDER TO IDENTIFY SHIFTS IN ATTENTION

Julian Brown

University of Bristol, School of Education

In the course of pilot studies for researching mathematics teacher pedagogy and classroom awareness, I have conducted interviews using different approaches to stimulated recall. In this methodological discussion, which draws on empirical data in order to illustrate methodological findings, I consider how the temporal constraints and activity-related context of the interviews interacts with the qualities of response from the participants in two such interviews. Reflections on two case studies suggest that where a stimulated recall interview follows on directly from the originating event, particular care must be taken to establish a descriptive frame for participant responses.

BACKGROUND CONTEXT

The context for this methodological report is an on-going investigation into shifts in awareness for classroom teachers of mathematics and the links to awareness in learners. In this context, I use “awareness” in two modes. It is used to describe a capacity to direct attention, consciously and deliberately, in response to encounters, with the prospect of recognising when and how students experience shifts in attention indicative of becoming “aware that what used to be attended to was only part of a larger whole” (Mason & Davis, 1988, p.488). It is also used to describe core actions or functions that must be present in order to learn (Mason, 2008), so that an awareness of counting squares covered by a shape might allow attention to be drawn to a definition of area and so on (Wheeler, 1975). Developing the powers to recognise these shifts of attention is problematic, since students may exhibit behaviours of a mathematician without possessing the associated awarenesses (Coles, 2016). Coles goes on to develop the ideas of Gattegno (1971) in order to identify the nature of such shifts, which might be accessible to all learners in a classroom. The teacher who, in turn, notices such shifts may have an awareness in the moment that attention might be drawn to a distinction in the context of classroom activity and offer something that acts as a further trigger for students. For instance, learners beginning to graph quadratics may work with tables of values and begin to attend to a reversal in the pattern of values, allowing the teacher to draw attention to the symmetry of a parabola; this might lead learners to attend to the location of the graph’s turning point, allowing the teacher to draw attention to its relationship with the points of intersection with the x -axis, and so on.

Through various pilot studies, I am seeking to track such triggering and re-triggering as a route to characterising teacher awareness of student awareness and, in turn, to develop a structure through which to explore differences in experiences of the mathematics classroom during a conscious and deliberate change in teaching approach. In this regard, the research project served by these pilot studies might usefully draw on the frame of ‘critical incidents’ and ‘turning points’ developed by Chapman (2017) in her study of mathematics teachers’ perceptions of significant changes in their approach to teaching. One theme drawn out in this study is the potential for student thinking to be a source of teachers learning about teaching. Whilst I am interested in a frame more explicitly attuned to shifts in attention, investigating the significance of these shifts as critical incidents for teachers and learners offers a valuable mechanism for exploring this “multiplicity and context specificity of processes” (Chapman 2017, p. 58). Work with teachers of mathematics in UK secondary schools (covering an age range of 11-18 years) has made use of stimulated recall interviews (SRIs) within differing protocols. The nature of responses in these SRIs has highlighted points of interest which may influence aspects of the on-going research design. These are illustrated through two case studies.

APPROACHES TO INTERVIEWS AND STIMULATED RECALL

As a research method, the interview brings an opportunity for the researcher to engage on a personal level with the participant (Brown & Dowling, 1998). The degree of structure incorporated in the interview design sets the location of control of process and content (Corben & Morse, 2003). The choice of mode for the interview is far from a matter of preference (Bryman, 2016). In an epistemology which admits shared experience, the role of the researcher as observer and interpreter and the essential role of the context in shaping what can be understood must be reflected in the frame established for the interview.

I take here an enactivist position, being concerned with the “dynamic co-emergence of knowing-agent and known-world” (Davis, 1995, p. 8) both in the mathematics classroom and, here, in interviews focusing on engagement with mathematics. Since the intention is to access participants’ reflection on their awareness in a situation already experienced, it is necessary to employ an approach that can follow participants as they offer their accounts. The interviews from which the case studies for this report are taken share an intention to operationalise processes of cognition associated with shifts in attention. In order to elicit accounts of cognitive processes, stimulated recall techniques involve replaying recordings of events to individuals or groups who participated in those events, providing a mechanism for re-entering and recreating the experience of those moments in order to explore new possibilities (Mason, 2002; Brown, 2015).

In his review of stimulated recall approaches in education research, Lyle (2003) concludes that a prominent feature of highly effective applications of the technique is close proximity to the original events in order to avoid a new layer of cognitive processing.

This immediacy is highlighted in some accounts of research designs in this and other fields that make use of stimulated recall (for example Schepens, Aelterman and Van Keer (2007); Dempsey (2010)), although the size of the gap between recording and interview often goes unreported.

CASE STUDIES

Two case studies are offered here. In each, interviews were conducted with teachers of mathematics with more than four years of experience of working with students aged between 11 and 18 years. The researcher was known to the participants from previous professional contact. In the first case, a lesson taught by the participant was observed and video recorded. This video record was transcribed by the researcher and used in the subsequent SRI. In this interview, the teacher was asked to watch several clips from the video and to describe the events. Descriptions were sought in two modes: accounts-of, which aim to capture the narrator's telling of what they have observed without entering into overt interpretation, and accounts-for, which relate interpretations to what was seen, trying out possible meanings and explanations (Watson & Mason, 2007). The focus of the interview was established in advance by the interviewer in the form of the prepared sections of video; control of the process shifted from the interviewer to the participant in the course of the interview.

In the second case study, the participant agreed to be recorded as they worked on their own selections from a prepared set of mathematical tasks, whilst being prompted to verbalise their thought processes, a form of clinical interview (Ginsburg, 1981). Immediately following this, the participant and researcher listened back to a recording, with the participant responding to episodes that were considered of interest in the context of a defined focus.

Certain key characteristics of the two case studies are summarised in Table 1. In each case, the participant was identified by convenience rather than by design. Durations and venues were established according to participant availability. In both cases, participants concluded the interview with unprompted declarations that they had found the process enjoyable and interesting. Each interview was transcribed by the researcher and utterances classified thematically according to the quality of response. Of particular relevance for the following discussion are distinctions between “accounts-of”, described above, “self-reporting”, where the participant reports information about their recall of their experience in the moment which goes beyond observable behaviours, and “self-analysis”, where the participant offers an interpretation that relates specifically to themselves of the observable behaviours or self-reporting comments.

Characteristic	Case Study 1	Case Study 2
Nature of originating event	90-minute classroom lesson	20 minutes working on mathematical tasks
Timing of interview	Two weeks after the video recording (dependent on availability of participant)	Within the same research encounter as the originating event
Duration of interview	60 minutes	65 minutes
Data collection	Audio recording followed by transcription	Audio recording followed by transcription
Participant's familiarity with techniques	Expert practitioner. Limited experience of reflection on video records of (own) lessons	Expert mathematician and communicator

Table 1: Some key characteristics of the case studies used.

OBSERVATIONS AND DISCUSSION

Before collecting the data, the substantial gap between data collection and SRI in the first case study was a concern. Indeed, this was illustrated by the participant's response to an initial question inviting general reflection. (In this and subsequent extracts from transcripts, the interviewer is represented by Interviewer and the participant by Participant *n* in Case Study *n*).

Interviewer: Do you recall any key moments in the lesson, when either you recognised students had made a significant step or when you recognised you made a significant decision about what was going to happen?

Participant 1: I don't recall, to be honest. Hopefully, seeing the video will help me resolve that!

In the second case study, the immediacy of the SRI led to a quite different response:

Interviewer: What's stuck in your mind about doing those two problems?

Participant 2: I found it a lot more difficult than I thought I would to say my thought process. I feel like I wanted an indeterminate amount of time just to sit and think. I think I felt, actually, quite a bit of pressure, not pressure so much, but I was quite keen to have a very systematic approach. And I was quite keen for that to be, thinking, it doesn't matter if I get it right or wrong, I just want to be nice and systematic. And then it was remarkable, both problems, that certainly wasn't something that I came in here thinking that's what I wanted to do but as soon as I sat down and I was faced with a question, immediately my approach was, well I just want to be systematic about it.

Participant 2 demonstrates access to an affective response to the mathematical experience that is entirely absent in Participant 1 before reviewing the recording, setting the scene quite differently for the remainder of the SRI. The sense of an emotional response to working on the mathematics during the clinical interview is still resonating for Participant 2 as the SRI begins and subsequent accounts continue to show a strong self-reporting element. It may be that this sense of emotional response would have returned if the SRI had taken place at a distance from the event; it is also possible that this first offering set a pattern that was absent for Participant 1, talking at some distance from the experience itself. Participant 1 responds to a request to focus on giving a descriptive account before entering into evaluative or interpretative comments with an observer's perspective:

Participant 1: OK, so I started off with the example on the board. It was grouped data, so I said "What do we use when have grouped data, because we can't just choose this number here. So [name], what would we use instead?" [Name] didn't know, so I asked "Does anyone else know?" and somebody said "Midpoint"...

The perspective for Participant 2 when commenting on the first section of the recording remained located in the activity itself, suggestive of a persistent experience rather than a re-lived experience:

Participant 2: I suppose I'm not sure if there's anything to note from just selecting the questions themselves. I'm not sure, even though I know you did not give me the instruction at all to do this, just from my point of view I knew what my method would be for doing that [mathematical problem] because I've done questions similar to them pretty recently, so I thought that just wouldn't be a very interesting way of thinking about my thought process because there's something I've definitely thought of before.

This characteristic remained present for Participant 2 throughout the responses to the recording, with repeated sequences of self-reporting and self-analysis offered in response to interventions intended to prompt descriptions:

Interviewer: What led up to that shift for you? That moment when you're thinking, actually I want to start this again in a different way. What was happening before I asked the question?

Participant 2: I think that was happening before you asked the question, because I think the reason why I couldn't get out what I was trying to say was because in my head, I was going, this is muddled and this is not the right way of doing this.

The response here is certainly not unreasonable, given the wording of the question. The persistent quality of autonomous insight, rather than stimulated recall, might in this case be seen as an artefact of the proximity of the experience to the telling about the experience. It also highlights the need for the interviewer to make explicit the quality of accounting that is being sought. In their account of a three-step design combining observation, stimulated recall and interview, Busse and Ferri (2003) make the inter-

esting observation that the nature of the data obtained from SRIs adheres more closely to the intended initial “account-of” quality as both interviewer *and* participant develop proficiency. Clearly, at early stages or in the case of a once-only participation, the responsibility for establishing and guiding the quality of responses lies with the interviewer. As a researcher, I experienced this as significantly more problematic when moving directly from the events to the SRI, particularly when these events were in a one-to-one context and the experiences were still being directly inhabited.

A second key difference associated with the proximity of event and SRI was in the identification of episodes of interest. In the first case study, the gap was used to transcribe the recording and to make selections which were presented to the participant. It would not, clearly, have been possible to review the 90-minute lesson in its entirety. The episodes selected originated in the 12th, 18th and 23rd minute of the recording and were unlikely to have been accessed without this intervention. In the second case study, each of the participant and interviewer chose, at different moments, to interrupt the recording in order to comment about and on points of interest. It is of note that Participant 1 asked for a recorded episode to be replayed in order to clarify the detail of what had been observed, something not present in the SRI with Participant 2. In fact, Participant 2 anticipated what was about to appear in the recording on more than one occasion:

Participant 2: I think [we’re] about to get onto another thing which I do, we’re about to get onto the next bit, [where] I think I started going along this question and I actually realised that I was getting in a bit of a muddle with it.

Participant 2: I think I ended up just stumbling, stumbling, stumbling, stumbling and then going, “Oh, well, no, here’s something else I’m going to do.” But yeah, I don’t know whether that’s the next bit.

A third area difference concerns the nature of the originating event. After working on mathematical tasks (case study 2), the SRI elicited responses with a much greater prevalence of self-reporting and self-analysis statements than when the participant was reflecting on an activity that was already part of their professional activity (case study 1). In combination with the immediacy of the SRI, there is a sense that Participant 2 was still taken by the mathematical activity itself, dwelling in the detail of the tasks and invested in discussing their strategies.

Reflection on these case studies suggests a tension between maintaining proximity to the event and eliciting responses at the intended level of reflection. This proximity appears to have had an impact not only on the quality of the participant’s responses but also on the ease and clarity with which the parameters of the SRI were established by the interviewer. The resolution, to be explored further, would appear to be a function of both the time separation of event and SRI and the guidance given and reinforced by the interviewer until such time as the participant has developed an understanding and proficiency in the required mode of reflection. The impact of filtering the original recording prior to the SRI might be mitigated in various ways, such as approaching a longer recording in a sequence of SRIs. If the participant were willing, this would have

the added benefit of establishing proficiency with the intended SRI methodology but would introduce further distance from the original event.

The widespread use of SRIs is suggestive of their value to educational researchers. There is, however, notable variation in their implementation. The experience of these two case studies indicates that close attention is warranted when matching the proximity of originating event and SRI, the nature of guidance given and the free or directed attention to episodes in the recording to a specific research design.

References

- Brown, A. & Dowling, P. (1998). *Doing Research / Reading Research: A Mode of Interrogation for Education*, London, UK: Falmer Press.
- Brown, L. (2015). Researching as an enactivist mathematics education researcher. *ZDM Mathematics Education*, 47(2), 185-196.
- Bryman, A. (2016). *Social Research Methods (Fifth Edition)*, Oxford, UK: Oxford University Press.
- Busse, A., & Ferri, R. B. (2003). Methodological reflections on a three-step-design combining observation, stimulated recall and interview. *ZDM Mathematics Education*, 35(6), 257-264.
- Chapman, O. (2017). Mathematics teachers' perspectives of turning points in their teaching. In B. Kaur, W. K. Ho, T. L. Toh & B. H. Choy (Eds.), *Proc. 41st Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 1, pp. 45-60). Singapore: PME.
- Coles, A. (2016). *Engaging in Mathematics in the Classroom: Symbols and experiences*. Abingdon: Routledge.
- Corbin, J. & Morse, J. (2003). The unstructured interactive interview: Issues of reciprocity and risks when dealing with sensitive topics. *Qualitative Inquiry*, 9(3), 335-354.
- Davis, B. (1995). Why teach mathematics? Mathematics education and enactivist theory. *For the Learning of Mathematics*, 15(2), 2-9.
- Dempsey, N. P. (2010). Stimulated recall interviews in ethnography. *Qualitative Sociology*, 33(3), 349-367.
- Gattegno, C. (1971). *What we owe children. The Subordination of Teaching to Learning*. London: Routledge and Kegan Paul Ltd.
- Ginsburg, H. (1981). The clinical interview in psychological research on mathematical thinking: Aims, rationales, techniques. *For the Learning of Mathematics*, 1(3), 4-11.
- Lyle, J. (2003). Stimulated recall: A report on its use in naturalistic research. *British Educational Research Journal*, 29(6), 861-878.
- Mason, J. (2002). *Researching your own practice: The discipline of noticing*. London: Routledge.

- Mason, J. (2008). Being mathematical with and in front of learners. In B. Jaworski and T. L. Wood (Eds.), *The mathematics teacher educator as a developing professional*, (pp.31-55). Rotterdam: Sense Publishers.
- Mason, J. & Davis, J. (1988). Cognitive and Metacognitive Shifts. In A. Barbas (Ed.), *Proceedings of PME-XII* (Vol. 2, pp.487-494). Vezprem: Hungary.
- Schepens, A., Aelterman, A., & Van Keer, H. (2007). Studying learning processes of student teachers with stimulated recall interviews through changes in interactive cognitions. *Teaching and Teacher Education*, 23(4), 457-472.
- Watson, A. & Mason, J. (2007). Taken-as-shared: a review of common assumptions about mathematical tasks in teacher education. *Journal of Mathematics Teacher Education*, 10(4), 205-215.
- Wheeler, D. (1975). Humanising mathematical education. *Mathematics Teaching*, 71, 4-9.

DECISION-MAKING IN NOTICING STUDENTS' PROPORTIONAL REASONING

Àngela Buform, Ceneida Fernández, and Salvador Llinares

Universidad de Alicante (Spain)

Research has shown that pre-service teachers and teachers have difficulties in proposing instructional decisions to foster students' understanding. In this research, we analyse the relationship between how pre-service primary school teachers identify the mathematical elements involved in the problem, how they recognise characteristics of students' understanding and the decisions they make according to students' understanding, in the specific domain of proportional reasoning. Results indicate that pre-service teachers who had identified the mathematical elements involved in a problem were more able to provide activities based on students' understanding.

THEORETICAL BACKGROUND

Noticing is a focus of interest for mathematics teacher educators. This competence allows teachers and pre-service teachers to identify relevant aspects of teaching and learning situations and interpret them to make instructional decisions (Mason, 2002; Sherin, Jacobs, & Philipp, 2010). A particular focus is noticing students' mathematical thinking. Jacobs, Lamb, and Philipp (2010) characterise this competence as a set of three interrelated skills: (i) attending to students' strategies that implies identifying important mathematical details in students' answers; (ii) interpreting students' mathematical reasoning taking into account the mathematical details previously identified; and (iii) deciding how to respond on the basis of students' reasoning.

Stahnke, Schueler and Roesken-Winter (2016), in their review of noticing research, conclude that studies considering decision-making skill show that it is the most challenging to develop in teacher education programs. Pre-service teachers and teachers showed deficits in terms of proposing instructional decisions to foster students' understanding that go beyond re-teaching (Cooper, 2009) or presenting students how to do it right (Son, 2013). As Choy (2013) pointed "the specificity of what teachers notice while necessary, is not sufficient for improved practices" (p. 187). Teachers can be very specific about what they notice without having a teaching decision in mind. Therefore, the relationship between how pre-service teachers interpret students' mathematical reasoning and decide how to respond on the basis of students' mathematical reasoning deserves further research.

Based on the review of Stanhke et al. (2016) "factors hypothesized to influence teachers' decisions were ranging from teachers' knowledge, beliefs to goals" (p. 23), we hypothesise that the identification of the mathematical elements involved in a problem (mathematical content knowledge) can play a significant role not only in in-

interpreting students' mathematical reasoning but also in deciding how to respond on the basis of students' reasoning.

Our study focuses on analysing the relationship between how pre-service teachers identify the mathematical elements involved in a problem, how they recognise characteristics of students' understanding and the decisions made by them according to students' reasoning, in the specific domain of proportional reasoning.

METHOD

Participants and task

Participants were 83 pre-service primary teachers (PTs) from the University of Alicante (Spain) enrolled in the third year of a degree to become a primary school teacher. In previous years, pre-service teachers had attended two courses focused on numerical and geometrical sense. In the third year, they were attending a mathematics methods course related to the teaching and learning of mathematics in primary school. One of the units of this course was about the teaching and learning of the fraction concept and proportional reasoning. The aim of this unit is pre-service teachers' development of noticing students' fractional and proportional reasoning. Data of this study were collected after this unit.

Lamon (2007) claimed that proportional reasoning is multifaceted and different types of thinking processes are needed to develop it. In our study, we consider three domains involved in the development of proportional reasoning: fraction's interpretations, ratio comparison situations, and discrimination between proportional and non-proportional situations (based on Lamon (2007) and Pitta-Pantazi, and Christou (2011)). Fraction's interpretations involve six sub-constructs: part-whole, measure, measurement, quotient, operator and reasoning up and down. Ratio comparison situations include the ideas of: covariance, ratio as a comparative index, unitizing process and relative thinking. Finally, discrimination between proportional and non-proportional situations includes missing-value proportional and non-proportional problems in order to discriminate between both situations. These sub-constructs are defined in Buform, Fernández, Llinares, and Badillo (2017).

The task consists of 12 primary school problems related to fraction's interpretations (6 problems), ratio comparison situations (4 problems) and discrimination between proportional and non-proportional situations (2 problems). Pre-service teachers had to interpret three students' answers to each problem with different characteristics of understanding, answering the four questions presented in Table 1.

Figure 1 shows the problem and the three students' answers to the *ratio as a comparative index* problem. In this problem, students had to compare three ratios and look for the one closer to 1 to know which loft is the squarest. The three primary students' answers show different characteristics of students' understanding: in the first answer, the student identifies the ratios between the sides and interprets that the loft whose ratio is closer to 1 will be the squarest; in the second answer, the student identifies the ratios

between the sides but provides a justification based on an additive relationship (“*The difference between 4.55 and 5.08 is the smallest*”); in the third answer, the student uses an additive strategy doing the subtractions between the lofts’ sides and choosing the loft whose difference is closer to 0.

Questions	Aim
a) What mathematical concepts must a primary school student know to solve this problem?	Identifying the mathematical elements of the problem
b) What are the characteristics of students’ mathematical understanding involved in each answer?	Recognising characteristics of students’ understanding
c) If a student does not understand the mathematical concepts, how would you change the problem to help the student understand these concepts?	Making-decisions based on students’ mathematical understanding in order to support their conceptual progression
d) If a student understands the mathematical concepts, how would you change the problem to help him progress in his understanding of these concepts?	

Table 1: Questions for pre-service teachers

Task 11. Ratio as a comparative index

A company is selling rectangular lofts in three different sizes in a new building:

a) 7.5m by 11.4m b) 4.55m by 5.08m c) 18.5m by 24.5m

Which loft is the squarest?

Answer 1

$$\frac{7.5}{11.4} = 0.65$$

$$\frac{4.55}{5.08} = 0.89 \rightarrow \text{Es el más cuadrado ya que es el número más cercano a 1.}$$

$$\frac{18.5}{24.5} = 0.75$$

Answer 2

$$\frac{7.5}{11.4} = 0.658 \quad \frac{18.5}{24.5} = 0.755$$

$$\frac{4.55}{5.08} = 0.896$$

En proporción 4.55 por 5.08
existe menor diferencia por lo que
para más cuadrado al tener lados más iguales.

Answer 3

* El cuadrado se caracteriza por tener los lados de igual medida, se parece más al cuadrado el que tengan menor diferencia de medidas, en decir:

$$\begin{array}{r} 11.4 \\ - 7.5 \\ \hline 03.9 \end{array} \quad \begin{array}{r} 5.08 \\ - 4.55 \\ \hline 0.53 \end{array} \quad \begin{array}{r} 24.5 \\ - 18.5 \\ \hline 06.0 \end{array}$$

* Es más cuadrado el segundo, porque sus lados son más similares en medida.

This is the squarest loft because the result is the closest to 1

The difference between 4.55 and 5.08 is the smallest so, this loft will be the squarest since the sides are more equal

All the sides of a square have the same size. Therefore, the squarest loft is the loft with the smallest difference. The second loft is the squarest.

Figure 1: Students’ answers to the ratio as a comparative index problem

Analysis

Data are pre-service teachers’ answers to the four questions (Table 1). Three researchers analysed pre-service teachers’ answers to each question, individually, identifying categories. Then, agreements and disagreements of the categories identified were discussed until we reached an agreement of the final categories. Respect to the first

question, we were interested in whether pre-service teachers identified the mathematical elements of the problem. Two categories emerged: pre-service teachers who identified the mathematical elements involved in the problem (it was coded with a 1), and pre-service teachers who did not identify them (it was coded with a 0). With regard to the second question, we focused on how pre-service teachers recognised characteristics of students' understanding using the mathematical elements identified to describe students' answers. Two categories emerged: pre-service teachers who recognised characteristics of students' understanding describing students' answers with the mathematical elements identified (it was coded with a 1) and pre-service teachers who provided general comments based on the correctness of the answer (it was coded with a 0). We carried out a Cluster Analysis using the SPSS. From this analysis, four profiles of pre-service teachers were identified that differ in how they had identified the mathematical elements involved in each problem and how they had used these mathematical elements to recognise characteristics of students' understanding. Characteristics of each profile are presented in the results section.

Regarding to questions (c) and (d), four categories emerged: activities focused on the mathematical element (these activities support students' conceptual progression); activities focused on characteristics of the problem such as the type of numbers, the context or the type of representation used; general teaching actions such as re-explain the content, and nonsense or blank answers. Table 2 shows examples of decisions provided by pre-service teachers in each category.

Categories	Examples of decisions provided by pre-service teachers
Activities focused the mathematical element	<i>"I would ask for a new loft squarer than the other three."</i>
General teaching actions	<i>"I would explain what ratio means and that the idea of the ratio between the sides of a square is 1."</i>
Activities focused on characteristics of the problem	<i>"I would use whole numbers instead of decimal numbers."</i>

Table 2: Categories and examples of decisions provided by pre-service teachers

RESULTS

Table 3 shows the characteristics of the profiles obtained in the Cluster Analysis (Buform et al., 2017). Only 70 out of the 83 pre-service teachers were grouped in the four profiles inferred from the Cluster Analysis. A characteristic of these profiles is that identifying the mathematical elements involved in the problem and recognising characteristics of students' understanding depend on the three domains involved in the development of proportional reasoning. These domains are progressively incorporated in the different profiles from P0 to P3. In fact, the mathematical elements of *fractional scheme* and the characteristics of students' understanding (except *reasoning up and*

down sub-construct) were identified and recognised easier by pre-service teachers than those related to the *discrimination between proportional and non-proportional situations* and, the identification of the mathematical elements and the recognition of characteristics of students' understanding in *ratio comparison situations* was a difficult task for them. Furthermore, identifying the mathematical elements of the problem makes pre-service teachers focused their attention on recognising characteristics of students' understanding. This claim is supported by the fact that pre-service teachers who did not identify the mathematical elements of the problem, could not recognise characteristics of students' understanding, providing general comments based on the correctness of answers. Therefore, the identification of the mathematical element(s) of the problem is required to be able to recognise characteristics of students' understanding.

Profile	Characteristics of each profile
P0 (20 PTs)	Pre-service teachers who do not identify the mathematical elements and do not recognise characteristics of students' reasoning
P1 (16 PTs)	Pre-service teachers who identify only the mathematical elements of the fractional scheme (except <i>reasoning up & down</i>) and start to recognise some characteristics of students' reasoning related to these problems
P2 (18 PTs)	Pre-service teachers who identify the mathematical elements of fractional scheme and the discrimination between proportional and non-proportional situations and recognise characteristics of students' reasoning related to these problems
P3 (16 PTs)	Pre-service teachers who identify the mathematical elements of fractional scheme, discrimination between proportional and non-proportional situations and ratios meaning in comparison situations and recognised characteristics of students' reasoning related to these problems

Table 3: Profiles of pre-service teachers

Regarding to the decisions provided by pre-service teachers, Table 4 shows the frequencies according to the three domains and the four profiles obtained (pre-service teachers can make more than one decision). We can underline three main results. Firstly, the difficulty of pre-service teachers in providing decisions focused on students' understanding. It is evidenced by the number of nonsense and blank answers along the profiles and the different domains. Although pre-service teachers had recognised characteristics of students' understanding, some of them did not provide activities that help students' progress in their reasoning (activities focused on the mathematical element). Secondly, data show a tendency: there is an increase in the number of activities focused on the mathematical element from P0 to P3. Therefore, when pre-service teachers had identified the mathematical element(s) involved in the problem, they were more able to provide activities focused on the mathematical ele-

ment. In the domain of *fraction's interpretation*, the number of pre-service teachers of P0 (pre-service teachers who did not identify and recognise characteristics of students' understanding in this domain) who proposed activities focused on the mathematical element in question c) is smaller than in P1, P2 and P3 (pre-service teachers of these profiles identified and recognised characteristics of students' understanding in this domain).

Domains		Fraction interpret.		prop. & non-prop		Ratio compar.		Total	
Decisions		c)	d)	c)	d)	c)	d)	c)	d)
P0 (20 PTs)	Focused on the element	21	27	1	15	8	3	30	45
	General teaching actions	16	0	0	0	4	0	20	0
	Focused on the characteristics of the problem	24	9	8	6	41	23	79	38
	Nonsense / blank answers	62	84	26	19	36	54	124	157
P1 (16 PTs)	Focused on the element	38	30	7	10	10	6	55	46
	General teaching actions	8	0	1	0	7	0	16	0
	Focused on the characteristics of the problem	22	12	13	6	35	20	70	38
	Nonsense / blank answers	38	61	15	17	24	42	77	120
P2 (18 PTs)	Focused on the element	40	37	10	22	16	5	66	64
	General teaching actions	11	0	4	0	3	0	18	0
	Focused on the characteristics of the problem	29	13	15	5	40	24	84	42
	Nonsense / blank answers	39	65	12	11	28	48	79	124
P3 (16 PTs)	Focused on the element	48	39	12	19	17	13	77	71
	General teaching actions	7	0	7	0	8	0	22	0
	Focused on the characteristics of the problem	26	24	10	7	39	30	75	61
	Nonsense / blank answers	23	38	5	7	8	22	36	67

Table 4: Decisions provided by pre-service teachers according to the three domains and the profiles obtained

In the domain of *discrimination between proportional and non-proportional situations*, the number of pre-service teachers who proposed activities focused on the mathematical element in P0 and P1 (pre-service teachers who did not identify and recognise characteristics of students' understanding in this domain) is smaller than in P2 and P3 (in these profiles, pre-service teachers identified and recognised characteristics of students' understanding in this domain). Finally, in the domain of *ratio comparison situations*, the number of pre-service teachers who proposed activities focused on the mathematical elements in P0, P1 and P2 (pre-service teachers who did not identify and recognise characteristics of students' understanding in this domain) is smaller than in P3 (pre-service teachers who identified and recognised characteristics of students' understanding in this domain). The same tendency can be observed in

question d) in each of the domains. Finally, results indicate that providing activities to help students progress in their understanding when they have understood the mathematical element of the problem (question d) is more difficult than providing activities to help students who have not understood the mathematical element of the problem (question c). This is observed because pre-service teachers provided more nonsense answers and less activities focused on the mathematical element when they have to help students who had understood the mathematical element (question d).

DISCUSSION AND CONCLUSIONS

Our results underline three aspects related to the skill of decision-making and its relationship with how pre-service teachers identify the mathematical elements involved in the problem and how they recognise characteristics of students' understanding. First of all, pre-service teachers had difficulties in providing activities focused on the mathematical element of the problem. This is evidenced by the number of nonsense or blank answers, and general teaching actions provided. This finding is in line with previous research showing that decision-making seems to focus on re-teaching (Cooper, 2009) or teaching students how to do it right (Son, 2013).

Secondly, our results show that when pre-service teachers identified the mathematical elements of the problem and recognise characteristics of students' understanding in these problems, they were more able to provide activities to support students' conceptual progression (i.e., activities focused on the mathematical element). This is evidenced by the increase of the number of activities focused on the mathematical element in accordance with the incorporation of the identification of the mathematical elements and the recognition of students' understanding of the different proportional reasoning domains in the profiles and with the decrease of the nonsense and blank answers. Therefore, these results indicate that the lack of ability to identify the mathematical elements in the problem fed the lack of ability to respond to students adequately (Son, 2013). However, the fact that some pre-service teachers were able to identify the mathematical elements and recognise characteristics of students' reasoning but proposed decisions that were not focused on the mathematical elements supports the claim that there are other factors that influence teachers' decisions such as beliefs or goals (Stanhke et al., 2016).

Finally, pre-service teachers had more difficulty in providing activities to consolidate students' understanding than to help students' progress in their understanding. In other words, it is more difficult to provide activities to support students who understand the mathematical element than activities to help students who do not understand the mathematical element in the problem. This result is in line with previous research showing that it is easier to propose instructional decisions for students who have difficulty solving problems than for those who do not have difficulties (Callejo, Pérez, Moreno, Sánchez-Matamoros, & Valls, 2017).

Our results seem to indicate that pre-service teachers who had identified the mathematical elements involved in the problem were more able to provide activities based on

students' understanding. Should this skill (identifying the mathematical elements involved in a problem) be part of the conceptualisation of noticing students' mathematical thinking?

Acknowledgments

This research was supported in part by the projects from the Ministry of Science and Innovation (MINECO, Spain) EDU2014-54526-R and EDU2017-87411-R and in part by the Grant for Pre-doctoral Contracts for Doctors' Training BES-2015-074424.

References

- Buform, A., Fernández, C., Llinares, S., & Badillo, E. (2017). Pre-service primary teachers' profiles of noticing students' proportional reasoning. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), *Proceedings of PME41* (vol. 2, pp. 193-200). Singapore: PME.
- Callejo, M. L., Pérez, P., Moreno, M., Sánchez-Matamoros, G., & Valls, J. (2017). A learning trajectory for length as a magnitude and its measurement: usage by prospective preschool teachers. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), (Eds.). *Proceedings of PME41* (vol. 2, pp. 201-208). Singapore: PME.
- Choy, B. H. (2013). Productive mathematical noticing: What it is and why it matters. In V. Steinle, L. Ball, & C. Bordini (Eds.), *Proc. 36th Annual Conference of Mathematics Education Research Group of Australasia* (pp. 186-193). Melbourne, Victoria: MERGA.
- Cooper, S. (2009). Preservice teachers' analysis of children's work to make instructional decisions. *School Science and Mathematics*, 109(6), 355–362.
- Jacobs, V. R., Lamb, L. C., & Philipp, R. (2010). Professional noticing of children's mathematical thinking. *Journal for Research in Mathematics Education*, 41(2), 169- 202.
- Lamon, S. J. (2007). Rational numbers and proportional reasoning: toward a theoretical framework. In F.K. Lester Jr. (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 629-668). NCTM-Information Age Publishing, Charlotte, NC.
- Mason, J. (2002). *Researching your own practice. The discipline of noticing*. London: Routledge Falmer.
- Pitta-Pantazi, D., & Christou, C. (2011). The structure of prospective kindergarten teachers' proportional reasoning. *Journal of Mathematics Teacher Education*, 14(2), 149–169.
- Sherin, M.G., Jacobs, V.R., & Philipp, R.A. (2010). *Mathematics teacher noticing: Seeing through teachers' eyes*. New York, NY: Routledge
- Son, J. W. (2013). How preservice teachers interpret and respond to student errors: ratio and proportion in similar rectangles. *Educational Studies in Mathematics*, 84(1), 49-70.
- Stahnke, R., Schueler, S., & Roesken-Winter, B. (2016). Teachers' perception, interpretation, and decision-making: a systematic review of empirical mathematics education research. *ZDM. Mathematics Education*, 48, 1-27.

CLUMPS OR CHUNKS? - CONTEXTUAL RELEVANCE OF STUDENTS' FEATURES OF THE DATA

Christian Büscher

TU Dortmund University, Germany

For reasoning on data, learners make use of features of the data in an intuitive and informal way. More insights however are needed into learners' processes of reasoning on data to identify conditions and reasons for learners to focus on particular features of the data. This study reports on results of a design research project on German 7th grade students' reasoning on data. The analysis shows how students' focus on features of the data follows their perceived contextual relevance induced by the context of a teaching-learning arrangement.

STUDENTS' FOCUS ON FEATURES OF THE DATA

The development of adequate reasoning on data is one of the main goals of statistics education and a central interest in statistics education research (Biehler, Frischemeier, Reading, & Shaughnessy, 2018). For doing so, learners need to be able to adopt an 'aggregate view' on data, perceiving data distributions not as unstructured collections of individual cases, but as holistic entities on their own with their own emergent properties like centre and spread (Konold, Higgins, Russell, & Khalil, 2015). Developing such an aggregate view on data however seems to be challenging for learners (Bakker & Gravemeijer, 2004).

In order to find approaches to develop learners' reasoning on data, statistics education research has identified a number of 'intuitive' or 'informal' *features of the data* that students seem to focus on. In a study by Konold et al. (2002), 7th and 9th grade students use *modal clumps* to summarize data of daily roadkill: small central ranges surrounding the mode of the data to represent the amount of 'typical' roadkill. Makar and Confrey (2003) find that preservice teachers use modal clumps to summarize data of student achievement and that they partition these data into *chunks* to represent groups of low, middle, or high achievement. In a newer study, Schnell and Büscher (2015) find that 8th grade students who compare daily temperature data also partition the data into chunks, but do so in a creative way according to their individual concepts instead of following a simple 'low-middle-high' partition.

Statistics education research commonly calls for building on learners' informal use of features of the data to develop their reasoning on data (e.g. Konold et al., 2002). However, although research has shown that learners commonly focus on features of the data such as clumps or chunks, little is known about whether such focus is 'natural' for students, or if it is influenced by other factors. More insight is needed into the

conditions and reasons for learners to focus on specific features of the data. This paper presents a contribution to close this gap of research.

THE SITUATIVE NATURE OF STUDENTS' ACTIONS

To provide a framework that can contribute to explaining students' use of specific features of the data, this study draws on the epistemological Theory of Conceptual Fields (Vergnaud, 1996). According to this theory, learners' actions follow possibly unconscious organizational invariants called *concepts-in-action* and *theorems-in-action*. Concepts-in-action are "categories (objects, properties, relationships, transformations, processes, etc.) that enable the subject to cut the real world into distinct elements and aspects [...] according to the situation and scheme involved" (ibid., p. 225), whereas theorems-in-action are "proposition[s] that [are] held to be true by the individual subject for a certain range of situation variables" (ibid., p. 225). Learning consists of expanding and connecting one's concepts- and theorems-in-action into increasingly complex *conceptual fields*.

Identifying relevant features of the data can provide an example action by learners in which they draw on concepts- and theorems-in-action. When investigating data, the learners from the study of Konold et al. (2002) use modal clumps to represent typical roadkill. Using the Theory of Conceptual Fields, this can be interpreted as drawing on the concept-in-action (indicated by $||...||$) of $||\textit{modal clumps}||$ to cut the data into relevant and irrelevant parts. Their theorem-in-action (indicated by $\langle... \rangle$) $\langle \textit{modal clumps represent the typical roadkill} \rangle$ then describes their use of the feature of $||\textit{modal clumps}||$. Thus, students' concepts- and theorems-in-action influence the features of the data held relevant by them.

Central to both, concepts- and theorems-in-action, is their situative nature. Concepts-in-action depend on the "situation and scheme involved", and theorems-in-action hold true for "a certain range of situation variables" (Vergnaud, 1996, p. 225). Thus, the specific situation at hand plays an important role in determining which features of the data are focused on by learners. Regarding the learning of statistics, such a situation can be introduced through the context of a statistics teaching-learning arrangement.

RESEARCH QUESTION

In order to develop learners' reasoning on data, instruction should make use of their intuitive use of features of the data, such as their use of modal clumps or chunks of data. However, more insights into the conditions and reasons for learners' focus on features of the data are needed in order to specifically support students' learning processes. The learning-theoretical background suggests a strong influence of the context of a teaching-learning arrangement for students' use of features of the data. Thus, this study concerns the following research question: *how does the context of a teaching-learning arrangement influence students' focus on features of the data?*

RESEARCH DESIGN

Design Research provides a research methodology suitable to evaluate the effects of the context of a teaching-learning arrangement. This section outlines the methodological considerations of this study.

Topic-specific Didactical Design Research as framework

This study is part of a larger research project in the framework of Topic-specific Didactical Design Research (Prediger & Zwetzscher, 2013; for the whole project see Büscher, 2018). Research conducted in this framework consists of iterative cycles of four interrelated working areas: (1) specifying and structuring learning goals and content; (2) developing the design; (3) conducting and analysing design experiments; and (4) developing local theories on teaching and learning processes. The methodological heart of Design Research consists of conducting design experiments (Gravemeijer & Cobb, 2006). Influenced by the design of a teaching-learning arrangement and controlled by a design experiment leader, design experiments do not simply aim at observing, but at actively initiating learning processes in order to investigate the effects of the design and to understand students' reasoning.

Participants, data collection, and data analysis

The larger research project consisted of five cycles of design experiments from 2014 to 2015, in total 34 participants in 32 design experiments. This study reports on results of the fifth cycle of design experiments in December 2015, consisting of 3x3 focus design experiments with 3x2 participants. Participants were assigned into pairs, and each pair took part in a design experiment series consisting of three consecutive design experiments few days apart. In each design experiment, one pair of 7th grade students from a German secondary school worked on a teaching-learning arrangement. The students volunteered and were sampled from one class by the mathematics teacher based on high communicative ability, but not based on high mathematics achievement.

The focus design experiments were videotaped and fully transcribed, resulting in 405 minutes of transcribed video data. The data corpus was reduced by partitioning the data into episodes corresponding to the phases of the design experiment manual (see below) and by choosing episodes that allowed insights into the students' use of features of the data. For these episodes, the students' concepts- and theorems-in-action were reconstructed in an interpretative step of basic analysis adopting Vergnaud's (1996) constructs and then following open coding and data-led category development (cf. Corbin & Strauss, 1990).

Design of a teaching-learning arrangement

Each design experiment of the series, introduces one problem to the students. Due to space restrictions, only the first two problems are sketched here, focusing on elements concerning features of the data (for a more thorough description see Büscher, 2018).

In the first design experiment, the students work on the *Antarctic Temperatures Problem*. In this problem, the students take the roles of advisors to researchers in Antarc-

tica. The students are given temperature data from a research station in Antarctica (Figure 1). The data concern the daily temperatures for the three months of July 2013, 2014, and 2015 and asked to give a prediction for ten days of July 2016. This task was chosen as to allow insights into which features of the data the students intuitively focus on when giving such predictions.

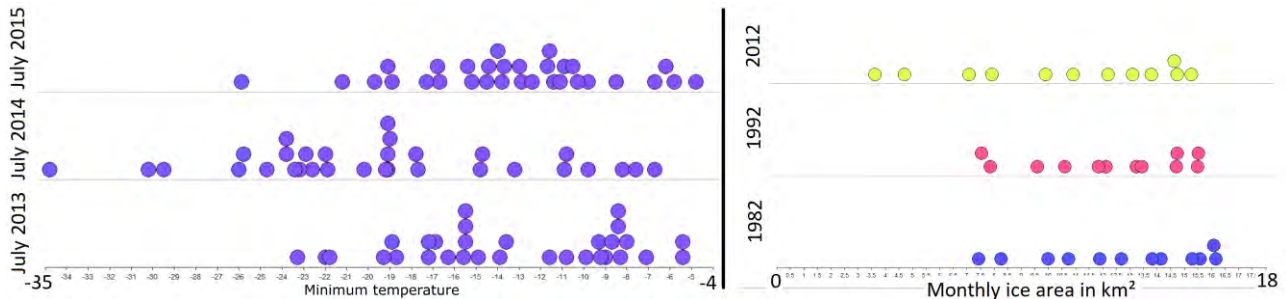


Figure 1: Data for the Antarctic Temperatures Problem (left side) and Arctic Sea Ice Problem (right side) (Translated from German)

After developing on ideas and measures, the students are given so-called filled-in *report sheets* (Figure 2). These report sheets are introduced to the students as short summaries produced by other students. The ‘sketch’-field of each filled-in report sheet illustrates a different feature of the data: the ‘Typical Report Sheet’ focuses on a chunk in the centre, the ‘Value Report Sheet’ on a modal clump, and the ‘MinMax Report Sheet’ on extreme values and overall spread.

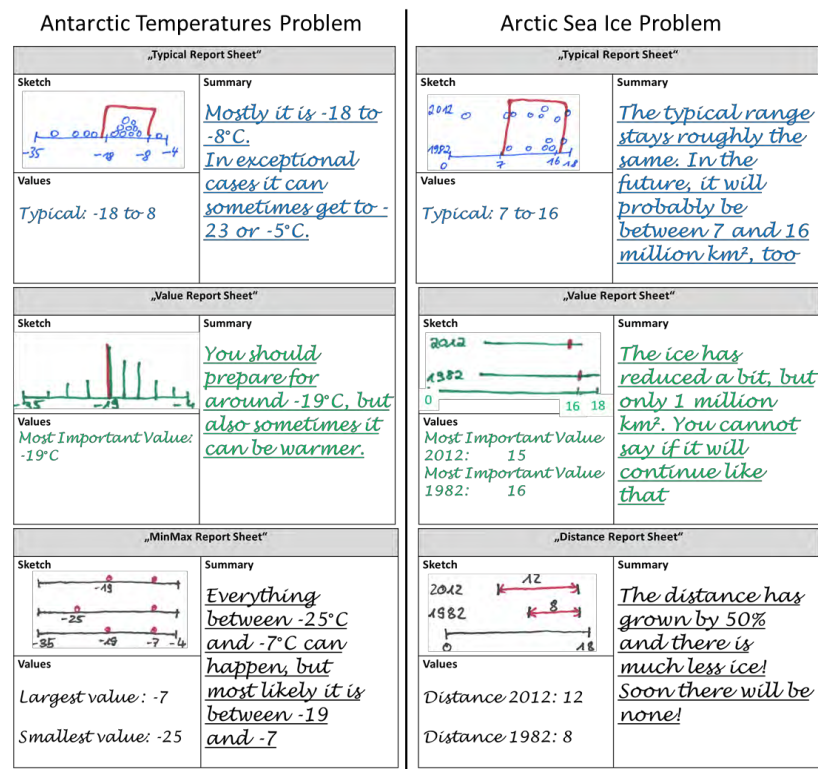


Figure 2: Filled-in report sheets for the Antarctic Temperatures Problem (left side) and Arctic Sea Ice Problem (right side) (Translated from German)

After the students are asked to evaluate the different filled-in report sheets, they are given an empty report sheet and prompted to create their own report sheet. This task allows to identify the features of the data preferred by the students at this point.

The second design experiment concerns the *Arctic Sea Ice Problem*. This problem follows a similar structure: this time, the students receive monthly Arctic sea ice data from 1982, 1992, and 2012 (Fig. 1, right side). They also again receive filled-in report sheets on Arctic sea ice (Fig. 2, right side), are asked to evaluate the report sheets, and create their own report sheet. This task allows this study to investigate whether a different context changes the students' focus on features of the data.

EMPIRICAL INSIGHTS INTO STUDENTS' FOCUS ON FEATURES OF THE DATA

This case study follows the students Jana and Mara during the first two design experiments. Empirical insights are provided in three steps of the development: students' initial predictions of Antarctic temperatures and their self-created report sheets for each of the problems.

Antarctic Temperatures Problem

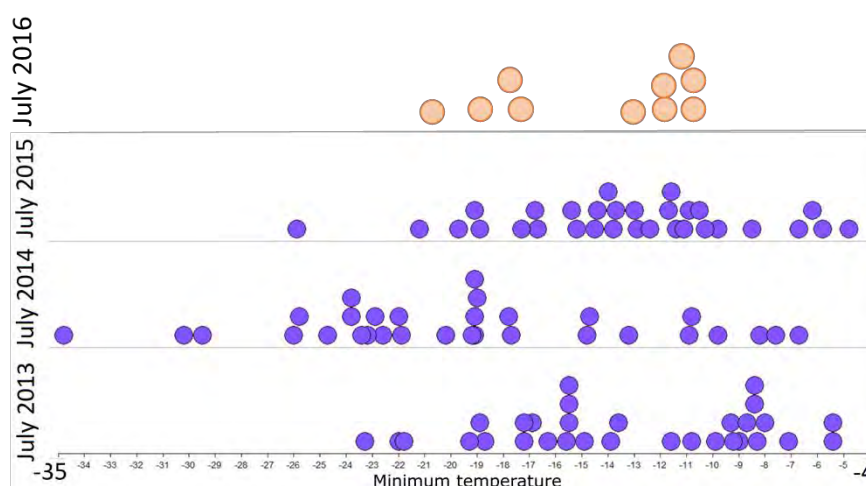


Figure 3: Jana and Mara's prediction for 2016

After giving a prediction for next year based on temperature data from three years mostly based on data from 2013 (Fig. 3), Jana (J) and Mara (M) explain their reasoning to the design experiment leader (DL).

63 J: Well, we took those where there are as many days as possible, because...

64 M: Because it will... it will most probably repeat itself, like...

65 J: Because that's, like, the normality.

[...]

70 M: [...] there it's like, that in the core there are really many, ehm, like, those days were really often, and those [extreme values] are really far away, so that they, like, maybe only were exceptional temperatures.

The students identify the features of the data of a central *||chunk of most data||* (“as many days as possible”, #63; the “core”, #70) and of the *||extreme values||* (which are “far away” from the “core”, #70). However, for predicting temperatures, the feature of the *||chunk of most days||* is more important, because *<the chunk of most data represents the normal temperatures>* (#65).

Later in the design experiment, the students create their own report sheet (Fig. 4). From this report sheet, it can be extrapolated that Jana and Mara did not focus on the features of the data emphasised by the Value Report Sheet and MinMax Report Sheet (Fig. 2), but instead continued to focus on the feature of the data of the *||chunk of most data||* (the “very frequent” days from -19 to -9, Fig. 3). Additionally, a short mention of the *||extreme values||* can be found.

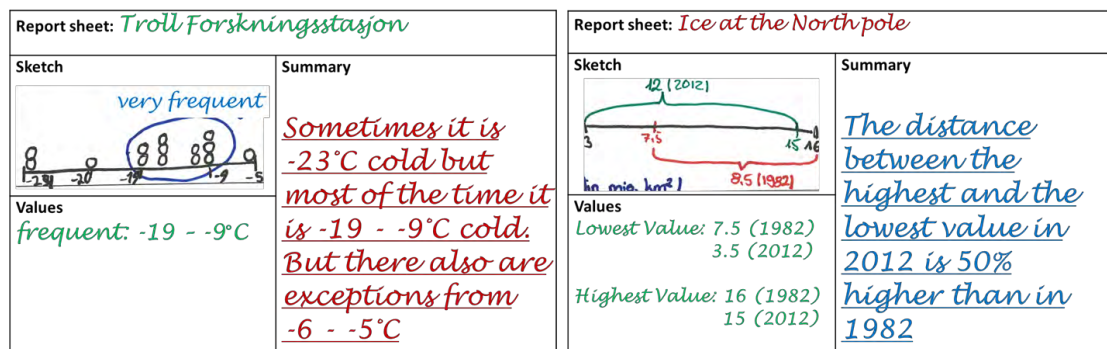


Figure 4: Jana and Mara’s report sheets for the Antarctic Temperatures Problem (left side) and Arctic Sea Ice Problem (right side) (Translated from German)

Arctic Sea Ice Problem

Towards the end of the Arctic Sea Ice Problem, Jana and Mara create their own report sheet (Fig. 4). This report sheet seems to focus on other features of the data: instead of focusing on a central chunk of data, the students adopt the Distance Report Sheet (Fig. 2) and show the features of the data of *||extreme values||* as well as the *||range||*. The design experiment leader asks the students to explain their reasoning.

- 232 J: Because we wanted to – like, with the sketch we wanted to accurately explain the distance.
- 233 M: And because the distance is larger, there also is less ice.
- [...]
- 240 DL Mhm, I understand. Last time you gave a range. You didn’t call it typical, but I don’t remember what exactly you called it. When you talked about temperatures. I think you used an area, like where there were many dots. Now you dropped that. Why wasn’t this as important to you now?
- 241 M: Ehm – because – because for the temperature we were supposed to look at the temperatures for which they should prepare the most – and uhm here it is more important to see, how the ice melted and not where the most – so when, in which period – so – uhm – how much ice at most – there is.

This excerpt shows how the students purposefully focus on the feature of the data of *||extreme values||* and the “distance” (#233) between them. The Arctic sea ice melted

more in 2012 than before, and *<the distance between extreme values represents the shrinking ice>* (#233). Challenged by the design experiment leader why they did not focus on the *||chunk of most data||*, they argue by means of the context that *<for Arctic sea ice, the extreme values are more important than the chunk of most data>* (#241).

Summary

The two excerpts show how the students identify the different features of the data of the *||chunk of most data||* and *||extreme values||*. Their focus however changes: whereas for Antarctic temperatures they focus on the *||chunk of most data||*, they focus on the *||extreme values||* for Arctic sea ice. As Mara explains, this is directly influenced by the context of the teaching-learning arrangement. The different features of the data show a different *contextual relevance* depending on the context in question: For Arctic sea ice, the *||extreme values||* are more relevant than the *||chunk of most data||*, and therefore, the students focus on this feature of the data. These phenomena of context-specific choice of features have also been found for the other focus students.

CONCLUSION

Research in statistics education has identified several features of the data ‘intuitively’ used by students (e.g. Konold et al., 2002). However, more insights into *why* students focus on specific features of the data are needed. This study shows how students’ focus on features of the data cannot be simply understood as a ‘natural’ or ‘intuitive’ form of reasoning on data, but is instead actively pursued by the students themselves according to the perceived *contextual relevance* of particular features of the data. This has important theoretical as well as practical implications: learners’ reasoning cannot be properly understood without paying attention to the context under investigation, and theoretical approaches should be chosen that are able to accommodate that fact – such as the Theory of Conceptual Fields (Vergnaud, 1996). Regarding the practical implications, this shows how the design of teaching-learning arrangements should explicitly acknowledge the possible contextual relevance of features of the data for the context at hand.

Context is already commonly held as especially important for the learning of statistics (e.g. Pfannkuch, 2011). This study adds another facet to the importance of context by illustrating possible influences of the context of a teaching-learning arrangement on students’ reasoning on data. For this study, the in-depth analysis however only allowed to closely examine parts of the learning processes of three pairs of students. Further research is necessary to compare the results to learning processes of other students, and to find ways the influence of the context can be deliberately utilized in the design of a teaching-learning arrangement in order to support students’ developing reasoning on data. This outlook is addressed in the overarching design research project (Büscher, 2018.)

References

- Bakker, A., & Gravemeijer, K. P. E. (2004). Learning to Reason About Distribution. In D. Ben-Zvi & J. Garfield (Eds.), *The Challenge of Developing Statistical Literacy, Reasoning and Thinking* (pp. 147–168). Dordrecht: Springer Netherlands.
- Biehler, R., Frischemeier, D., Reading, C., & Shaughnessy, J. M. (2018). Reasoning About Data. In D. Ben-Zvi, K. Makar, & J. Garfield (Eds.), *International Handbook of Research in Statistics Education* (pp. 138–192). Cham: Springer International Publishing.
- Büscher, C. (2018, in prep.). *Designing for Mathematical Literacy in Statistics [working title]* (Dissertation). TU Dortmund University, Dortmund.
- Corbin, J. M., & Strauss, A. (1990). Grounded Theory Research: Procedures, Canons, and Evaluative Criteria. *Qualitative Sociology*, 13(1), 3–21.
- Gravemeijer, K., & Cobb, P. (2006). Design Research from the Learning Design Perspective. In J. van den Akker, K. Gravemeijer, S. McKenney, & N. M. Nieveen (Eds.), *Educational Design Research: The Design, Development and Evaluation of Programs, Processes and Products* (pp. 45–85). London: Routledge.
- Konold, C., Robinson, A., Khalil, K., Pollatsek, A., Well, A., Wing, R., & Mayr, S. (2002). Students' Use of Modal Clumps to Summarize Data. In B. Phillips (Ed.), *Proceedings of the Sixth International Conference on Teaching Statistics: Developing a Statistically Literate Society. [CD-ROM]*. Voorburg, The Netherlands: International Statistical Institute.
- Konold, C., Higgins, T., Russell, S. J., & Khalil, K. (2015). Data Seen Through Different Lenses. *Educational Studies in Mathematics*, 88(3), 305–325.
- Makar, K., & Confrey, J. (2003). Clumps, Chunks, and Spread out: Secondary Preservice Teachers' Reasoning about Variation. In C. Lee (Ed.), *Proceedings of the Third International Research Forum on Statistical Reasoning, Thinking, and Literacy (SRTL-3). [CD-ROM]*. Mount Pleasant, Michigan: East Michigan University.
- Pfannkuch, M. (2011). The Role of Context in Developing Informal Statistical Inferential Reasoning: A Classroom Study. *Mathematical Thinking and Learning*, 13(1-2), 27–46.
- Prediger, S., & Zwetschler, L. (2013). Topic-specific Design Research with a Focus on Learning Processes: The Case of Understanding Algebraic Equivalence in Grade 8. In T. Plomp & N. Nieveen (Eds.), *Educational Design Research - Part A: An Introduction* (pp. 409–423). Enschede, the Netherlands: SLO.
- Schnell, S., & Büscher, C. (2015). Individual Concepts of Students Comparing Distribution. In K. Krainer & N. Vondrová (Eds.), *Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education* (pp. 754–760). Prague, Czech Republic: Charles University in Prague, Faculty of Education and ERME.
- Vergnaud, G. (1996). The Theory of Conceptual Fields. In L. P. Steffe (Ed.), *Theories of Mathematical Learning* (pp. 219–239). Mahwah, N.J.: L. Erlbaum Associates.

A LARGE SCALE CASCADE MODEL IN THE CONTEXT OF MATHEMATICS CURRICULUM REFORM: INTERACTIVE FACTORS OF INFLUENCE ON MULTIPLIERS' WORK

Ana Paula Canavarro^{1,2} and Leonor Santos²

¹Universidade de Évora and UIDEF, ²Instituto de Educação, Universidade de Lisboa

This study aims to know the influences identified by the multipliers on their work with schools, in order to understand the development of the cascade model implemented in the context of a Portuguese national large-scale programme for support mathematics teachers in mathematics curriculum change. 80 multipliers were surveyed with an open-ended questionnaire and the data were analysed with an inductive approach. Multipliers revealed they were affected by interrelated factors from different contexts: their colleagues, the scientific commission, the mathematics teachers and their schools and the Minister of Education. This suggests that the cascade model do not develop in a top down way, neither it is bottom up defined. Instead, it accommodates contrasting influences and its dynamics evolves as conditions change.

INTRODUCTION

In 2006, the Portuguese Ministry of Education (ME) invited public schools to propose school projects aiming the improvement of the results of their students' mathematics achievement (grades 7-9). More than 1000 schools, involving 12500 mathematics teachers, corresponded to this invitation. A national large-scale programme—the Mathematics Plan (MP)—was created to provide schools support (Ferreira et al., 2010).

In 2007, a new mathematics curriculum for the basic education (grades 1 to 9) was approved, proposing new contents (topics and mathematical process) and an inquiry-based approach to mathematics teaching (Maaß & Artigue, 2013). It was to be progressively implemented by the schools. So, from 2007/2008, the MP oriented its action for providing professional development focused on the new mathematics curriculum guidelines, concerning mathematical contents and classroom practices, recognizing the importance of having well-prepared teachers for the success of students mathematical learning (Sowder, 2007). In 2009/10, 37% of the schools voluntary implemented the new mathematics curriculum in some classes, without having textbooks. In 2010/11 begun its compulsory generalization. In 2011, a new government was elected and radical changes in educational policy were announced. The Mathematics Plan ended at the end of 2011/2012.

The Scientific Committee (SC) responsible for MP development, a group of eight teacher educators and mathematics teachers (including this paper authors), adopted a cascade model (Maaß & Artigue, 2013) to implement at national scale. This cascade

counted with multipliers (Krainer, 2015) who provided professional development to teachers in schools, based on the education they received from the SC.

This is a possible approach to scaling up professional development (Maaß et al., 2015) that remains of interest for the research on mathematics teacher education, namely for learning “how fruitful initiatives can be sustained” (Lerman & Zehetmeier, 2008, p. 17). One of the major concerns of this model is the question of how much can actually be handed down the cascade (OCDE, 1998).

Hearing from the voices of key players in the scaling up process as the multipliers can be worthwhile for the development of a more complete scenario of what counts as important in a large scale and time extended cascade model for its effectiveness and sustainability (Krainer, 2015).

In this study, we aim to know the influences identified by the multipliers for the development of their work in PM cascade model, in order to understand how the different contexts of the cascade interact and affect the development of PM programme. For so, we formulated the following research questions:

- What were the fostering factors most valued by the multipliers for the development of their work with schools?
- What were the hindering factors perceived by the multipliers for the development of their work with schools?

THEORETICAL FRAMEWORK

One of the well-known strategies for scaling-up professional development (PD) it is the so-called “cascade model. Here, multipliers are trained, who in turn train other teachers. This model can range from being top-down and also bottom up in parts initiatives” (Maab & Artigue, 2013, p. 785).

Decisive elements are the multipliers. They usually are supported by specialist and their education requires intensive efforts so they can acquire the knowledge they are expected to deliver to the teachers, in which they need to trust in order to transfer it to the continuous professional development courses they are responsible for (Roesken-Winter, Schüler, Stahnke, & Blömeke, 2015). This is of particular concern in times of curricular change, when usually exists a potential contrast between the current practice at schools and the reform practice intended by those financing PD.

Multipliers’ education must also discuss the role assumed by the multiplier. It can range from the outsider “transmitter of knowledge” to the “autonomous facilitator” of the teaching practice that helps teachers to critically reflect on their practice and to establish connections and compromises between the current practice and the intended reform practice (Krainer, 2015, p. 144). The exchange of experiences with colleagues and the meetings focused on themes recognized as relevant for day-to-day teaching are two factors of effectiveness for professional development initiatives (Maaß & Artigue, 2013) and the multipliers should be prepared to deal with them.

One important issue for the development of multipliers' work is the dynamics of schools social support in what concerns collaboration. Raising the likelihood and sustainability of success of a PD programme involves more than the teachers in an individualized way. "Scaling up PD is to reach all teachers, designing adequate school development processes are necessary—a good school is more than the sum of single good mathematics teachers" (Krainer, 2015, p.144). Colleagues need to share their knowledge and their teaching experiences and to reflect on their practice and how they can innovate in the context of a reform context. Supporting teachers' work in communities and networks is a strategy with a great potential: "Community building and networking represent the core factors fostering sustainable impact of professional development programmes" (Lerman & Zehetmeier, 2008, p. 17). Teacher networks can be seen as groups of colleagues providing social support in the development of demanding instructional practices. "This affords time built into the school schedule for collaboration among mathematics teachers and access to colleagues who have already developed relatively accomplished instructional practices" (Zehetmeier, 2015, p. 119). So, the organizational level concerning the institutions enrolled in the scaling up process (e.g. schools or ministries) need to be considered for support the individuals and their communities in their efforts to learn and bring about change. "A reform project needs several channels of networking between the autonomous triangle's domains practice, research and policy" (Krainer & Zehetmeier, 2013, p. 884). A final remark goes to the importance of the contexts, that must provide high level and good balance of internal and external resources and support: "internal and external resources and support are needed, but there is no direct and time continuous interconnection between internal and external interests" (Krainer & Zehetmeier, 2013, p. 884).

METHODOLOGY

The participants of this study are the group of 80 multipliers responsible for delivering professional development concerning new mathematics guidelines in schools. They were selected from the 450 responding to a national call and they were chosen by their experience and curriculum relevance concerning mathematics education, mathematics teacher training and development of projects. They were organized in teams covering all the regions of the country. Each one of the teams was monitored by one or two members of the SC, depending on its dimension.

The multipliers got support from the SC along all the PM development. SC provided annually intensive courses of two weeks focused on curricular, mathematical and didactical knowledge concerning the new mathematics curriculum, including both mathematical contents (e.g., mathematical reasoning, statistical literacy, ...) and methodological ones (e.g., inquiry-based learning, tasks design, digital resources, ...). Besides, SC also provided continued support to multipliers by a monthly regular one-day meetings aiming to update and deepen their knowledge concerning the new guidelines and to support them in dealing with the specific problems they faced in their regular work with the schools. These meetings focused on specific strategies of involving teachers in terrain in reflection about their mathematics teaching practice, like

discussing students' mathematical productions, appreciating the potentialities of different mathematical tasks, exchanging experiences from classroom, etc. Each multiplier could choose, from the topics of these meetings, the agenda of the follow up meetings he/she had to convey to the schools under his/her responsibility.

Each multiplier was responsible for a group of schools (up to 17) organized by four or five subgroups according to geographical proximity in order to facilitate the schools interactions and collaboration. A multiplier had to spend one day (Tuesday) per month with each subgroup, promoting meetings with the teachers responsible for the mathematics curricular development in all the schools belonging to the subgroup. These meetings took part in different schools, aiming to foster also the participation of the mathematics teachers of the school "receiving" the meeting. In order to allow the participation of the teachers in these meetings, school directors were expected to respect the Ministry of Education's recommendation of releasing the Tuesday afternoon from classes to the involved teachers. This recommendation lasted to 2011 but was not assumed by the Ministry of Education in 2011/12 and some of the schools maintained the Tuesday free but others did not.

We followed an interpretative methodological approach, capturing the meanings of the multipliers from their responses to four selected open questions of an open-ended questionnaire. This questionnaire was elaborated by the SC and was used from 2007 to 2012, maintaining the same questions along the six years. The 80 multipliers completed the questionnaire at the end of each scholar year. The principles of confidentiality and informed consent were guaranteed to all (AERA, 2011).

The open questions considered for this study required the multipliers' reflection about the influences for the development of their work and asked for: Appreciation of the support by SC; Appreciation of the conditions of the schools; Strategies they used for preparing for work with schools; Difficulties faced in their work in schools.

The data were analyzed and classified with an inductive approach. Inspired by theory, we considered three a priori broad categories. These are the influences that the multipliers identified to their work concerning (1) the education assured by SC, (2) the dynamics of the schools contexts, and (3) the conditions provided by the Ministry of Education. In the course of the analysis, we added a new category that has come up with a strong expression: the influences of collective work among the multipliers themselves. From the analysis of these influences, we obtained the fostering factors and the hindering factors of the multipliers' work.

Quotations from the multipliers were selected from the questionnaires for illustration of the interpretation of data and its categorization. The reference (M "year") means a quotation extracted from a response of a multiplier on the questionnaire of the scholar year ending in "year" (e.g., M 2010 is a quotation from 2009/10).

RESULTS

Fostering factors valued by the multipliers

The multipliers revealed a positive opinion about the support they got from SC every year. In particular, the regional regular meetings were perceived by them as fundamental for their development of self-confidence required to act as multiplier in schools. They valued the contents focused and also the way they were approached. They also appreciated the materials provided by the SC, namely the theoretical references concerning the knowledge they needed to deliver to the schools:

The topics I discussed at the follow-up meetings (with my schools) were mainly proposed following the guidelines and work of the meetings with the SC, which were extremely useful and adequate and had theoretical support, which gave me self-confidence to work in the follow-up meetings. (M 2012)

Another aspect valued by the multipliers related with the SC support concerned strategies for approaching the new curricular topics with the teachers without assuming the role of “transmitter of knowledge” but more like a facilitator of curricular development. The SC suggestion of beginning the school meetings with the sharing of experiences from teachers’ classrooms was perceived as a positive influence for fostering reflection about mathematics teaching practice and collaboration among teachers:

One of the strategies that has been very effective in the past few years is to promote the sharing of ideas, materials and reflections both within each group of schools and between schools belonging to different groups — (as we do with the committee). (M 2012)

Other fostering factor of multipliers work was the support they get from each others. In fact, this was not planned in advance and was an initiative of the multipliers themselves. They develop periodical regional forums for sharing experiences and materials and for joint preparation of their meetings with the schools:

This group of multipliers meets once or twice a month in a rotating way between Braga, Esposende and Famalicão. We prepare the agenda, the subjects to be worked, the materials to be presented. We carry out a continuous work of sharing, reflecting before and after the meetings, expanding this work with the exchange of emails. (M 2012)

This happened in all the country, being intensified along the years, and was valued as a fundamental contribution for a good adequacy of the meetings with schools:

The collaborative work carried out by the multipliers of the region (...) was fundamental because it helped to make more sensible decisions and to make fewer mistakes. (M 2009)

Other fostering factors that multipliers identified come from the schools and their teachers. Two different aspects can be acknowledged. One was their consideration of the schools’ suggestions about topics to be focused on the school meetings:

Colleagues suggested some topics to be explored at follow-up meetings, such as: assessment, isometries, dynamic geometry (Geogebra), quadratic functions, data analysis, and tasks (elaboration and sharing). (M 2012)

It allowed the multipliers to better meet schools interests and needs:

It is very important that the mathematics topics meet teachers' expectations. (M 2010)

Other aspect considered by the multipliers as a positive contribution for their work was the interest and enthusiasm they perceived in schools, particularly at the PM starting, due to the satisfaction with the collective dynamics of teachers and to their disposition to improve mathematics teaching practice:

It was encouraging to see a large number of teachers interested in listening to what others wanted to share and also decided to embark on a gradual change in their teaching practice. (M 2007)

Hindering factors valued by the multipliers

Multipliers referred to same factors that constituted difficulties of the effective development of their work with schools and these varied along the time. One has to do with the pervasive curricular school culture. In many schools, multipliers had to deal with a sentiment of resistance that arouse significantly in the phase of progressive generalization of the new mathematics curriculum, namely when the teachers were expected to concretize the new guidelines with the respective students:

I feel that teachers have difficulties in taking over the New Program, there is a certain resistance and distrust in the students' learning with some tasks, and a little critical about some aspects of the program. This change of mentalities takes time. (M 2010)

At that time, PM was extended to the grades 1 to 9 and some of the teachers were teaching from the old program and other with the new one. This diversity added complexity to multipliers work, namely because some of the teachers did not recognized relevance to the topics approached in the meetings at schools:

The fact is that the meetings are joint: schools with the new program and schools without it. The colleagues from schools who do not have the new program felt that it made no sense to be working on a topic they do not need yet to teach in their school. (M 2010)

Another hinder that multipliers felt in the last year of PM has to do with the decreasing of conditions (time and budget) that was imposed by the Ministry of Education due to economic restrictions, affecting the work that multipliers could do in schools:

During this school year, the greatest difficulty was the lack of availability of teachers on Tuesday afternoon to participate in the meetings. Also the lack of free time in common of the teachers of each school was a constraint mentioned by some coordinators for the collaborative work in their schools. (M 2012)

The new Ministry discourse about the mathematics program being implemented was very critical and he announced severe changes in the guidelines that teachers were experiencing and beginning to become comfortable with. All these changes contributed to a pervasive general feeling of disappointment and accommodation by many teachers, with consequences for the decreasing of the participation of teachers in the follow-up meetings and required an increased effort of the multipliers:

This situation (reduction of time for collaborative work), along with others (reduction of time for consultancies, supports, etc.), contributed a lot to the demotivation and demobilization of the teachers that I felt this year in the follow-up meetings. It became a factor of resistance to the proposed work and was not easy to overcome. (M 2012)

CONCLUSIONS

The MP multipliers identified several influences for the development of their work. These influences aroused in different contexts. From SC, multipliers valued the direct support for the development of the meetings with the teachers in schools, consisting in opportunities for the acquisition and deepening of knowledge and strategies for PD development. From their colleagues, multipliers valued the support for the preparation of the follow-up meetings with teachers, taking profit of the collaboration with peers, not planned in the MP structure. From the schools, multipliers received incentive for the development of their work but also met difficulties to achieve their objectives concerning curricular changes. From the educational policies, multipliers found both fostering and hindering factors concerning, respectively, the provision or the withdrawing of logistic conditions and the promotion of a positive climate of investment in schools or the disruptive interruption of teachers' work for the improvement of mathematics teaching.

So, the multipliers were affected not only by the SC (as could be supposed in a top-down cascade model) but also by the other interacting actors. This suggests that the development of a cascade model depends of the consideration and articulation of all involved contexts. As Krainer (2015) points out, key players interact in the context of scaling up PD and it is important to understand how they can take profit of each others. The SC was perceived by the multipliers of a major importance, what suggests that a cascade programme benefits from a strong support of the entity that is scientifically responsible for it, which is expected to deliver knowledge and orientations for the PD' processes. But the multipliers also considered themselves as a relevant resource for the development of their work and this suggests, in line with Krainer & Zehetmeier (2013), the need of assuring conditions for the multipliers collaboration in a cascade model.

A remark goes to the perceived effects of the changes in political conditions along the years and also in the attitude of schools and teachers. This suggests that cascades that develop in long time period need to acknowledge its dynamic nature. "Large-scale and long-term projects need both flexible plans and the use of windows of opportunity" (Krainer & Zehetmeier, 2013, p. 884).

References

- American Educational Research Association (2011). Code of Ethics. *Educational Researcher*, 40(3), 145-156.
- Ferreira, R., Santos, L., Pinheiro, A., Santos, E., Amado, N., Pires, M. & Canelas, R. (2010). The plan of mathematics: a national project to support mathematical learning. *PME 34 International Group for the Psychology of Mathematics Education*, 4, 353.
- Krainer, K. (2015). Reflections on the increasing relevance of large-scale professional development. *ZDM Mathematics Education*, 47, 143-151.
- Krainer, K., & Zehetmeier, S. (2013). Inquiry-based learning for students, teachers, researchers, and representatives of educational administration and policy: reflections on a nation-wide initiative fostering educational innovations. *ZDM Mathematics Education*, 45, 875-886.
- Lerman, S., & Zehetmeier, S. (2008). Face-to-face communities and networks of practicing. In T. Wood (Series Editor) & K. Krainer (Volume Editor), *International handbook of mathematics teacher education* (Vol. 3). *Participants in mathematics teacher education: individuals, teams, communities, and networks*. Rotterdam: Sense Publishers.
- Maaß, K., & Artigue, M. (2013). Implementation of inquiry-based learning in day-to-day teaching: a synthesis. *ZDM Mathematics Education*, 45, 779-795.
- Maaß, K., Barzel, B., Törner, G., Wernisch, D., Schäfer, E., & Reitz-Koncebovski, K. (Eds.) (2015). *Educating the educators: international approaches to scaling-up professional development in mathematics and science education*. Münster: Verlag.
- OCDE (1998). *Pathways and participation in vocational and technical education and training*. OCDE Publishing: Paris.
- Roesken-Winter, B., Schüler, S., Stahnke, R., & Blömeke, S. (2015). Effective CPD on a large scale: examining the development of multipliers. *ZDM Mathematics Education*, 47, 13-25.
- Sowder, J. (2007). The mathematical education and development of teachers. In F. Jr. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 157-223). Reston, VA: NCTM.
- Zehetmeier, S. (2015). Sustaining and scaling up the impact of professional development programmes. *ZDM Mathematics Education*, 47, 117-128.

RESPONSE PROCESS VALIDITY EVIDENCE: A PROPORTIONAL REASONING EXAMPLE

Michele B. Carney and Katie Paulding

College of Education, Boise State University

Providing validity evidence to support the interpretation of scores from diagnostic assessments is a critical component of validation (Ercikan & Pellegrino, 2017). This study provides an example of examining response process validity through analysis of cognitive interviews with middle school students on a proportional reasoning item type embedded in the Diagnostic Assessment of Proportional Reasoning. Our findings indicate students' reasoning when solving the contextual equation item type differed in important ways from what we had assumed and highlight the critical need for assessment developers to provide, and assessment users to expect, the provision of response process validity evidence as the norm within the mathematics education research community.

INTRODUCTION

Diagnostic assessments are intended to be used by educators to rapidly assess the individual mathematical knowledge, skills, and/or ability (KSA) of large groups of students. Diagnostic assessments developers typically make claims related to how scores can be interpreted in relation to students' KSA to make instructional decisions. It is important developers provide evidence to support the validity of these claims (Ercikan & Pellegrino, 2017). The *Standards for Educational and Psychological Testing* (AERA, APA, & NCME, 2014) refer to this aspect of test validation as response process. Our purpose is to provide an example of examining response processes to determine if students demonstrate the anticipated understanding when the item type is solved correctly. If so, this provides evidence to support claims related to student understandings when engaging with this item type. This is just one small aspect of examination of response process validity, but a critically important type of assumption to verify. Padilla and Benítez (2014) highlight the lack of focus in the assessment development community on providing response process validity evidence and present cognitive interviews as the primary methodology for gathering such evidence. Thus, we find value in providing an example for the mathematics assessment development community, and for users of mathematics assessments to encourage them to expect this type of evidence when assessments are being considered for use.

Our example examines an item type embedded in an assessment of proportional reasoning designed to assess students' understanding of the multiplicative comparison relationship. The claim related to the item type is that students who solve this item correctly are likely have some understanding of the constant multiplicative relationship

between the two quantities in the ratio. We selected this particular item type to serve as our example because we have found it a particularly hard concept to assess via a pencil and paper test (Carney, Smith, Hughes, Brendefur, & Crawford, 2016).

THEORETICAL FRAMEWORK FOR PROPORTIONAL REASONING

Due to the wealth of research on proportional reasoning and the various perspectives taken and terms used, it is helpful to present the conceptual framework we use for proportional reasoning related to the mathematical relationships and associated student conceptions. We use this framework in large part due to its ease of understanding for teachers (summarized in Lobato, Ellis, & Charles, 2010).

Mathematical Relationships: Scalar and Functional

In order for students to reason proportionally – rather than procedurally - they need to be able to make use of and eventually be able to generalize their understanding of important mathematical relationships. More specifically one important aspect is the ability to fluently and flexibly make use of the scalar and functional relationships that exist within ratio, rate and proportion situations (Ellis, 2013). The scalar relationship describes the common (scale) factor each quantity in the ratio can be multiplied by to generate an equivalent ratio. For example, given the ratio situation of a constant pace of hiking 12 miles in 3 hours, the 12:3 relationship may be scaled up or down to create equivalent ratios by multiplying each quantity in the ratio by the same factor (see Figure 1 for an example of scaling by a factor of 3). The functional relationship describes the constant multiplicative relationship that exists between the two quantities in the ratio. For the situation involving a constant pace of 12 miles in 3 hours, the constant multiplicative relationship of 4 times the hours (as seen in Figure 1) or $\frac{1}{4}$ of the miles can be used to describe and generalize the ratio relationship.

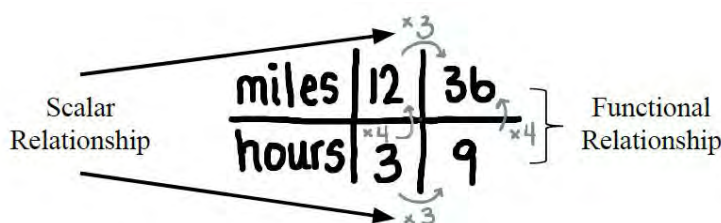


Figure 1: Example of scalar and functional relationships.

Student Conceptions: Composed Unit and Multiplicative Comparison

Students' ability to fluently and flexibly make use of the scalar and functional relationship involves coordination of understandings related to ratios, proportions and rates. In missing value situations (e.g., *If Jane hikes at a constant rate of 12 miles in 3 hours, how many hours will it take Jane to hike 36 miles?*) students tend to demonstrate understanding of ratios as a composed unit – two quantities that must be coordinated in conjunction with one another – making use of the scalar relationship (Lobato et al., 2010). This understanding is often initially demonstrated through the ability to double

and/or halve the quantities in the ratio or additive scaling to generate equivalent ratios. This can eventually develop into demonstrating understanding of the scalar multiplicative relationship by treating the initial ratio as a composed unit that may be scaled by multiplying (or dividing) by a single scale factor.

There is evidence students tend to have difficulty developing and/or expressing understanding of the multiplicative comparison within the functional relationship (Simon & Placa, 2012; Steinhorsdottir & Sriraman, 2009). The multiplicative comparison conception involves understanding and using the constant multiplicative relationship that exists between the two quantities in a ratio. Table 1 provides examples of expressions of composed unit and multiplicative comparison conceptions.

Conception	Description	Example Student Verbalization
Composed Unit	Understanding two quantities in a ratio must be coordinated in conjunction with one another through use of the scalar relationship	<i>If Jane hikes at a constant rate of 12 miles in 3 hours, and I know Jane has hiked 2 miles... then I divide 12 by 2, to get 6. Then I also have to divide 3 by 6, to get $\frac{1}{2}$ which is the hours.</i>
Multiplicative Comparison	Understanding and making use of the constant multiplicative relationship that exists between the two quantities in a ratio	<i>If Jane hikes at a constant rate of 12 miles in 3 hours, and Jane has hiked 2 miles... I see that the hours are $\frac{1}{4}$ of the miles and $\frac{1}{4}$ of 2 is $\frac{1}{2}$. So it is $\frac{1}{2}$ hours.</i>

Table 1: Example student verbalizations for the item prompt - If Jane hikes at a constant rate of 12 miles in 3 hours, how many hours will it take Jane to hike 2 miles?

DIAGNOSTIC ASSESSMENT OF PROPORTIONAL REASONING

The item type examined in this study comes from the Diagnostic Assessment of Proportional Reasoning (DAPR). The DAPR is designed to assess students' understanding of composed unit and multiplicative comparison conceptions. There are three equivalent test forms with 20 items per form. A common ratio relationship stem and a set of five item types designed to assess different aspects of students' proportional reasoning KSA are presented as a problem set, with four problem sets per form. The item types are designed to assess if a student possesses certain conceptions based on which item types they solve correctly. Table 2 provides an example of the items types and their associated conceptions. On a particular assessment, students will see each item type repeated four times with different contexts and ratio relationships.

Common Ratio Relationship Item Stem

5 sugar cookies for \$4

If the relationship between cookies and cost (\$) remains the same no matter how many cookies you buy, complete the following statements about the relationships between cookies and cost (\$).

Item Types	Example	Conception	
Small Single-Digit Multiplier	Callie bought 5 sugar cookies for \$4. How many cookies can she buy with \$8?	Informal Reasoning	
Double-Digit Scalar Multiplier	It would cost _____ to buy 75 cookies.		Composed Unit
Unit Rate Situation	One cookie costs _____.		
Contextual Equations	number of cookies = _____ * cost		
Generalizing	The cost is always _____ times the number of cookies.	Multiplicative Comparison	

Table 2: Example problem set from the DAPR with each item type represented.

Students' responses are scored as correct or incorrect, and interpreted in relation to composed unit and multiplicative comparison conceptions. For instance, students who correctly respond to items designed to assess the multiplicative comparison conception (i.e., contextual equations and generalizing) are interpreted as likely to have generalized some understanding of the multiplicative comparison relationship.

Our long-term goal related to response process validity evidence involves developing individual student profiles for each student interviewed related to multiplicative comparison and composed unit conceptions. The profiles will be analysed in conjunction with the students' DAPR test score and the draft interpretation statements for test scores. For example, if a student score on the DAPR falls within the score range for the following interpretation - *It is likely the student has some composed unit conceptions but likely does not have understanding of the multiplicative comparison* - the degree of

alignment between that interpretation and the interview profile will be examined. We suspect we need to add additional nuance to the score interpretation.

For the purposes of this study we are examining student responses related to the contextual equation item type on the DAPR assessment. We selected this item type because we have found the multiplicative comparison conception difficult to assess. This is one of several analyses used to inform our student profiles.

METHODS


As recommended by Padilla and Benítez (2014) we engaged students in a think-aloud protocol involving the assessment item types. The interviews were transcribed and analysed for evidence of student solution strategies and understandings. Phase 1 involved developing a coding rubric for the solution strategies and understandings exhibited by students during the interviews. Phase 2 involved applying the coding rubric to the interview data. Phase 3 involved analysing the coded data for consistency with item type interpretations related to solution strategies and student understanding.

STATEMENT OF ASSUMPTIONS

The general response process assumption underlying the DAPR assessments is that students' responses to items reflect their proportional reasoning understanding related to composed unit and multiplicative comparison conceptions. For the purpose of this investigation we are focused on examining students' responses on item types designed to assess multiplicative comparison conceptions. We assume students who correctly solve multiple contextual equation item types possess some understanding of the multiplicative comparison conception.

Participants. We conducted cognitive interviews with 33 students in grades 6-8 (ages 11-14) at three different schools. Students were selected based on their recent performance on the DAPR. The goal was to have a range of knowledge and skills.

Interview Items. The interview protocols used ratio scenarios from the DAPR assessment items, and the five associated item prompts. Two interviews protocols, A and B, were developed (see Table 3 for the specific contextual equation item prompts).

Identifier		Ratio Scenario	Contextual Equation Item Prompt
A	1	6 ounces of red paint to 3 ounces of white paint	ounces of white paint = _____ • ounces of red paint
	2	Buys 5 sugar cookies for \$4	number of cookies = _____ • cost
B	3	Bikes 12 miles in 3 hours	number of hours = _____ • number of miles
	4	Hikes 5 miles in 2 hours 	number of hours = _____ • number of miles

Note: • indicates multiplication was provided under the item prompt

Table 3: Ratio scenario and associated contextual equation item prompt in interviews.

ANALYSES AND RESULTS

We first identified the instances of students who correctly solved the contextual equation item types across the 33 students and 66 possible instances (students received two contextual equation item types per interview). Five students correctly solved both and six students correctly solved one of the two contextual item types. This resulted in 16 out of 66 (24%) instances of the contextual item types being solved correctly. Given that this is the hardest item type on the assessment, this was not unexpected. Next we developed a coding framework to describe the solution strategies and associated conceptions exhibited by students who correctly solved the contextual equation items, and to identify exemplars (see Table 4). We then coded the 16 instances using this framework. Our final step was to determine the likelihood a student who solved a contextual equation item correctly demonstrated understanding of the multiplicative comparison relationship. We found that of the 16 instances, seven instances (44%) indicated an understanding of the multiplicative comparison relationship, and nine (56%) did not (see Table 4 on next page).

We anticipated students who correctly solved a contextual equation would express a multiplicative comparison conception of the ratio relationship. Instead a number of students inserted the ratio quantities into the equation and solved for the missing value. We suspected this approach would be used but were surprised by the frequency.

DISCUSSION AND CONCLUSION

The purpose of this paper was two-fold. First, we investigated our assumption related to student understanding of the multiplicative comparison when correctly solving a contextual equation item type. Second, we provided an example situated in mathematics education of an investigation into response process validity. Regarding our assumption, the findings indicate that while almost half the time students who correctly solved the contextual equation item type did provide evidence of multiplicative comparison thinking; however, more than half of the students inserted the quantities from the original ratio into the equation and solved for the missing value. While this approach does not preclude students from possessing a multiplicative comparison conception, we need to further examine our assumption. Our sample size is a significant limitation of our findings. Future research will involve interviewing a larger sample of students who performed well on the DAPR (a score of 15 or higher) to ensure we have enough instances of students correctly solving the item type to generalize. In addition, further investigation of other item types and the development of student profiles are likely to reveal patterns in understandings that are difficult to discern when examining responses at the item level.

Solution Strategy: Description	Conception & Frequency	Student Interview Exemplar [Item Identifier]
MC: determined multiplicative comparison, used to determine the missing value	MC 6 (37.5%)	<i>So I got for the number of hours equals one fourth times the number of miles because for this [ratio] three is a fourth of twelve. [B3]</i>
Mixed Strategies: Expressed unit rates, expressed multiplicative comparison, and checked answer by solving the equation (with unit rate)	CU & MC 1 (6.5%)	<i>S: ... since the ratio is two to one, and then that can also be represented as a fraction... two over one...that means there is two for every white and then one half means that there is one half... for every red, I think. I: So tell me, why the one half right there? S: One ounce of white paint, let's say, and then we'll just use the two for the ounces of red paint, since there is one half the amount of white paint that there is used for red paint, and the ratio, then if you multiplied two times one half then you get the one ounce of white paint that you get over here. [A1]</i>
Solved equation: Inserted original ratio quantities into the contextual equation and solved for the missing portion	P 7 (44%)	<i>So what do you have to times by the amount of red paint to get the ounces of white paint so I'm going to use the original one I got - six ounces of red paint to three ounces of white paint - so I'm going to put that in equation form. Three equals x times six and x would have to be one half because three is one half of six. So that answer would be one half. [A1]</i>
Unit Rate: Inserted the unit rate and checked by solving the equation	CU 2 (12.5%)	<i>The number of cookies equals something times cost. I think it'd just be one of those [pointing to paper]. So if the cost is, so then it would be one point two five. No. Yes. Right? Because then you have one point two five cookies for the cost of one dollar. And if it's two dollars, then it's one point two five times two, which is two and a half. Yeah, I think so. [A2]</i>

MC = Multiplicative Comparison, CU = Composed Unit, P = Procedural

Table 4: Student solution strategies, conceptions, frequencies, and exemplars for the contextual equation item type.

In terms of providing an example of response process validity evidence situated in mathematics education, our investigation highlights the need for assessment developers to provide and for assessment users to request response process validity evidence. The assumption that a particular score on an assessment reveals information about student KSA is critically important to investigate. It is vital that evidence of response process become the norm with the mathematics education research community. This will ensure valid score interpretations and uses, and more importantly ensure educators have high-quality diagnostic assessments that accurately reveal the KSA of students.

References

- AERA, APA, & NCME. (2014). Chapter 1: Validity *Standards for educational and psychological testing*: American Educational Research Association.
- Carney, M., Smith, E., Hughes, G., Brendefur, J., & Crawford, A. (2016). Influence of Proportional Number Relationships on Item Accessibility and Students Strategies. *Mathematics Education Research Journal*. doi:10.1007/s13394-016-0177-z
- Ellis, A. (2013). *Research Brief: Teaching Ratio and Proportion in the Middle Grades*. Retrieved from Reston, VA: <http://www.nctm.org/news/content.aspx?id=35822>
- Ercikan, K., & Pellegrino, J. W. (2017). *Validation of Score Meaning for the Next Generation of Assessments: The Use of Response Processes*: Taylor & Francis.
- Lobato, J., Ellis, A. B., & Charles, R. I. (2010). *Developing essential understanding of ratios, proportions, and proportional reasoning for teaching mathematics in grades 6-8*. Reston, VA: National Council of Teachers of Mathematics.
- Padilla, J.-L., & Benítez, I. (2014). Validity evidence based on response processes. *Psicothema*, 26(1).
- Simon, M., & Placa, N. (2012). Reasoning about Intensive Quantities in Whole-Number Multiplication? A Possible Basis for Ratio Understanding. *For the learning of mathematics*, 32(2), 35-41.
- Steinhorsdottir, O. B., & Sriraman, B. (2009). Icelandic 5th-grade girls' developmental trajectories in proportional reasoning. *Mathematics Education Research Journal*, 21(1), 6-30.

MATHEMATICAL INDUCTION AT THE TERTIARY LEVEL: LOOKING BEHIND APPEARANCES

Gemma Carotenuto¹, Cristina Coppola¹, and Pietro Di Martino²

¹Dipartimento di Matematica, Università degli Studi di Salerno, Italy

²Dipartimento di Matematica, Università degli Studi di Pisa, Italy

The relevance of inductive proofs in Mathematics is beyond question and the research in Mathematics Education has widely documented the students' difficulties in understanding and applying mathematical induction, both at secondary school level and at university level. In this paper, we present a qualitative study involving third year Mathematics degree students aimed at investigating the solidity/fragility of mathematical induction comprehension. The results highlight that mathematical induction is a very hard topic also in this context, in which are involved mathematical competent students. We argue the need to design non-standard activities able to get the misconceptions emerge, in order to support a deep understanding of the topic.

INTRODUCTION AND THEORETICAL PERSPECTIVE

The process of reasoning called “mathematical induction” has had a long history as almost the whole of the mathematical concepts. We can already recognize the germ of mathematical induction (MI) in the Euclidean proof of the infinity of prime numbers, but the name “mathematical induction” was used for the first time in 1838 by Augustus De Morgan in his article “Induction (Mathematics)” (Cajory, 1918). From a didactical point of view, as Ernest (1984) underlines, the word “induction” introduces an ambiguity between the heuristic method for arriving at a conjecture (starting by a finite number of examples) and the mathematical rigorous form of deductive proof.

The relevance of MI in Mathematics is related to the foundation of natural numbers – the modern formal version of mathematical induction was stated in Peano Arithmetic (Peano, 1889) – and it is well summarized by Poincaré's opinion that MI is the “mathematical reasoning *par excellence*” (Poincaré, 1906). On the one hand Poincaré considers MI as the affirmation of a property of the mind itself, on the other hand he is aware of the complexity of a principle that “contains, condensed, in a unique formula, an infinity of syllogisms” (Poincaré, 1906). It exists a clear parallel between the relevance of MI in Mathematics and its relevance from an educational point of view. The students' understanding of MI allows to reflect on inductive reasoning and on the fundamental properties of natural numbers (Palla, Potari & Spyrou, 2011). This is one of the main reason why it has been argued that MI should be introduced in upper secondary school (NCTM, 2000).

The research in Mathematics Education has widely documented the (secondary school and university) students' difficulties in understanding and applying MI (Ernest, 1984;

Harel, 2002; Nardi & Iannone, 2003; Palla, Potari & Spyrou, 2011). On the basis of the research results, we can identify three kinds of difficulties: the problematic relationship between inductive argumentation and proof by induction (this is related to the students' difficulty to conceptualise the need for a proof), technical difficulties in developing proof by MI and conceptual difficulties in understanding the structure and use of MI.

Pedemonte (2007) has carefully studied the cognitive difficulties related to the transition from an inductive argumentation to a proof by MI. She finds that students construct inductive proofs of a sentence only when they are able to generalize the process leading to the solution of the problem in inductive argumentation. Further upstream, this transition is problematic because several students see inductive argumentation as perfectly convincing, not recognizing the need for a rigorous proof by MI (Stylianides & Stylianides, 2009). Harel (2002) underlines that the standard instructional treatments introduce the formal principle of MI without ensuring that the students develop an intellectual need for it. On the other hand, Weber (2010) underlines that spurring students to see the limitations of empirical arguments can help to share with them the need for a rigorous proof, but it does not imply improving their comprehension and ability in producing mathematical proofs.

Nardi and Iannone (2003) have exactly studied the students' difficulties with MI when the necessity of proof seems to have been recognized. They carefully describe the technical difficulties emerging in the proof of the inductive step $P(n) \rightarrow P(n+1)$, especially in contexts where predicate P involves non-symmetrical, one-way relationships such as algebraic inequalities. In some cases, also the explicit writing of the $P(n+1)$ term in the inductive step can be hard for undergraduate students.

For example, in the task: Prove that: for any n ($n > 0$) positive numbers $x_1, x_2, x_3, \dots, x_n$, if $\prod_{i=1}^n x_i = 1$, then $\sum_{i=1}^n x_i \geq n$; a common mistake is the use of the same variables $x_1, x_2, x_3, \dots, x_n, x_{n+1}$ in the $P(n+1)$ part (Avital & Libeskind, 1978).

On the conceptual level, the pioneering studies of Ernest (1984) and of Fischbein and Engel (1989) discuss the difficulties related to the complex use of quantifiers in MI and to the awkward distinction in dealing with the inductive step $P(n) \rightarrow P(n+1)$ between the inductive hypothesis $P(n)$ and the thesis, generally expressed as $\forall n P(n)$. Also Sfard (1992), within her theory of reification, describes the conceptual difficulties related to MI. In the passage from the operational to the structural comprehension of MI, the learner reifies MI in an object – the global structure of natural numbers – that can be manipulated (Palla et al., 2011). In the passage from elementary to advanced mathematics, the level of complexity of proving by MI develops from the operational to the structural level, and the related difficulties from the technical level to the conceptual one.

An interesting general concept developed in the field of Mathematics Education – just in a research focused on students' understanding of MI – is the “fragility” of personal knowledge (Movshovitz-Hadar, 1991, pp. 41-42):

A human being's knowledge is fragile as long as this person can be put in a cognitive conflict [...] The ultimate goal of the process of [...] mathematical learning in particular, is to develop one's understanding [...] that is robust enough to withstand any attempt to create a cognitive conflict, to inject contradictions into it.

Movshovitz-Hadar coins the term “knowledge fragility” to describe a knowledge “in the intermediate stage of development, when understanding is yet to be negotiated” (Movshovitz-Hadar, 1993, p. 266). In the context of mathematics, it seems particularly interesting to study this dimension (solidity/fragility) of knowledge also when the understanding of a complex concept has already been negotiated: in Sfard's term, when a conceptual understanding seems to have been developed.

In this framework, we developed a qualitative study involving a particular sample composed by third year Mathematics degree students (from three different cohorts). Our aim was to investigate the solidity/fragility of MI-comprehension of apparently *MI-competent* students, i.e. students that: i) surely recognize the need for a deductive proof for a mathematical statement; ii) seem to have no technical difficulties in complex algebraic manipulations and explicit writing of the $P(n + 1)$ term in the inductive step; iii) seem to have developed a structural comprehension of MI.

PROCEDURE AND RATIONALE

The work described in this report is based on the analysis of the qualitative data collected in three consecutive academic years during the course Elementary Mathematics from an Advanced Standpoint: Arithmetic. The course – held by the third author – is an optional course for the third year of the Mathematics degree and it deals with the topic of number systems. The reflection about natural numbers is an important part of the course program and, in particular, MI is the focus of five structured class meetings (CM in the following).

The first CM is used to get an idea of students' knowledge about the formulation of MI and its equivalent forms, and also to test their practice and ability in technical manipulations involved in the application of MI. For this aim, we use recursion problems that result insidious from a technical point of view. In particular, according to Nardi and Iannone's analysis (Nardi & Iannone, 2003), we use problems involving non-symmetrical, one-way relationships such as algebraic inequalities. Many of them were formerly faced by the students in the first-year exam of the Algebra course (see for example Figure 1).

Consider the following series: $\begin{cases} a_0 = 2; a_1 = 3; a_2 = 5 \\ a_{n+1} = a_n - a_{n-1} + 2a_{n-2} \quad \forall n > 1 \end{cases}$

Prove that $a_{n+1} > a_n$ for each n .

Figure 1: Example of a task given in the first CM.

The remaining four CMs are organized into two pairs: the first CM of each pair is a problem solving section, the second CM takes place three days later and it is organized as a discussion group focused on the previous problem solving section.

The first pair of CMs deals with the relationship between conjecturing and proving. In particular, in these two meetings the aim is to observe the behaviour of the students when they are not successful in proving a conjecture achieved by empirical arguments: do they mistrust the developed conjecture or their capabilities in proving it by induction? During these CMs, two geometrical tasks are proposed to the students (Figure 2).

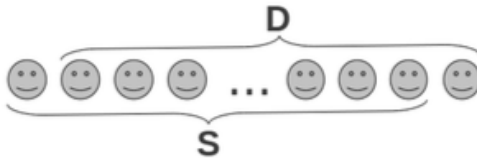
Task 1: How does the number of diagonals of a convex polygon P vary, varying the number n of its sides?	Task 2: Determining the maximum number of pieces in which it is possible to divide a circle by n points on the circumference, joined by all the possible chords.
---	--

Figure 2: Examples of tasks used in CM 2 and CM 3.

The second pair of CMs is focused on conceptual understanding of MI. Similar to what was done by Movshovitz-Hadar (1993), the first one of these meetings starts with a discussion about the following two related issues: a) to rate the difficulties of mathematical induction as proof method; b) to rate the expected success in solving a problem which calls for a proof by mathematical induction. At the end of the discussion, the “Children’s eye colour” problem is proposed (see Figure 3).

Theorem: All newborns have eyes of the same colour.

Proof: For $n = 1$ the thesis is trivially true. Suppose $P(n)$ holds (that is, in every set of n babies, all babies have eyes of the same colour). Consider now a set of $n + 1$ babies. The following graphical scheme proves that: if $P(n)$ holds, then $P(n + 1)$ holds.



What is wrong with this proof?

Figure 3: The children’s eye colour problem.

The data of our study consist in the students’ written responses to several mathematical tasks and in the notes taken by the third author during the class discussions following the work on the mathematical tasks. Forty-seven students participated overall in the three editions of the course. All of them had already encountered MI in two different courses of the first university year (that is, Analysis and Algebra) and they were used to prove rather complex arithmetical facts by MI, especially in the Algebra course.

RESULTS AND DISCUSSION

CM 1. The first meeting, focused on tasks involving explicitly the use of induction to prove mathematical facts, is aimed at evaluating the technical competences of the sample participating in the study. The collected data confirm (for all the three involved cohorts) that the considered students have a stabilized familiarity with the use of MI. No technical difficulties in complex algebraic manipulations or in explicit writing of the $P(n + 1)$ term in the inductive step emerge.

CM 2 – CM 3. In the second meeting the students have to face Task 1 and Task 2, which are similar at first glance and both belonging to a geometrical context, but actually very different. According to the terminology introduced by Harel (2002), the first task elicits a ‘process pattern generalisation’: all the students arrive quickly to the correct conjecture (the diagonals’ number of a convex polygon of n sides is $n(n-3)/2$) starting by empirical proves on polygons with few sides and then focusing on regularity of the process of adding a single vertex to a given polygons with n vertexes (see Fig. 4). They observe that $n-2$ diagonals can be drawn from the new vertex V_{n+1} , and a new diagonal joins the two vertexes V_1, V_n that in the n -side polygon were consecutive. The generalization of this process pattern coincides with the inductive step proof.

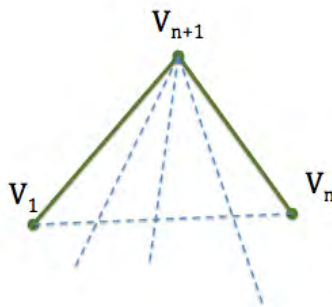


Figure 4: The typical student’s sketch supporting the process pattern generalization.

Task 2, instead, elicits what Harel calls a ‘result pattern generalization’, that is a pattern based on regularity in the results. For n varying from 1 to 5, the result is 2^{n-1} , and this led all the students of the three cohorts to conjecture that the result holds for every n . In fact, they typically stop in their results’ control process exactly when $n = 5$. The interesting aspect is that Task 2 has not a generalizable solution by simple algebraic formulas, since for $n = 6$ the maximum number of pieces is unexpectedly 31, instead of 32. It is only during CM3, that the failure in generating a proof leads to questioning the developed conjecture.

It is clear that the following clause of the didactic contract plays an important role in this situation: if students have to produce a generalization conjecture then the final statement can be express by a simple algebraic formula. On the other side, the interesting phenomenon is the comeback to the empirical arguments after the first failures of the proof by induction. Students spent a lot of time for the case $n = 6$, that is anything but trivial to graphically manage. The students that rightly finding 31 as the maximum number of pieces convince themselves that they get an inaccurate and wrong drawing

or, alternatively, they believe that the disposition of the points on the circumference does not maximize the number of pieces. Then, they try with a bigger drawing to avoid the chords' overlapping or with a new arrangement of the points on the circumference.

This behaviour shows how also for those students who are undoubtedly aware that proving a statement for a finite number of cases is not enough to prove it for infinite cases, the trust in a beautiful conjecture, based on a finite number of arguments, strongly survives.

CM 4 – CM 5. The students' answers to questions a) and b) show an interesting overlap between the shared high confidence in their own capabilities in applying MI and the judgment about MI as an absolutely simple proof method. This phenomenon occurs despite the fact that many students can remember how hard it had been to manage MI in the first years' courses of the master degree. This phenomenon highlights the issue – which we only sketch – regarding the poor attitude of high-achievers in Mathematics in understanding the difficulties that they do not have anymore. To the question “which difficulties can characterize a proof by induction?”, the collected answers refer only to the particular tasks that: i) request some “specific tricks” (for example, specific minorizations) to carry out the inductive step; or ii) require to consider *particular subsequent terms* (for example conjecture involving sequences in which the even terms are defined by formulas different from the ones for odd terms).

The most interesting part of the two last meetings is doubtlessly the discussion about the Children's eye colour problem. This problem is well-known in different formulations, that can all be summarized in the (obviously false) result that every equivalence relation in a countable set is a total relation. The version used in this study is especially meaningful, since it is really evident that the conclusion is false.

There are few students (under the 10% of the whole sample) highlighting the error in the proof, that is, grasping the no validity of the induction step for every n : the underlying reasoning in the presented drawing (Figure 3) clearly does not work for $n = 2$. Indeed, in this case, set S and set D have empty intersection and it is not possible to use the relation's transitivity to conclude that the only element in S is equivalent to the element in D.

The explanations of the students which do not notice the error highlight the knowledge fragility with respect to MI and let the main misconceptions emerge.

In particular, from the analysis of the three cohorts' collected answers, we can classify the explanations regarding what does not work in the presented proof (reminding that the conclusion is surely false) into three different and interesting categories.

The first explanation *contests* the field of application of MI, in terms of: the considered set: “*MI is applicable to numerical sets, not to different sets (such as sets of persons)*”; the considered property: “*MI is not applicable to physical characteristics, only to numerical properties*”; not recognizable order: “*MI is applicable to events' sequences, but in this case the events are not subsequent*”.

These argumentations can be easily demolished by a reformulation of the problem's statement, with $P(n)$, to be proved, becoming: "In every set of n newborns there is only an eye colour".

The second category of explanation regards the conceptual difficulty related to the use of quantifiers in MI. What is contested, in fact, is that the inductive step hypothesis is used in two different sets with n elements: "*In the proof the inductive step is applied to different sets*"; "*In the proof I have to suppose that the property holds for a specific set of n newborns, not for all the sets of n newborns*".

The third category of explanation regards the already discussed conceptual complexity about the difference between inductive hypotheses and thesis. There is the inclination to think that MI presumes that the premise $P(n)$ of the implication $P(n) \rightarrow P(n + 1)$ is true and so to wrongly believe that MI use as hypotheses what has to be proved: "*In the proof of the theorem about the eye colour $P(n)$ is supposed true, but it has not been proved*".

As Fischbein e Engel (1989, p. 282) affirm:

The induction step requires a proof on its own (as a temporarily autonomous implicative statement). The idea that one has to prove an implication $p \rightarrow q$ for which the problem of the objective truth of each of the two components, p and q , is totally irrelevant (in the realm of the induction step) seems to be intuitively unacceptable. This situation is complicated by the fact that the antecedent p includes the theorem to be proven.

CONCLUSIONS

The results of our study lead to some considerations. First of all, we can observe how MI remains a very hard topic, also in the case where there is a shared awareness of the necessity of proving by a deductive process the empirical conjectures and also when students have excellent technical capabilities. More in general, we would like to highlight that, to evaluate the knowledge fragility/solidity, it is necessary to develop and to propose activities able to create 'crisis' moments' for the students, activities which undermine their convictions. Tasks like the Children's eye colour problem seem to be adequate for pursuing this aim in relation to MI. So, it could be interesting to project other activities of this kind, not only for MI, but, more in general for other basic mathematical concepts. Indeed, we are convinced that finding out the knowledge fragility about some issues is surely interesting from the point of view of the research but it is also a crucial point for the educational practice.

References

- Avital, S. & Libeskind, S. (1978). Mathematical induction in the classroom: Didactical and mathematical issues. *Educational Studies in Mathematics*, 9, 429–438. ^[1]_{SEP}
- Cajori, F. (1918). Origin of the name 'mathematical induction', *The American Mathematical Monthly*, 25(5), 197–201.

- Ernest, P. (1984). Mathematical induction: A pedagogical discussion. *Educational Studies in Mathematics*, 15, 173–189.
- Fischbein, E. & Engel, H. (1989). Psychological difficulties in understanding the principle of mathematical induction. In G. Vergnaud, J. Rogalski & M. Artigue (Eds.), *Proc. of the 13th Conference of the IGPME*, vol. 1 (pp. 276–282). Paris: France.
- Harel, G. (2002). The development of mathematical induction as a proof scheme: A model for DNR-based instruction. In S. Campell & R. Zaskis (Eds.), *Learning and teaching number theory: Research in cognition and instruction* (pp. 185–212). New Jersey: Ablex Publishing.
- Movshovitz-Hadar, N. (1991). The falsifiability criterion and refutation by mathematical induction. In F. Furinghetti (ed.), *Proc. of the 15th Conference of the IGPME*, vol. 3 (pp. 41–48), Assisi: Italy.
- Movshovitz-Hadar, N. (1993). The false coin problem, mathematical induction and knowledge fragility. *The Journal of Mathematical Behavior*, 12, 253–268.
- Nardi, E. & Iannone, P. (2003). The rough journey towards a consistent mathematical proof: The $P(n) \rightarrow P(n + 1)$ step in mathematical induction. In A. Gagatsis & S. Papastavridis (Eds.), *Proc. of the 3rd Mediterranean Conference on Mathematical Education* (pp. 621–628). Athens: Hellenic and Cyprus Mathematical Societies.
- National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics.
- Palla, M., Potari, D. & Spyrou, P. (2011). Secondary school students' understanding of mathematical induction: structural characteristics and the process of proof construction. *International Journal of Science and Mathematics Education*, 10: 1023–1045.
- Peano, G. (1889). *Aritmetics Principia. Nova methodo exposita*. Roma: Fratres Bocca Editore.
- Pedemonte, B. (2007). How can the relationship between argumentation and proof be analysed? *Educational Studies in Mathematics*, 66, 23–41. ^[1]_{SEP}
- Poincaré, H. (1906). *La science et l'Hypothese*. Paris: Flamorion.
- Stylianides, G. & Stylianides, A. (2009). Facilitating the transition from empirical arguments to proof. *Journal for Research in Mathematics Education*, 40, 314–352.
- Weber, K. (2010). Mathematics majors' perceptions of conviction, validity, and proof. *Mathematical Thinking and Learning*, 12, 306–336.

AN INNOVATIVE QUALITATIVE VIDEO ANALYSIS INSTRUMENT TO ASSESS THE QUALITY OF POST-SECONDARY ALGEBRA INSTRUCTION

Anne Cawley¹, Irene Duranczyk², Angeliki Mali¹, Vilma Mesa¹, April Ström³, Laura Watkins⁴, Patrick Kimani⁴, and Dexter Lim²

¹University of Michigan, ²University of Minnesota, ³Scottsdale Community College,
⁴Glendale Community College

Evaluating the Quality of Instruction in Post-secondary Mathematics (EQIPM) is a video analysis instrument designed to capture the interactions between instructors, students, and content in post-secondary (tertiary-type B) algebra courses. The instrument evolved from two existing instruments that assess the quality of instruction in K-12 settings—the Mathematical Quality of Instruction (MQI) (Hill, 2014) and the Quality of Instructional Practices in Algebra (QIPA) (Litke, 2015). This paper describes codes from the EQIPM instrument that address three aspects of instruction and presents findings from a pilot study involving 15 class sessions taught by six mathematics instructors. We highlight the potential of the instrument for capturing instances high-quality instruction that leads to meaningful student learning.

THEORETICAL FRAMEWORK WITH SUPPORTING LITERATURE

In this study, we shed light into various aspects of instruction at community colleges through the use of the *Evaluating the Quality of Instruction in Post-secondary Mathematics* (EQIPM) instrument for the analysis of algebra class sessions. Community colleges are U.S. post-secondary institutions covering theoretical foundations with an emphasis on practical, technical, or occupational skills. Community colleges also offer courses in the first two years of university study, so they can be considered as tertiary-type B institutions (OECD, 2017). Algebra courses at community colleges are prerequisites for advanced mathematical courses (e.g., calculus), which are fundamental for all science, technology, engineering, and mathematics degrees (Mesa, Wladis, & Watkins, 2014). Community colleges serve about half of all undergraduate mathematics students in the U.S. and they offer tertiary-type B education to students pursuing fulfillment of general education requirements, continuing education, and certification in various fields. With their open-access philosophy, community college offer affordable post-secondary education to students who could not otherwise pursue a college degree, such as students from low socio-economic backgrounds (Attewell & Domina, 2008). In spite of the importance of community colleges in tertiary education, research on the quality of instruction at those institutions is still scarce (Mesa, Ullah, Mali, & Diaz, 2017). We assume that teaching and learning are phenomena that occur among people enacting different roles—those of instructor or student—aided by resources of different types (e.g., classroom

environment, technology, knowledge) and constrained by specific institutional requirements (e.g., completing preset mathematical content or having a set length of time for each class; see Chazan, Herbst, & Clark, 2016; Cohen, Raudenbush, & Ball, 2003). In this paper, we focus on instruction, one of many activities that can be encompassed within teaching (Chazan et al., 2016), and define it as the interactions that occur between instructors and students with mathematical content (Cohen et al., 2003). The definition of instruction is embedded in the specific content and is fundamental in understanding teaching practices in mathematics. First, we believe that the experiences of instructors and students while interacting with mathematical content have a significant impact on what students are ultimately able to demonstrate in terms of knowledge and understanding. Second, we conjecture that it is possible to identify different levels of qualities of the instruction that are enacted in mathematics classrooms. Empirical evidence from primary classrooms indicates that high quality instruction is positively correlated with student performance on standardized tests (Hill et al., 2008; Hill, Rowan, & Ball, 2005). Sikko and Pepin's (2013) study with first year students in tertiary mathematics in Norway indicated that the most important learning takes place when students are "working with the mathematics themselves, and in particular when they are working together with their [classmates] in small groups" (p. 2452). In our study, we seek to describe meaningful instances of high-quality instruction in order to understand what quality of instruction means in the context of post-secondary education. To this end, we develop the EQIPM instrument that assesses the quality of instruction at community colleges. This paper addresses the following research question: *What is the relationship between the characteristics of mathematical instruction and students' learning gains and performance in community college algebra courses?*

RESEARCH METHODS

In the pilot phase of the larger research study, we video-recorded 15 class sessions in introductory, intermediate, and college algebra classes from three different community colleges in three states during the Fall 2016 semester. The class sessions ranged in duration between 45 and 120 minutes, and were taught by six different instructors (two part-time and four full-time). The mathematical topics taught were linear, rational, and/or exponential equations and functions, chosen because they offer opportunities to analyse instruction on key algebraic concepts and to attend to key ways of thinking about equations and functions (e.g., preservation of solutions after transformations; covariational reasoning), which are foundational algebraic ideas that support more advanced mathematical understanding.

The EQIPM instrument was modelled after the process used by Hill et al. (2008) and Litke (2015) in their development of MQI and QIPA, respectively. Their instruments describe and qualify (via a score from 1 to 5) instructional practices from video-recorded class sessions deemed representative by rating all individual 7.5-minute segments in each class session. EQIPM evolved through several iterations of segment and whole-class session coding, and discussions with a subset of segments. In the final

phase of development, all 141 segments in the data corpus were double-coded by 10 researchers using an earlier version of EQIPM. First, each researcher independently coded three class sessions, then, in pairs, held calibration meetings to discuss codes with a discrepancy in ratings greater than one point.

We theorize that the instrument addresses four dimensions of instruction: (a) Features of the Segment, (b) Quality of Instructor-Student Interaction, (c) Quality of Instructor-Content Interaction, and (d) Quality of Student-Content Interaction (see Figure 1). The instrument also encompasses two crosscutting codes: mathematical errors/imprecisions and mathematical explanations. In this paper, we describe three codes, *Instructors Making Sense of Procedures*, *Student Mathematical Reasoning and Sense-Making*, and *Instructor-Student Continuum of Instruction* to illustrate and show their usefulness in understanding how sense-making in the classroom is advanced while attending to the qualities of instructor-student interaction. We selected these three codes as they correspond to a certain quality of interaction in Figure 1.

EQIPM: Evaluating Quality of Instruction in Post-secondary Mathematics			
Features of Segment	Quality of Student-Content Interaction: Building Understanding through Connections	Quality of Instructor-Content Interaction: Teaching Procedures	Quality of Instructor-Student Interaction: Developing "Student Autonomy"
Mathematics is Focus of Segment	Student Mathematical Reasoning and Sense-Making	Instructors Making Sense of Procedures	Instructor-Student Continuum of Instruction
Procedure Taught	Connecting across Representations	Supporting Procedural Flexibility	Classroom Environment
Modes of Instruction	Situating the Mathematics	Organization in the Presentation of Procedures	Inquiry/Exploration
	Mathematical Errors and Imprecisions in Content or Language		
	Mathematical Explanations		

Figure 1: Dimensions and codes for the EQIPM instrument.

The *Quality of Instructional Practices in Algebra* (QIPA) instrument defined a procedure as "instructions for completing a mathematical algorithm or task" (Litke, 2015, p. 160). With *Instructors Making Sense of Procedures*, a code originally from QIPA, we sought to identify ways in which instructors used mathematical relationships or properties to motivate a mathematical procedure. Such work includes activities that attend to features such as the type of solution generated by a procedure and its interpretation or to the conditions of the problem that may suggest what procedure to apply and where in the process to use it. This code seeks to capture all instances in which instructors make salient mathematical properties, relationships, and connections embedded in a specific mathematical procedure. Making sense of procedures helps students to understand the underlying logic of the procedure and how to get from one step to the other rather than merely reproducing the work from a textbook example. We

hypothesize that when the instructors engage in sense-making with procedures, their students have opportunities to engage more substantively with the mathematics. In our coding, only three segments did not attend to teaching a procedure.

Student Mathematical Reasoning and Sense-Making is characterized by how students are participating in the classroom to deepen their understanding. Mathematical reasoning involves drawing logical conclusions, providing conjectures, counter-claims, reasoning and engaging cognitively in problem solving. Sense-making involves “developing an understanding of a situation, context, or concept by connecting it with other knowledge” (NCTM, 2009). There must be clear evidence of students engaging in such practices through verbal utterance(s) and/or through written work.

When we consider the code *Instructor-Student Continuum of Instruction*, we are capturing the degree to which students make evident their investment in their own learning and development of mathematical understanding by expressing ideas that advance their learning. Evidence of their investment comes from their verbal or written contributions about significant development of mathematical ideas.

The evidence in each segment of the class sessions was rated on a scale of 1 to 5. A rating of 1 in the *Instructor Making Sense of Procedures* code indicates that the instructor does not engage in sense-making while teaching a procedure; a 5 indicates that the instructor consistently engages in sense-making throughout the segment. A rating of 1 in the *Student Reasoning and Sense-Making* code indicates that students do not participate in mathematical reasoning or sense-making; a 5 indicates that students engage in sustained reasoning and sense-making such that their ideas contribute to the development of the mathematics covered. A rating of 1 in *Instructor-Student Continuum* code indicates that the instructor is the only person contributing to development of the mathematics; a 5 indicates that students are mainly advancing the development of the mathematics.

FINDINGS

Table 1 shows the distributions for *Instructors Making Sense of Procedures*, *Student Mathematical Reasoning and Sense-Making*, and *Instructor-Student Continuum*. In the pilot sample of class sessions, there were no segments with a rating of 5 on Student Mathematical Reasoning and Sense-Making and Instructor-Student Continuum of Instruction. We discuss each of the three codes in more detail with examples.

Instructor Making Sense of Procedures

Out of 138 segments in which a procedure was taught, we only identified one in which no instructor sense-making was present; 59 segments (43%) had a rating of 3, and 55 segments (39%) had a rating of 4 or 5 (30% and 9%, respectively) (see Table 1). Thus, across the class sessions, we were able to provide evidence for each of the ratings, which suggests that the instrument allows for differentiation of the role of instructor sense-making in the classroom. In most cases, we can say that instructors were making

a genuine effort to assist students in making sense of the procedures taught during the video-recorded sessions.

EQIPM Ratings	Instructor Making Sense of Procedures (n=138)	Student Mathematical Reasoning & Sense-Making (n=141)	Instructor-Student Continuum (n=138)
1	12%	36%	35%
2	15%	45%	36%
3	46%	23%	20%
4	17%	11%	4%
5	10%	0%	0%
Total	100%	100%	100%

Table 2: Distribution of ratings across three codes of EQIPM.

For example, in a class session on linear functions, instructor 0613 presented a word problem in which students were asked to model the value of a copy machine, v , as a function of time, x . The instructor asked students to consider how to write a linear function v as a function of x . Students contributed three answers: $f(x)$, $f(v)$, and $v(x)$. The instructor reasoned through all three responses using the information in the problem to make sense of the appropriate way to notate the function as $v(x)$ (0613_L1, 2016, 26:22). Later in the segment, the instructor asked, “What does the value of \$120,000 mean in this problem? What does a slope of negative 12,000 mean in this problem?” (0613_L1, 2016, 29:00). The subsequent conversation detailed the meaning of the value of the vertical intercept and the slope for this specific context. In this example, the instructor covered more than one problem, and this problem lasted for 3 of the 7.5 minutes. For this reason, the segment was rated as a 3.

In contrast to the previous example, Instructor 0112 demonstrated sense-making that was rated as a 5 when working with a growth problem modelled by $y = 3(2)^x$. The instructor asked students to think about the meaning of the general equation $y = ab^x$ with a concrete example that used paper folding to demonstrate the meaning of doubling and how to model this pattern algebraically, where x was the number of times a piece of paper was folded in half and y was the number of layers generated by the folds. Students were required to reason that one fold created two layers, two folds created four layers, three folds created eight layers, and so on (0112_E1, 2016, 30:00). This segment was rated as a 5 because sense-making was the focus of the segment due to the instructor’s continued prompting of students to think about the resulting pattern within the context.

Student Mathematical Reasoning and Sense-Making

We did not find much evidence in the 141 segments of student mathematical sense-making or reasoning. Approximately 66% of the ratings were 1 or 2, indicating

that students rarely contributed to the class session orally or in writing. Approximately half of the instructors devoted class session time to bring all students to the whiteboards at the same time so the students could solve exercises. In only 11% of segments evidence was rated as 4. In a class session from instructor 0412, students explained their solution paths to solving a rational equation to their peers while also comparing their paths to the instructor's work. Since there were times in this segment students led the discussion the code was rated as 4 (0412_R1, 2016, 1:07:30). We also found a handful of segments with evidence of students' written work at the boards.

In a different analysis that investigated the quality of questions during class sessions (Mesa et al., 2017), we identified missed opportunities for instructors to advance student sense-making through the use of inquiry approaches. There were instructor questions (average of 5 questions every 4 minutes) that could elicit student engagement in mathematical thinking; however, the minimal wait time reduced the opportunity for students to verbally engage in mathematical reasoning or sense-making. Average length of student responses varied from one to eight words with the latter being present in highly interactive class sessions. Class size does not play a role in the reduced student participation, because the number of students in these class sessions ranged from 25 to 35. Student hesitation to contribute meaningfully to the class sessions may be attributed instead to the classroom environment, as students may have been socialized to wait for the instructor to do all the mathematical work.

Instructor-Student Continuum

As can be seen from Table 1, the instructor was the main person who contributed to the development of mathematical ideas (74% of segments rated as a 1 or 2). For example, for a class session on exponents, Instructor 0313 received a rating of 1 for all 14 segments of class; students did not contribute to the class session in any way. In contrast, in a class session on graphing rational functions, Instructor 0112 discussed the behaviour of the function $A(x) = (10 + .10x)/x$ as the value x gets larger. After one student said that the output would get smaller as x gets bigger, the instructor asked all students to find a space at the whiteboard and to graph the function using a table and the x -values of 1, 5, 10, and another large number of their choosing. This segment received a rating of 4 because the students explored and demonstrated graphically what happens to the function, as the input gets large (0112_R1, 2016, 7:30).

There were not many segments in which students contributed to developing an understanding of abstract concepts, formulas, notation, definitions, concrete examples, pictorial examples, and rules/properties *with minimal contribution from the instructor*. Overall, we see that the instructor-student continuum verges towards an instructor-centred environment, where instructors are the ones contributing to developing understanding of abstract concepts, formulas, notation, definitions, examples, and rules/properties. This is congruous with our findings that students do not regularly engage in meaningful reasoning and sense-making of mathematical content during the class session.

CONCLUSIONS

We have illustrated three codes from the video analysis instrument that attends to what we believe are different components of instruction. Together these codes help describe the extent to which the instructors support students in making sense of the mathematics and present mathematical procedures with attention to key features of the procedures, as well as how both instructors and students contribute to the development of mathematical ideas. In the sample of class sessions we analysed, we were able to identify variation and other patterns that help to fully describe the quality of instruction that students experience.

Studying instruction can provide focused, meaningful professional development for tertiary mathematics instructors (Winsløw, Gueudet, Hochmuth, & Nardi, 2017). From the analysis of pilot data it appears there is still much work to be done so that post-secondary instructors can provide opportunities for students to make meaning and sense of the mathematics. Using segments rated at 4 and 5 in an EQIPM code, we plan to use evidence from our data to inform professional development workshops and discuss different ways instructors engage students with the mathematics in meaningful ways.

Acknowledgment

Funding for this work was provided by the National Science Foundation Award EHR #1561436. Any opinions, findings, and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

References

- Attewell, P., & Domina, T. (2008). Raising the bar: Curricular intensity and academic performance. *Educational Evaluation and Policy Analysis*, 30(1), 51-71.
doi:10.3102/0162373707313409
- Chazan, D., Herbst, P., & Clark, L. (2016). Research on the teaching of mathematics: A call to theorize the role of society and schooling in mathematics instruction. In D. H. Gitomer & C. A. Bell (Eds.), *Handbook of research on teaching* (5th ed., pp. 1039-1097). Washington DC: American Educational Research Association.
- Cohen, D. K., Raudenbush, S. W., & Ball, D. L. (2003). Resources, instruction, and research. *Educational Evaluation and Policy Analysis*, 25, 119-142.
- Hill, H. C. (2014). *Mathematical Quality of Instruction (MQI)*. Retrieved from Harvard, MA: [https://hu.sharepoint.com/sites/GSE-CEPR/MQI-Training/Shared Documents/MQI 4-Point.pdf](https://hu.sharepoint.com/sites/GSE-CEPR/MQI-Training/Shared%20Documents/MQI%204-Point.pdf)
- Hill, H. C., Blunk, M., Charalambous, C., Lewis, J., Phelps, G., Sleep, L., & Ball, D. L. (2008). Mathematical Knowledge for Teaching and the Mathematical Quality of Instruction: An exploratory study. *Cognition and Instruction*, 26(4), 430-511.

- Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, 42, 371-406.
- Litke, E. G. (2015). *The state of the gate: A description of instructional practice in algebra in five urban districts*, (PhD). Harvard University, Cambridge, MA.
- Mesa, V., Ullah, A., Mali, A., & Diaz, L. (2017). *Authenticity of instructor and student questions in algebra instruction at community colleges: An exploratory study*. University of Michigan, Ann Arbor, MI.
- Mesa, V., Wladis, C., & Watkins, L. (2014). Research problems in community college mathematics education: Testing the boundaries of K-12 research. *Journal for Research in Mathematics Education*, 45(2), 173-193.
- OECD. (2017). *Education at a glance 2017: OECD Indicators*. Paris: OECD Publishing.
- Sikko, S. A., & Pepin, B. (2013). Students' perceptions of how they learn best in higher education mathematics courses. In B. Ubuz, C. Haser, & M. A. Mariotti (Eds.), *Proceedings of the 8th Congress of the European Society for Research in Mathematics Education* (pp. 2446-2455). Ankara, Turkey: Middle East Technical University and ERME.
- Winsløw, C., Gueudet, G., Hochmuth, R., & Nardi, E. (2017). *Research on university mathematics education*. Paper presented at the 10th Congress of the European Society for Research in Mathematics Education, Dublin, Ireland: DCU Institute of Education and ERME.

THE PREDICTIVE ROLE OF DIFFERENT REASONING FORMS TO STUDENTS' EARLY ALGEBRAIC THINKING ABILITIES

Maria Chimoni and Demetra Pitta-Pantazi

Department of Education, University of Cyprus

The development of early algebraic thinking has become a goal for many curricula, yet important questions remain regarding the nature of students' early algebraic thinking. This study aims to clarify the relationship between early algebraic thinking and different reasoning forms. To this end, 9, 10, 11, and 12-year-old students were tested in three tests: (i) an algebraic thinking test which involved a range of early algebraic tasks, (ii) the Naglieri Non-Verbal Ability Test (NNAT) which allows the examination of analogical reasoning, inductive reasoning, and spatial reasoning, and (iii) a deductive reasoning test. The quantitative analysis of the data yielded insights into the predictive role of different reasoning forms on students' early algebraic thinking abilities in the age groups under investigation.

INTRODUCTION

In the last two decades, numerous studies examined the integration of early algebra in elementary mathematics (e.g. Kaput, 2008). At the core of this research two paramount subjects can be identified: (i) the characteristics of early algebraic thinking, and (ii) students' ability and readiness to develop early algebraic thinking from the beginning of their mathematical learning. Regarding the first subject, several researchers seem to agree that a crucial characteristic of early algebraic thinking is structural awareness in all domains of mathematics (e.g. Blanton et al., 2011). Regarding the second subject, empirical data advocated that young students are able to develop early algebraic thinking that is not exclusively based on alphanumeric symbolism within a variety of activities. For example, it has been shown that elementary school students are able to treat arithmetic operations as functions of one or two variables (e.g. $a+5$, $a+b$), rather than simply produce computations (Carragher & Schliemann, 2015). They are also able to describe generalizations about the terms of a sequence and their position, rather than simply extend the sequence based on repeated counting (Warren & Cooper, 2008).

Despite the significant advances in the field, many research questions remain open. Further research is still needed regarding the constituent parts of algebraic thinking and the way in which algebraic thinking emerges in young students and becomes more robust and formal over time (Radford, 2012). The current study aims to clarify the relationship between early algebraic thinking and reasoning forms. Specifically, this study examines whether different reasoning forms predict students' early algebraic thinking abilities, and as a consequence support its emergence and advancement.

THEORETICAL FRAMEWORK

Content strands, concepts, and processes characterizing algebraic thinking

The notion of early algebraic thinking has been approached through various perspectives which focused on the identification of content strands that involve algebraic thinking, the specification of algebraic concepts and objects, and the description of important algebraic processes. According to Kaput (2008), there are three core algebra content strands from K-12: (i) generalized arithmetic which refers to the identification of structure and relationships regarding number properties and properties of operations, (ii) functional thinking which refers to the expression of relationships between co-varying quantities, and (iii) the application of modeling languages which is related to the expression of regularities that are implicitly presented through problem situations. These content strands are associated with a variety of algebraic concepts and objects, such as equality and the equals sign, variable, equation, expression, tables, graphs, diagrams, and alphanumeric symbols (Kieran et al. 2016). Blanton et al. (2011) pointed to the significant role of several processes while students deal with early algebraic tasks, such as noticing, generalizing, representing, and justifying. All of these processes facilitate (i) the search for similarities and differences and (ii) validation which aims to change the epistemic value of a mathematical narrative (e.g. from likely to more likely) (Jeanotte & Kieran, 2017). Especially the process of generalization is considered a core algebraic process that is related to students' ability to notice "the same and the different" (Radford, 2008). For example, pattern generalization arises when students notice a commonality among particular terms, extend this commonality to all subsequent terms, and finally extract a direct expression that allows the representation of any term of the pattern (Radford, 2008).

Reasoning forms related to algebraic thinking

Jeanotte and Kieran (2017) suggested that both processes and reasoning forms are present while students deal with mathematical tasks and are related dialectically. In this line of thought, the specification of forms of reasoning that are related to early algebraic thinking is essential in order to better depict its nature and characteristics. A number of research studies investigated this idea and supported that different reasoning forms, such as analogical, inductive, deductive, and spatial reasoning, lead students to different types of inference while dealing with algebraic tasks. Analogical reasoning enables the extraction of inferences regarding similarity and difference relations. According to research from the field of psychology, analogical reasoning transfers meaning about relations between objects or concepts, rather than individual objects or representations, while children are sensitive to similarity and difference relations from infancy (Demetriou, Spanoudis & Mouyi, 2011). English and Sharry (1998) found that analogical reasoning is a mental source that facilitates students to uncover the structure of algebraic expressions. Inductive reasoning develops in three main levels during the age span of 6 to 12 years old. At the first level students can identify patterns and formulate generalizations based on a single dimension or relation. At the second level, inductive reasoning enables the identification of hidden or implicit relations. Finally, at

the third level, inductive reasoning is based on theoretical assumptions (Mouyi, 2008; in Demetriou, Spanoudis & Mouyi, 2011). Jeanotte and Kieran (2017) also pointed to the association of inductive reasoning to the process of generalization. Rivera and Becker (2007) showed that in pattern activities, inductive reasoning triggers the formulation of an expression that represents the general structure of the pattern. Deductive reasoning results to inferences where meaning is transferred from general premises to specific premises and enables the systematic search for the relations suggested by the premises of an argument; this ability appears for the first time around the age of 5 to 6 years old and progresses until the age of 11 to 12 years old (Demetriou, Spanoudis & Mouyi, 2011). Deductive reasoning is mainly linked to the process of proving (Jeanotte & Kieran, 2017). Ellis (2007) suggested that deductive reasoning enables students to achieve more accurate generalizations. For example, after the construction of a general pattern rule, students might attempt to establish their generalization through increasingly deductive justifications. Spatial reasoning enables 9 to 10-year-old students to represent familiar persons and objects, and these abilities are improved by the age of 11 to 12 years old where students become able to imagine non-real objects (Demetriou, Spanoudis & Mouyi, 2008). Previous research has shown that spatial abilities enable students in manipulating visual-spatial representations mentally, such as graphs (Tolar et al., 2009). Mason and Sutherland (2002) suggested that a basic feature of algebraic thinking is the formulation of generalizations in both numerical and spatial contexts. Taking into consideration these findings, it seems that various reasoning forms occur while students study mathematical structure and relationships, and they are linked to students' abilities to develop algebraic processes and understand basic concepts.

AIM OF THE STUDY

This study aims to further investigate the relationship between different reasoning forms and early algebraic thinking abilities through empirical data. Specifically, we hypothesize that different reasoning forms might predict students' achievement in solving different types of early algebraic tasks at different grades.

METHODOLOGY

Participants

The participants were 684 students that were selected by convenience. The students were divided in four age groups: 170 were students of Grade 4 (9-year-olds), 164 were students of Grade 5 (10-year-olds), 184 were students of Grade 6 (11-year-olds), and 166 were students from Grade 7 (13-year-olds).

Measurement tools

The test on early algebraic thinking consisted of 18 tasks which were categorized according to Kaput's (2008) framework of three core algebra content strands. Table 1 presents examples of the tasks. The first group (generalized arithmetic) involved tasks that required the use of properties of numbers and operations, and solution of equations

and inequalities. The second group of tasks (functional thinking) involved the identification of the n^{th} term in patterns, the interpretation of graphs, and description of co-variational relationships. The third group of tasks (modeling) required the generalization of relationships in problem contexts. The internal consistency for the algebraic thinking test was satisfactory (Cronbach's $\alpha = 0.88$).




Content strand	Concept	Example task	Algebraic Processes
Generalized Arithmetic	Properties of numbers	Is the sum $245676 + 535731$ an odd or even number? Explain your answer.	Generalizing the property of adding an even and an odd number. Justifying whether the sum will be odd or even.
Functional thinking	Pattern, Variables,	   Fig.1 Fig.2 Fig.3 Bill is arranging squares. How many squares there will be in the 16th figure?	Generalizing the relationship between the number of squares in each term with the position of the term. Formulate an expression. Justifying the number of squares in the 16th figure.
Modeling languages	Variables, Equation	Joanna will take computers lesson twice a week. Which is the best offer? Justify your answer. A: €8 for each lesson B: €50 for the first 5 lessons of the month and then €4 for every additional lesson	Generalizing the relationship between an independent variable (number of lessons) and a dependent variable (cost of lessons). Representing the relationship using equations. Justifying the best offer.

Table 1: Examples of tasks included in the early algebraic thinking test

In order to investigate reasoning forms, we used the Naglieri Non Verbal Ability Test (NNAT), which measures cognitive ability (Naglieri, 2003). The NNAT is a matrix reasoning type of exam, which contains patterns formed by shapes that are organized into designs. All tasks are multiple choice and students are asked to choose the answer that completes the pattern. The NNAT includes different categories of questions that reflect different reasoning forms: (a) Reasoning by Analogy questions which require students to identify relationships between a series of drawings by focusing on their similarities and differences; (b) Serial Reasoning questions which require students to conceive the way a set of drawings are placed in a series, horizontally and vertically, and then generalize the rule that guides the placement of drawings in another series. Serial reasoning shares common features with inductive reasoning; (c) Spatial visualization questions which request students to visualize the way a shape will look after a transformation or the way two shapes will look after they are combined. The internal consistency for the NNAT test was satisfactory (Cronbach's $\alpha = 0.84$).

Moreover, we used a deductive reasoning test, which involved tasks that were adapted from a previous research conducted by Watters and English (1995). These researchers examined the relationship among scientific problem solving and deductive reasoning in students of approximately the same age as the participants in the current study. The tasks represented 10 syllogisms. Students had to analyze premises that described formal truth relationships, without reference to the empirical or practical truth value of the premises and extract a logical result or consequence. The internal consistency for this test was satisfactory (Cronbach's $\alpha = 0.79$).

Analysis

Descriptive Statistics and Multivariate Analysis of Variance (MANOVA) were conducted to describe students' performance in the early algebraic thinking test and possible differences in the performance of students from different grades. Multiple Regression Analysis was also conducted since this kind of analysis informs on the way one or more independent variables predict the variance of a dependent variable. Specifically, for each age group, a regression model was investigated to define which reasoning forms predict early algebraic thinking abilities. All assumptions for multiple regression analyses (e.g., multicollinearity, $VIF < 10$ and $\text{tolerance} > 0.20$; homoscedasticity, linearity, independent errors, $1 < \text{Durbin Watson} < 3$) were met.

RESULTS

Table 2 presents the means and standard deviations of students' performance in the three groups of early algebraic tasks. The highest mean score of fourth graders appears in generalized arithmetic tasks. Students of both Grades 5 and 6 have almost equal mean scores in generalized arithmetic and functional thinking tasks.

	Grade 4 (<i>n</i> =170)		Grade 5 (<i>n</i> =164)		Grade 6 (<i>n</i> =184)		Grade 7 (<i>n</i> =166)	
	M	SD	M	SD	M	SD	M	SD
Generalized Arithmetic	.500	.262	.556	.264	.582	.265	.669	.209
Functional Thinking	.373	.323	.514	.337	.604	.325	.609	.342
Modelling	.254	.258	.364	.320	.407	.255	.513	.257

Table 2: Students' performance in early algebraic tasks test by grade level

Their lowest mean score appears in modeling tasks. The highest mean score of seventh graders appears in generalized arithmetic tasks, followed by their mean scores in functional thinking, and then in modeling tasks. The results also show that students' mean scores in all types of tasks increase from grade to grade. MANOVA analysis indicated that these differences are statistically significant (Pillai's $F=1148.548$, $p < .01$). Table 3 presents the results of Multiple Regression Analysis, where the performance of students in each grade in the early algebraic thinking test is explained by their performance in the questions corresponding to the four reasoning forms. The B

coefficients show the change in the outcome resulting from a unit change in the predictor, whereas the beta coefficients (β) are the standardized version of the B coefficients. As the R-squares show, in each grade level, a percentage of 31-36% of the variance in achievement in early algebraic thinking can be explained by the independent variables (analogical, spatial, inductive, and deductive reasoning).

	Grade 4			Grade 5			Grade 6			Grade 7		
Early Algebraic thinking	B	SE	Beta	B	SE	Beta	B	SE	Beta	B	SE	Beta
Analogical reasoning	,240	,083	,266**	-,002	,053	-,002	,111	,066	,138	,096	,095	,099
Spatial reasoning	,089	,097	,098	,196	,080	,209*	,306	,089	,302**	,292	,107	,261**
Inductive Reasoning	,199	,100	,211*	,364	,085	,375**	,153	,077	,173*	,136	,100	,144*
Deductive Reasoning	2,174	,608	,226**	,950	,604	,112*	2,164	,700	,215**	2,223	,634	,268**
	R ² =,361			R ² =,309			R ² =,354			R ² =,335		

*p<.05 , **p<.01

Table 3: Regression analysis of students' performance in the four types of reasoning forms with dependent variable the performance in early algebraic thinking by grade

Although in all age groups reasoning forms explain a respectable proportion of the variance in achievement in early algebraic thinking, the combination of reasoning forms that appear as statistically significant predictors is not the same for all age groups. Specifically, different model equations are generated in order to describe statistical associations between early algebraic thinking and different reasoning forms in each grade. Inductive and deductive reasoning appear as significant predictors of early algebraic thinking abilities in Grade 4 ($\beta=.211$ and $\beta=.226$ respectively), Grade 5 ($\beta=.375$ and $\beta=.112$ respectively), Grade 6 ($\beta=.173$ and $\beta=.215$ respectively), and Grade 7 ($\beta=.144$ and $\beta=.268$ respectively). However, analogical reasoning appears as a significant predictor of early algebraic thinking only in Grade 4 ($\beta=.375$) and disappears from the regression models in Grades 5, 6, and 7. Furthermore, spatial reasoning appears as a predictor of early algebraic thinking in Grades 5, 6, and 7 ($\beta=.209$, $\beta=.302$, and $\beta=.261$ respectively).

DISCUSSION AND CONCLUSIONS

In this paper we investigated the relationship of early algebraic thinking with reasoning forms. The results indicate the predictive role of various reasoning forms on early algebraic thinking abilities in all of the grade levels under investigation. Moreover, different models are described in each grade level, implying variations of the reasoning forms that predict algebraic thinking from grade to grade. In grade 4, analogical, inductive, and deductive reasoning seem to predict students' early algebraic thinking abilities. Specifically, analogical reasoning was a significant predictor of students'

performance in the algebraic thinking test only in Grade 4. It seems that analogical reasoning supported fourth graders' efforts to solve some types of early algebraic tasks. For example, in generalized arithmetic tasks, analogical reasoning might have facilitated them to transfer pre-acquired arithmetical knowledge to uncover the relational meaning of concepts like equations, equals sign, properties of numbers and operations. As aforementioned, analogical reasoning develops from infancy and allows children to transfer meaning about similarity and difference relations (Demetriou, Spanoudis & Mouyi, 2011). Previous studies (e.g. English & Sharry, 1998) also showed the effect of analogical reasoning on students' abilities for examining the structure of algebraic expressions. However, in Grade 5 analogical reasoning does not appear as a statistically significant predictor of early algebraic thinking. At the same time, while inductive and deductive reasoning continue to appear as predictors of algebraic thinking, spatial reasoning emerged at Grade 5 as a predictor for the first time. This structure of the model is repeated for Grades 6 and 7. This finding is aligned with findings from psychological research that indicate the advancement of spatial reasoning by the age of 11 to 12 years old (Demetriou, Spanoudis & Mouyi, 2011). This result might explain the fact that as students become older, they become more successful in solving functional thinking and modeling tasks which are linked to concepts, such as variable and function, and objects, such as tables and graphs. This reasoning form seems to facilitate students to examine figural patterns, tables, and graphs where students have to identify both numerical and spatial relationships (Mason & Sutherland, 2002). Inductive reasoning appeared in the regression models of all grade levels. This type of reasoning might have facilitated younger students in extending sequences based on one single relation (Demetriou, Spanoudis & Mouyi, 2011). As students grow older, inductive reasoning seems to enable the identification of implicit relations in patterns based on two dimensions (e.g. the term and its position), while in modeling tasks inductive reasoning allows the extraction of implicit relationships between variables described in problem situations. Previous findings also pointed to the significance of inductive reasoning in generalization activities with patterns (e.g. Rivera & Becker, 2007). Deductive reasoning was also a significant predictor of early algebraic thinking in all grades. In the context of the current study, deductive reasoning seems to have facilitated students in building viable generalizations regarding co-variational relationships (Ellis, 2007), analyzing problem situations which described quantitative relations, testing conjectures, and justifying. The overarching results of the present study depict the profound and diverse connections of early algebraic thinking with various reasoning forms. This relationship seems to vary with age, and its variations seem to be reflected in students' early algebraic thinking abilities. It is implied that, with age, there is an ongoing shift from forms of early algebraic thinking that are more intuitive to forms that are more formal and abstract. Future longitudinal studies might further investigate this issue and examine whether changes in the relationship between early algebraic thinking and reasoning forms are expressed through particular changes in the quality of students' understanding of algebraic concepts and application of algebraic processes.

References

- Blanton, M., Levi, L., Crites, T., & Dougherty, B. (2011). Developing essential understanding of algebraic thinking for teaching mathematics in grades 3-5. In B.J. Dougherty & R.M. Zbieck (Eds), *Essential understandings series*. National Council of Teachers of mathematics: Reston, VA.
- Carraher, D.W., & Schliemann, A.D. (2015). Powerful ideas in elementary school mathematics. In L.D. English & D. Kirshner (Eds), *Handbook of International research in mathematics education* (3rd ed., pp. 191-218). New York: Taylor & Francis.
- Demetriou, A., Spanoudis, G. & Mouyi, A. (2011). Educating the Developing Mind: Towards an Overarching Paradigm. *Educational Psychology Review*, 23(4).
- Ellis, A.B. (2007). Connections between generalizing and justifying students' reasoning with linear relationships. *Journal for Research in Mathematics Education*, 38(3), 194-229.
- English, L.D. & Sharry, P.V. (1996). Analogical reasoning and the development of algebraic abstraction. *Educational Studies in Mathematics*, 30(2), 135-157.
- Jeannotte, D., & Kieran, C. (2017). A conceptual model of mathematical reasoning for school mathematics. *Educational Studies in Mathematics*, 96(1), 1-16.
- Kaput, J. (2008). What is algebra? What is algebraic reasoning? In J. Kaput, D. W. Carraher, & M.L. Blanton (Eds), *Algebra in the early grades* (pp. 5-17). New York: Routledge.
- Kieran, C., Pang, J., Schifter, D., & Ng, S.F. (2016). Early algebra: Research into its nature, its learning, its teaching. In Kaiser, G., *ICME-13 Topical Surveys*. Springer Open.
- Mason, J., & Sutherland, R. (2002). *Key aspects of teaching algebra in schools*. UK: QCA.
- Naglieri J.A. (2003). *Naglieri Nonverbal Ability Tests*. In R.S. McCallum (Eds) *Handbook of Nonverbal Assessment*. Springer, Boston, MA.
- Radford, L. (2008). Iconicity and contraction: A semiotic investigation of forms of algebraic generalizations of patterns in different contexts. *ZDM*, 40(1), 83-96.
- Radford, L. (2012). On the development of early algebraic thinking. *PNA*, 6(4), 117-133.
- Rivera, F., & Becker, J. (2007). Abduction – Induction (generalization) processes of elementary majors on figural patterns of algebra. *Journal of Mathematical Behavior*, 26 (2), 140-155.
- Tolar, T.D., Lederberg, A. R., & Fletcher, J. M. (2009). A structural model of algebra achievement: Computational fluency and spatial visualisation as mediators of the effect of working memory on algebra achievement. *Educational Psychology*, 29(2), 239–266.
- Warren, E., & Cooper, T. (2008). Generalising the pattern rule for visual growth patterns: Actions that support 8 year olds' thinking. *Educational Studies in Mathematics*, 67(2), 171–185.
- Watters, J.J. & English, L.D. (1995). Children's application of simultaneous and successive processing in inductive and deductive reasoning problems: Implications for developing scientific reasoning skills. *Journal for Research in Science Teaching*, 32(7), 699-714.

PEER OBSERVATION AS A TOOL TO FACILITATE MATHEMATICS TEACHERS' SELF-REFLECTION IN A PROFESSIONAL DEVELOPMENT INTERVENTION

Brantina Chirinda and Patrick Barmby

University of the Witwatersrand

The paper reports on how peer observation of teaching (POOT) was used as a tool to assist grade 9 South African mathematics teachers' self-reflection in a classroom-based design research project. Two grade 9 mathematics teachers at a public secondary school in Gauteng, South Africa acted as observer and observed with reciprocation of roles after each observation. Findings indicated that the observer developed increased awareness of how mathematical problem-solving was being taught and learned. The observed valued feedback from the observer and became aware of their mathematical problem-solving pedagogy and revised their own teaching strategies. The authors advocate that there are clear benefits when POOT is used as a professional development strategy to support mathematics teachers' self-reflection.

INTRODUCTION AND BACKGROUND

The study was conducted in the second cycle of a three-cycle design based research (DBR) project. The focus of the large project was to design and evaluate a professional development (PD) intervention for teachers' mathematical problem-solving pedagogy. In the second cycle of the project we decided to use POOT as a PD strategy to support participant teachers' self-reflection, because of the usefulness that had emerged from teachers observing practice in videos in the first cycle (Chirinda & Barmby, 2017). In the first cycle, participant teachers were shown videos from the USA with high school mathematics teachers teaching problem-solving.

POOT is teachers observing each other's practice and learning from one another. Teachers provide descriptive feedback to one another without necessarily judging each other (Chamberlain, D'Artrey & Rowe, 2011), the overall goal being to advance teaching (Hendry & Oliver, 2012) through reflective practice. Chamberlain et al. (2011) state that peer observation is intended for PD and frequently for quality assurance and reward, for example in the Integrated Quality Management System (IQMS). IQMS (Department of Basic Education, 2010) is a quality and performance instrument that was introduced in 2003 to measure and improve the quality of teaching and learning in South African schools. POOT has been used successfully at universities in Asia and Australia (Bell & Cooper, 2013; Bolt & Atkinson, 2010; Engin & Priest, 2014; Hendry & Oliver, 2012). However, not much is documented in the literature on POOT in South African institutions, although it is a component of the IQMS. There is, therefore, a dearth of research in South Africa on how POOT can be used to

support mathematics teachers' PD, hence the significance of the study. The study contributes to the body of knowledge on POOT, teachers' self-reflection and PD to support teachers in the teaching of mathematical problem-solving. The study was guided by the following research questions:

- What are the benefits of peer observation in a professional development intervention for teachers' mathematical problem-solving pedagogy?
- To what extent did peer observation augment or hinder the development of teachers' mathematical problem-solving pedagogy?

THEORETICAL PERSPECTIVE

This study was grounded in teachers' self-reflections. Wilson, Shulman and Richert (1987) define reflection as what the teacher does when he or she looks back at the teaching and learning that has occurred and reconstructs the events, emotions and experiences of the situation. We carried out semi-structured reflective interviews with participant teachers who we required to reflect on their experiences of observing and being observed and the feedback received during the post-observation discussion.

METHODOLOGY

Research participants and data sources

Two grade 9 mathematics teachers aged between 25-30 years participated in this qualitative study. Pseudonyms were assigned and they were known as Mr. M and Ms. N. Mr. M was an experienced teacher with a diploma in mathematics education and six years' experience teaching high school mathematics. Ms. N was a novice with a master's degree in mathematics education and only one year's experience of teaching high school mathematics. Pairing experienced and a novice teacher is beneficial since it holds the promise of merging expertise and knowledge transfer (Bolt & Atkinson, 2010). Data was collected through classroom observations, semi-structured reflective interviews, teachers' field notes from the observations, and audio recordings from the post-observation feedback sessions and semi-structured reflective interviews.

The intervention

POOT is a powerful vehicle for teacher professional development (Bell & Mladenovic, 2008) that focuses on teachers' unique needs. It provides the observer and the observed with the opportunity to learn from one another's practice and to offer constructive feedback to one another (Bell, 2002). Bell and Mladenovic (2008) note that POOT is beneficial if conducted in well-planned and supportive contexts. In this study, the POOT process was appropriately structured and carefully thought out. During the first PD workshop, we explained and clarified the process and objectives of POOT to the participant teachers. We clarified to the teachers that with POOT, the feedback is non-judgmental, non-evaluative, descriptive, formative, and solely for teachers' mathematical problem-solving pedagogy advancement (Bell & Mladenovic, 2008). As suggested by Engin and Priest (2014), we explained to the participant teachers that the

observer was rather supposed to evaluate, self-correct and reflect on their own mathematical problem-solving teaching in the light of the observation.

Participant teachers understood that they were carrying out the observations neither to criticize nor to discover if they had more advanced mathematical problem-solving pedagogy than the observed. Bell and Cooper (2013) recommend that the relationship between the observer and the observed should be that of critical friends, where feedback is given as a dialogue and not a judgment, and this we encouraged with the teachers. It is essential that teachers should freely participate in the POOT process (Engin & Priest, 2014) and not feel that the process is one more aspect of the PD intervention they have to do. To avoid this we explained to the teachers that the process was an enriching part of the PD intervention and not an evaluative process. After understanding the process the two teachers agreed to participate in the POOT programme. A buddy system was employed and the teachers acted as the observer and observed with reciprocation of roles after each observation. Each teacher observed or was observed once in two months in a 6-month period. At the end of the 6-month cycle the teachers had been observed or had observed three times. The POOT process involved four phases: a pre-observation discussion, observation, post-observation feedback and reflection (Bell, 2002).

Pre-observation discussion

Participant educators met for a short time before each observation episode to establish objectives for each observation and to agree on the date and time. Both teachers were teaching grade 9 mathematics, therefore the content was the same; however, the observed was required to provide a brief background and context of the lesson.

Observation

The focus of the observation was on the teachers' mathematical problem-solving pedagogy. During observation, the observer was required to take notes on the mathematics content, mathematical problem-solving pedagogy, and learners' problem-solving processes. The observer was cautioned neither to disrupt the learning process nor to interact with the observed and their learners during the observed lesson. The objective was to observe in a natural rather than an intrusive manner (Bolt & Atkinson, 2010). In the second and third observation sessions, the observer was required to check for transformation in the observed's practice as per the previous post-observation discussions.

Post-observation feedback

A post-observation discussion, which was audio recorded, occurred immediately after the observed lesson. The observer was required to share their recorded notes with the observed. The observed also shared their experience of being observed. The purpose of the post-observation discussion was to identify the strengths and weaknesses of observed lesson and to share ideas on how both teachers could improve their mathematical problem-solving pedagogy and learners' problem-solving processes.

Reflection

It was essential that the participant teachers reflected on their mathematical problem-solving pedagogy in light of the feedback from the observer and their experience of observing. During the semi-structured reflective interviews, we reflected with the teachers on their experiences with the POOT process. Three reflective semi-structured interviews were carried out with participant teachers once every two months during the 6-month cycle. To assist teachers' self-reflection we asked them the following umbrella questions:

- What did you observe that you would like to incorporate in your teaching of mathematical problem-solving? Why?
- What was the observed doing that seemed central in enhancing learners' understanding and mathematical problem-solving processes? Why?
- What have you learnt from participating in the POOT process?
- What is working with the POOT process? What can we do to help?
- How is POOT affecting learners' learning of mathematical problem-solving?
- How are you going to implement what you have gained from the POOT process in your next lessons?

Validity and Reliability

The post-observation feedback sessions and semi-structured reflective interviews were audio recorded to increase the reliability of the process. Semi-structured reflective interview questions were pilot tested with a pair of grade 9 maths teachers at a neighbouring school to increase the validity of the data collection process. Transcripts of audio recordings from the semi-structured reflective interviews were taken to the participant mathematics teachers for checking.

RESULTS

The reflections from the semi-structured interviews were transcribed and analysed using grounded theory techniques with constant comparison (Glaser & Strauss, 1967), and direct quotations were used to give insight into when some remarks were made and to make participant teachers' voices audible. Themes that emerged from the POOT process were professional development in mathematical problem-solving pedagogy, non-threatening process and safe environment, valuable and necessary process, increased awareness of mathematical problem-solving pedagogy and valuable feedback.

Professional development in mathematical problem-solving pedagogy

During the semi-structured interviews teachers reflected that their knowledge of the teaching of mathematical problem-solving had increased by observing another teacher in action. Teachers acknowledged that by merely observing their colleague teaching, they were able to identify areas of their own pedagogy that needed improvement. This was explained by Ms. N during one semi-structured reflective interview:

As a beginner teacher, POOT is the best way to improve myself and to improve my teaching of mathematical problem-solving. I learnt new skills from Mr. M, not only about teaching problem-solving, but about planning, classroom control and discipline which is a major challenge at our school.

Teachers reflected that observing their colleague teaching mathematical problem solving reinforced what they had learnt in the PD workshops as reflected by Mr. M:

It helped me to see Ms. N encouraging and helping her learners to look back and evaluate their answers after solving given problems. We had done this in the workshop but to see her doing it in the classroom facilitated my understanding.

Participant teachers reflected that they learnt that each teacher interacts with learners in unique ways. Mr. M professed that he observed Ms. N teaching functions using a game and was inspired to see that games can be incorporated into the teaching and learning of functions successfully. Mr. M also noted that he learnt new skills about teaching learners how to understand a given problem:

At the professional development workshop we learnt how to make our learners understand a given problem. It was worthwhile to see Ms. N practically incorporating the different ways we had learnt in the workshop to make learners understand the given problems. I will now let learners read and translate given problems to each other in their own languages during my next lessons.

Mr. M's reflections indicated that he was going to incorporate translanguageing in his lessons. We had done translanguageing in the PD workshops but it turned out that Mr. M became more comfortable in integrating the process in his lessons after observing Ms. N practicing it in her lessons.

Non-threatening process and safe environment

Participant teachers reflected that the POOT process was non-threatening but rather beneficial. However, they acknowledged that POOT has a potential of being nerve wracking if done by a teacher that they are not comfortable with. Below are the responses from the participant teachers.

Ms. N: As a beginning teacher it was initially unsettling to be observed, however, it was a great privilege to work along with my colleague. To have real conversations with him before and after the observations benefitted me professionally in that he is familiar with the subject, CAPS curriculum and problem-solving teaching strategies.

Mr. M: The way Ms. N conducted her observations was considerate and non-intrusive.

During the interview we probed Mr. M to elaborate what he implied by being considerate and non-intrusive. Below is how he responded:

She always came in quietly such that I did not feel her presence. However, I work with Ms. N frequently since we teach the same grade such that I wonder if I would have felt the same way if it was a different teacher.

Valuable and necessary process

Mr. M acknowledged in one semi-structured interview that POOT is a valuable and necessary process:

It is always helpful to see the teaching of mathematical problem-solving by someone else. I found it valuable to observe my colleague teaching mathematical problem-solving to her learners. I think POOT is necessary and should be done by all teachers in the school.

Ms. N appreciated the POOT process and indicated that it made her think carefully about every step in the teaching of mathematical problem-solving.

Increased awareness of mathematical problem-solving pedagogy

Mrs. N acknowledged that observing Mr. M in action made her more flexible as a maths teacher. She indicated that after observing Mr. M she went home and reflected on and reassessed her own mathematical problem-solving pedagogy.

I began to fully understand myself as a teacher and the way I teach mathematical problem-solving. I became flexible in incorporating Polya's problem-solving steps in my teaching. By observing Mr. M teaching I realized that learners must reflect on their answer after solving a problem. Each day I observed Mr. M I went home and reflected on what I had learnt by observing him.

Valuable feedback

Participant teachers treasured the feedback they received from their peers and indicated that it improved their mathematical problem-solving pedagogy. This was acknowledged by Mr. M:

I had never received feedback on my teaching for the 6 years that I have been teaching. It really made me see areas in my teaching that required improvement. I liked that I received the feedback within a short time just after completing my lesson.

DISCUSSION

From the findings, it can be concluded that the POOT process seemed to be non-threatening, useful, valuable and a source of professional development for teachers' mathematical problem-solving pedagogy. The observer developed an increased awareness of how mathematical problem-solving was being taught and learned. By observing one another, participant teachers not only learnt new ideas that they could apply in their own teaching of mathematical problem-solving but also aspects of planning and classroom management. This finding resonates with what Bolt and Atkinson (2010) noted in their study on academics who participated in a peer review of face-to-face teaching. Bolt and Atkinson found that the process of POOT increases the teaching skills of the observer as they observe another teacher in action. This also concurs with what Bell and Mladenovic (2008) found in a peer review program, that POOT is valuable to the participant teachers and gives them the opportunity to emulate the teaching techniques of others. Engin and Priest (2014) assert that POOT has an

immediate impact on teachers' practices in that the observer may go to their next class and immediately implement strategies from the observation.

In this study, the observed became aware if their teaching of mathematical problem-solving was clear and understandable to learners and revised their own teaching strategies. This finding is in agreement with Bell and Cooper (2013) who unearthed that participants in their study improved their teaching skills and confidence in teaching. It was of particular interest that early career academics in the study benefitted the most from the process. This implies that it is worthwhile to pair the experienced with the novice for POOT. For this study Ms. N, who had only 1 year experience teaching mathematics felt that she benefitted the most by observing Mr. M.

Participant teachers valued feedback from their peers and felt that the feedback improved their performance. This finding matches with what Hendry and Oliver (2012) found in their study of Foundations lecturers that participated in a POOT process that was part of a formal teacher-development programme. Lecturers in the programme found the feedback from the observer generally useful and learnt new skills by observing a peer teach. The findings from this study suggest that POOT reinforced and enhanced the correct way of teaching mathematical problem-solving that the teachers were learning in the PD workshops, which in turn resulted in an increased focus on learners' problem-solving processes during the lessons. This finding is in agreement with an Indonesian study by Zwart, Wubbels, Bolhuis and Bergen (2008) which showed that POOT results in the growth of teacher learning.

The limitations of POOT are that the observer can be unintentionally subjective (Engin & Priest, 2014); the presence of the observer can distract the learners and the observed can feel very uncomfortable and fail to teach in their normal way. As per the teachers' reflections, in this study learners were not distracted because they knew the observer as teaching the other grade 9 class. Teachers were both grade 9 teachers and usually planned together and were attending the PD intervention together; therefore they were comfortable with observing each other and understood the importance of being objective during the observations and when giving feedback. We noted that for POOT to be successful, it needs to be non-evaluative and non-judgmental; otherwise teachers become anxious and view it as a daunting task and may refuse to participate. We also noted that if participant educators lack training and experience in observation and feedback this can cause drawbacks and ineffectiveness of the process. For this study, teachers were trained during the PD workshops and understood the POOT process.

CONCLUSION AND IMPLICATION FOR FURTHER STUDIES

POOT was implemented in the second cycle of a large DBR project and the results seem to indicate that it is a valuable PD strategy that can be used to support grade 9 South African teachers' mathematical problem-solving pedagogy. The study having been done with two teachers because of limited resources, it is unjustifiable to generalise the findings to a larger population. However, the findings fill in a gap in the literature on POOT since there is limited literature on successful POOT experiences

with grade 9 South African maths teachers. Other studies can be done in the future with a large number of South African maths teachers that are a representative sample and the results can be compared with our findings. Further study is also required that considers how teachers can be trained to give constructive and non-judgemental feedback to the observed. It can be recommended that PD interventions in South Africa that support the teaching of mathematical problem-solving should include POOT to assist teachers' self-reflection. However, the PD interventions must take into account the time constraints of the South African curriculum.

References

- Bell M. (2002) Peer observation of teaching in Australia. *LTSN Generic Centre*.
- Bell, M. & Cooper, P. (2011). Peer observation of teaching in university departments: A framework for implementation. *International Journal for Academic Development*, DOI:10.1080/1360144X.2011.633753.
- Bell, A. & Mladenovic, R. 2008. The Benefits of Peer Observation of Teaching for Tutor Development. *Higher Education*, 55(6): 735–752.
- Bolt, S. & Atkinson, D. (2010). Voluntary peer review of face-to-face teaching in higher education. In M. Devlin, J. Nagy and A. Lichtenberg (Eds.) *Research and Development in Higher Education: Reshaping Higher Education*, 33 (pp. 83–92). Melbourne: Australia.
- Chamberlain, J.M., D'Artrey, M. & Rowe, D-A. (2011). Peer observation of teaching: A decoupled process. *Active Learning in Higher Education*, 12(3), 189-201.
- Chirinda, B. & Barmby, P. (2017). The development of a professional development intervention for mathematical problem-solving pedagogy in a localised context. *Pythagoras*, 38(1), a364.
- Department of Basic Education. (2010). *Your Integrated Quality Management System (IQMS)*. Pretoria, South Africa: Government Printing Works.
- Engin, M. & Priest, B. 2014. Observing Teaching: A Lens for Self-reflection. *Journal of Perspectives in Applied Academic Practice*, 2(2), 2-9.
- Glaser, B. G., & Strauss, A. L. (1967). *The discovery of grounded theory; Strategies for qualitative research*. Chicago, IL: Aldine.
- Hendry, G. D. & Oliver, G. R. (2012). Seeing is believing: *The benefits of peer observation*. *Journal of University Teaching & Learning Practice*, 9(1), 1-9.
- Wilson, S. M., Shulman, L. S., & Richert, A. E. (1987). “150 different ways” of knowing: Representations of knowledge in teaching. In J. Calderhead (Ed.), *Exploring teachers' thinking* (pp. 104–124). London: Cassell.
- Zwart, R.C., Wubbels, T., Bolhuis, S., & Bergen, T.C.M. (2008). Teacher learning through reciprocal peer coaching: An analysis of activity sequences. *Teaching and Teacher Education*, 24(4), 982-1002.

A SOCIAL EXPRESSION OF NUMBER USING TOUCHCOUNTS

Sean Chorney

Simon Fraser University

This research reports on an episode involving Kindergarten and grade one students using the iPad application TouchCounts. This episode highlights a moment when the children's improvised and shared gestures become a material resource to make sense of number. These social gestures are generative for further mathematical exchanges giving rise to both personal and social meaning-making.

INTRODUCTION

With the proliferation of touchscreen devices, new forms of interaction are becoming a focus of interest in the learning of mathematics (Davis et al., 2015). In particular, different patterns of participation and collaboration can emerge (Hegedus and Penuel, 2008), particularly with a mobile and multi-touch device such as the iPad. While social patterns can be seen as being epiphenomenal to mathematical practices by some theories of learning, they are seen as constitutive of learning by others (Roschelle, 1992).

In this research report, the mathematical application *TouchCounts* (Sinclair & Jackiw, 2011) is used to engage early learners with aspects of number. *TouchCounts* is a free touchscreen mathematical application (see <http://touchcounts.ca/>) for the iPad. The application uses the iPad's multimodal affordances, thus enabling learners to engage with tangible, aural, visual and symbolic aspects of number.

Early mathematics curriculum commonly begins with counting and number patterns (counting by 2s, skip counting). These topics have been shown to be very important in future mathematical abilities such as reasoning in upper elementary and middle school (Jordan et al., 2009). This paper reports on an episode that highlights the manner in which the sharing of constructed resources is an integral part of the resulting social and individual mathematical meaning-making. In particular, Roschelle's (1992) theoretical approach of learning is elaborated and extended to highlight the learning of number amongst four children using *TouchCounts*.

BRIEF DESCRIPTION OF TOUCHCOUNTS

TouchCounts offers a unique connection between touch and number. The touchscreen of the iPad provides direct contact between finger(s) and number discs. When the screen of the iPad is touched, a disc appears with a number symbol and an audible voice speaks the number word. The first tap will produce a disc with the numeral '1' displayed on it and subsequent taps will construct discs with successive numbers displayed. When the screen is touched with two or more fingers at once, that number of discs appears and that number is audible spoken. The activity in this research report

uses a shelf (a horizontal line across the screen). If a user touches below the shelf, the discs will fall downward and disappear off the bottom of the screen. If a user touches about the shelf, the disc(s) will remain on the shelf. In Figure 1, the discs numbered ‘1’, ‘2’, ‘3’ and ‘4’ are falling downward off the screen and will disappear. The ‘5’, however, will remain on the shelf (Figure 1).

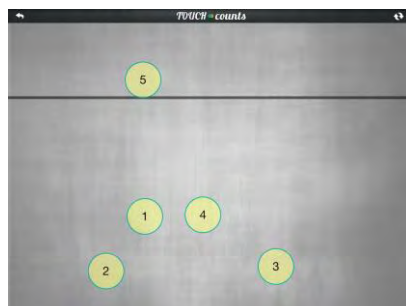


Figure 1: ‘1’, ‘2’, ‘3’, ‘4’ fall off the screen and ‘5’ remains on the shelf

FRAMEWORK

de Freitas and Sinclair’s (2014) inclusive materialism is helpful in looking at interactions with new technologies. Their framework challenges the representational approach to mathematical learning so that the focus is not about how children internalize knowledge but how they materially engage with tools and with each other. Inclusive materialism adopts the inseparability of concept and matter. The practice of mathematics is a material engagement of student, movement, and tool. The concept is not an invisible mathematical idea but partakes of the physical world and all gestures and social interactions become an essential aspect to the concept in a materially framed way. While concepts of number can be seen to be abstract, the use of material resources and social interactions supports the meaning-making of number and its relevance in different situations accessible and tangible to children. This is especially important when using technology. In an inclusive materialist approach, technology is not an add-on device or a tool that enhances the mathematics, it fundamentally transforms the interactions and the subsequent conceptions of the student’s mathematical understandings.

In addition, I draw on Hegedus and Penuel (2008), who identify forms of participation within technological digitally connected classrooms. They view digital technology tools as active, rather than passive, contributors to the dynamics of the classroom activity. While their interest is in argumentation and discourse, I draw from their elaboration of participant structures and focus on the construction of shared resources to support inquiry and knowledge. In the episode presented in this paper, I identify socially shared metaphorical gestures as “... mathematical objects of their own joint creation” (p. 175) and how “...students identify themselves with the object or the mathematical attributes of the object (embodying the mathematical idea as a personal expression)” (ibid, p. 182). The participation of the children through their contempla-

tion, consideration and the joint mathematical objects they create, are fused together, supporting an inclusive material approach.

Lastly, Roschelle (1992), describes learning as a collaborative interaction amongst students. Based in a scientific paradigm and working with how scientific ideas are communicated and subsequently negotiated, Roschelle notes that socially shared meanings does not emerge from a social experience but develops when the experience becomes a resource for participants. The participants interpret and achieve convergent relational meanings and these relational actions of participants are the essential units of how learners come to understand. Roschelle uses a conversational analysis approach but I extend his approach of convergence to include materials, in particular, *TouchCounts*. Roschelle advocates for a democratic participation by noting that knowledge cannot be based on individual idiosyncratic understandings but must emanate from socially shared experiences and resources.

In this research report, I attend to the participation structures that emerge between the children and articulate how these structures become material expressions that converge to shared mathematical meanings. I suggest that the students' social and shared physical expressions echo, as well as determine, the mathematical ideas employed. The socially shared material expression and the mathematics are in unity. The children are not necessarily thinking number but acting number. The theoretical frameworks elaborated above account for how mathematical meanings are constructed through participation, sharing and, finally, convergence. This study explores how the children in an episode, outlined below, enact convergent meaning-making.

CONTEXT

The episode reported took place in a daycare that hosts children from Kindergarten, grade 1 and grade 2 afterschool. A research team of three people including myself went to the daycare on Tuesday afternoons for 10 weeks. During each visit, we worked with three or four children and engaged in mathematical activities using *TouchCounts*. All episodes were videotaped and ethnographic methods were employed to detail all activities. This included transcribing each video as well as referring to the observational notes that were taken by the third researcher.

In this particular episode four boys, two in Kindergarten and two in grade 1, were gathered at a table with a single iPad with *TouchCounts* open. The boys were requested to 'put 10 on the shelf' which involves touching the screen 9 times with a single finger below a horizontal line ('the shelf') and once above. Since these boys had experience using *TouchCounts* from our earlier visits, there was no difficulty in fulfilling this task. The iPad was passed to each boy, one at a time, in a turn-taking manner and each boy did the same thing; they tapped nine single finger times (in an ordinal manner) below the shelf and once above (Figure 2). Each of the turns was very rhythmic. The time it took and the way they touched the iPad was similar for each of the boys.

There were numerous forms of participation. It was common for a boy to count while another boy was tapping. For example, Andrew counted out loud along with both Brent and Harry's tapping. Both Brent and Harry's tapping was very rhythmic so it was easy for Andrew to count along. When Brent was tapping, Andrew, in an attempt to help, said 'up' instead of 10, indicating that Brent should tap above the shelf on his tenth tap. Another example involved Walter, who, holding his hand close to his body, would mimic the tapping gesture with his outstretched forefinger along with Andrew's tapping of the iPad (Figure 3).



Figure 2: Putting 10 on the shelf



Figure 3: Walter (far left) tapping in the air



Figure 4: Piano playing on the iPad

Another interesting engagement occurred between turns. Once the task of putting 10 on the shelf was completed by each boy, they became interested in quickly tapping on the iPad so that the numbers got higher and higher (at times, with both hands on the iPad, like they were playing the piano) (Figure 4). This is noteworthy because of how the screen became a shared space for activity. Each boy put 10 on the shelf in an ordered turn-taking approach but when the iPad was passed from student to student, the in-between time was a multi-touch and multi-person affair. At one point, all four of the boys were tapping on the iPad. The screen was a place of shared attention and also shared action. In general, the space on and around the iPad, both physically and verbally was a shared arena for collaborative activity.

AN EPISODE

When the researcher (me) asked for an alternative way to put 10 on the shelf (that is, different from the strategy of using single touches), Andrew thrust his one hand out into the middle of the group (above the table) with five fingers extended. Brent followed by putting his hand out with two fingers extended (Figure 5, 6). Walter revealed 10 fingers but kept his hands closer to his body (Figure 7). The fourth boy, Harry (not shown), revealed three fingers. This movement from one boy to the next: Andrew, then Brent, then Walter, then Harry happened within a second. There was no time for the boys to decide to act, the expressions of fingers indicate an improvisational act, not intended.

The gestural expression was ordinal in the sense that it started with one boy and then moved around the group but emergent and supported by social collaboration. At this point the boys were each looking at each other's fingers and the verbal expressions of

number that followed did not correspond to the number of their own outstretched fingers. Andrew, the first to show five said, ‘how about three?’ (he was using Harry’s finger-expressed number); Brent, looking at Andrew’s five fingers (Figure 7) who put two fingers forward, said, “how about fo...five”.

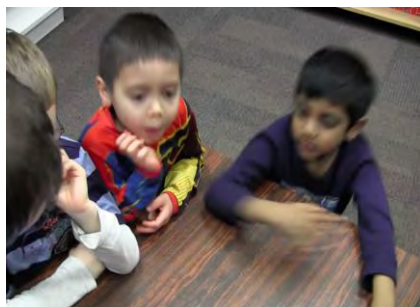


Figure 5: Moving hands to the centre of the table



Figure 6: Cardinal expression from left to right



Figure 7: Full expression of number

Brent looked at Andrew’s five fingers (Figure 7). He then opened up his five fingers (Figure 8) and drew his hand a little closer towards his body. He looked at his five fingers, and said ‘how about four?’ but the four was never fully articulated, and the words trailed off and became inaudible. He then opened both hands, revealing five fingers on each hand (Figure 9). As he spoke ‘five’, he placed his five fingers from one hand in a one-to-one correspondence with his fingers from the other (Figure 10) After another round of turn-taking, in his next attempt to put 10 on the shelf in a ‘different’ way, Brent first used five fingers all-at-once (Figure 11) so that 5 discs appeared simultaneously and TouchCounts said “five”. He then touched with one finger to make 6, two fingers all-at-once to make 8, and finally two single taps, placing 10 on the shelf. It is important to note that Brent implemented the representation of ‘five’ in this turn after seeing Andrew’s expression of five. Brent used Andrew’s finger expression as his own.



Figure 8: Making five



Figure 9: Bringing hands together

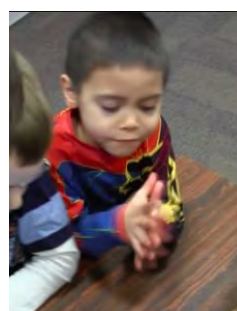


Figure 10: 1 to 1 correspondence

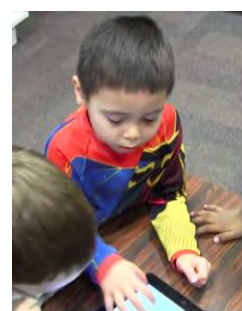


Figure 11: Using five fingers

DISCUSSION

I suggest that the episode described above is an example of a socially shared experience, which provided an opportunity for meaning-making. From Roschelle’s (1992) perspective, “meanings are taken to be relations among [...] gestural actions” (p. 235).

The outstretched fingers expressed by each child became a shared resource that was a gestural manifestation of number. These were expressed in the ‘centre’ space where they could be accessed by all of the boys.

The gestures in this episode not only take on a material role of actually making a number, but also play a communicative role. When Brent opened his hand, increasing his fingers from two to five, he was using a resource from the group. The cardinal form of ‘five’ was borrowed and personalized in the one-to-one correspondence of his own fingers, along with his verbal articulation of ‘five’. He could be seen negotiating his own fingers (increasing the number of fingers from two to five) and his own words (beginning to say ‘four’ but finally saying ‘five’). The cardinality of the number expression ‘five’ became a one-to-one connection to Brent’s personal knowing and implemented in his next round with TouchCounts.

It is important to note that when the researcher requested an alternative method of ‘getting ten on the shelf’, possibilities were not physically visible until the children displayed numbers on their fingers. The request of choosing a different method is not simple. Andrew, for example, at another time, had chosen to use different fingers and tap different parts of the screen to satisfy the ‘different way’ of getting ten. The expressions of cardinal numbers on their fingers, however, set a precedent of how to approach the problem and also offered possibilities of cardinal numbers to use.

The act of showing a physical display of fingers along with the verbal expression was unique in that a physical display of fingers was not explicitly requested in the mathematical task. I suggest that because TouchCounts supports the use of fingers, the expression of the children’s fingers emerged because they were using TouchCounts. This is particularly relevant since the verbal number expressed and the physical display of fingers did not necessarily match as with Andrew and Brent. The fingers were thrust out spontaneously. The display of fingers was so quick that there was no time for mental contemplation of what to display. The inability to verbally articulate what the fingers expressed support this point. The numbers were improvised in the moment. One may argue that the fingers were informing the mind. de Freitas and Sinclair’s (2014) inclusive materialism supports this idea that the material, TouchCounts in this case, fundamentally transforms the interactions and the subsequent conceptions of the children’s mathematical understandings. This is interesting, particularly because the expression of number was very important in the way that it provided a resource for further meaning-making.

It is also interesting to note that when Brent was examining his hand making five with five fingers that his palm was face down as if he were about to use TouchCounts. Typically, when one examines and looks closely at the number of fingers they are expressing, the palm is face up. This is important to note for the participation structure is not only influenced by the children but also by the technology. In this way, we see TouchCounts influencing Brent’s manifestation of number.

During the overall activity, TouchCounts provided a space for multiple forms of engagement: the counting of another's tapping; the mimicking of another's tapping; and the quick tapping in between the turn-taking. These were forms of participation supported by the active role of TouchCounts. Hegedus and Penuel (2008) argue that technology is active and contribute to the dynamics of participation. TouchCounts, also by its very nature of providing a single screen helped create a shared space. Once this space was created, the later contributions of finger numbers could become a shared resource. These resources were important in this environment because they became mathematical objects that the boys could use to further act and communicate—as can be seen in Brent's intervention. The children have an opportunity to “embod[y] the mathematical idea”, as Hegedus and Penuel describe it (p. 182).

The social nature of the activity with TouchCounts in this episode gives rise to different aspects of meaning-making that go beyond a linear model of acquiring mathematical ideas. The attention to shared expressions of different cardinal manifestations of fingers presented a situation where the material creates possibilities for thinking about number. Roschelle (1992) argues that the collaboration in a situation such as this episode is a way to achieve convergent relation meanings. His account of learning notes that a ‘mathematical’ activity does not in and of itself lead to understanding, but in the reflection and contemplation of the results of constructed collaborative resources, as in Brent's case, becomes the place of learning. The resulting shared resources emerging from these new structures lend themselves to how learning becomes a convergence. Brent manifested this process when he brought his newly created gesture of five fingers from the table, the shared space, back to his own personal space where he made a one-to-one correspondence, articulating “five” audibly as TouchCounts would do. He further expressed his appropriation when he used ‘five’ to put ‘ten on the shelf’ differently. The sharing of resources and ideas affords a convergence of shared meanings, and this process is what creates a common understanding.

TouchCounts supports the engagement of number with one's fingers. While it is sometimes conveyed in mathematics education research that finger use and finger-based counting may be secondary to abstract notions of number, recent neuroscience research indicates an important connection between finger and number. In particular, Moeller et al. (2011) state that there is “a functional and beneficial interrelation between fingers-based numerical presentations and numerical/arithmetical development in terms of embodied numerosity representation” (p. 77).

It is important to have multi-modal opportunities for young children to express themselves since they do not often have the ability to verbally articulate what they would like. Seen in this way, TouchCounts provides new approaches that support new social structures of participation and new ways of mathematical understanding.

CONCLUSION

The episode outlined in this research report opens up an opportunity to see the importance of social groupings, task sharing activities and the use of digital technology.

The collaborative technology of the iPad and the app *TouchCounts* supports early learning of number sense by drawing on different elements of interaction including the social environment it affords. I have drawn upon the notion of shared resources to highlight the social element of how a group of children engaged with *TouchCounts* and, in the sharing and borrowing of these resources, how meanings were created. Mathematical meanings are not invisible ideas but material expressions: the ways these manifestations were jointly expressed enabled opportunities that would have been very differently expressed had the child worked alone.

The episode described here provides an example of some of the activities that emerge when children are given the opportunity to explore number in its many forms in a shared, collaborative environment. This episode elaborates two notions: the sharing of resources within a socially shared space and the *generativeness* of resources when they enable for further exploration and meaning-making.

References

- Davis, B., & Spatial Reasoning Study Group. (2015). *Spatial reasoning in the early years: Principles, assertions, and speculations*. New York: Routledge.
- de Freitas, E. and Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the classroom*. Cambridge University Press.
- Hegedus, S. J. & Penuel, W. R. (2008). Studying new forms of participation and identity in mathematics classrooms with integrated communication and representational infrastructures. *Educational Studies in Mathematics*, 68:171–183.
- Jordan, N. C., Kaplan, D., Ramineni, C., & Locuniak, M. N. (2009). Early math matters: kindergarten number competence and later mathematics outcomes. *Developmental Psychology*, 45, 850–867.
- Moeller, K., Martignoon, L., Wessolowski, S., Engel, J. & Nuerk, H. (2011). Effects of finger counting on numerical development—the opposing view of neurocognition and mathematics education. *Frontiers in Psychology*, 2, 326, 75–78.
- Roschelle, J. (1992). Learning by collaborating: Convergent Conceptual Change. *Journal of the learning sciences*, 2(3), 235–276.
- Sinclair, N. & Jackiw, N. (2011). TouchCounts [software application for the iPad]. <https://itunes.apple.com/ca/app/touchcounts/id897302197?mt=8>

THE DESIRED TEACHER REFLECTED IN RESEARCH ARTICLES ON PRACTICUM

Iben Maj Christiansen and Lisa Österling

Department of Mathematics and Science Education, Stockholm University

Starting from questions about what is privileged in mathematics teacher education, we conducted a systematic review of research on practicum. One element was to interrogate the notion of the desired mathematics (student) teacher reflected in existing research. Selecting peer reviewed, empirically based articles for 2001-2017 resulted in the inclusion of 51 articles. Our findings suggest the desired teacher implied in papers to have content knowledge, MKT/PCK, positive beliefs and attitudes, and the ability to reflect on teaching. Teachers who can exercise reasoned judgement were more frequently valued than teachers who can implement specific practices.

INTRODUCTION

To develop desired teacher education practices, it is important, we claim, to understand current practices. Our research focuses on what is privileged/foregrounded in mathematics teacher education globally (cf. Christiansen, Österling & Skog, forthcoming); how this is reflected in content, organisation and pedagogy of mathematics teacher education programs; and how student teachers and novice teachers engage with the content, goals and practices of their education. We are therefore interested in notions of the desired teacher or desired teaching reflected in research on teacher education. We believe that this is most strongly reflected in relation to the school based part of teacher education, and hence to focus this paper, we have concentrated on research on practicum.

Research on the outcomes of teacher education always rest on assumptions of what constitutes teacher knowledge, good teaching and teacher learning. Learning outcomes for teacher education may include declarative knowledge, practical knowledge, or beliefs; it can concern content knowledge, mathematical knowledge for teaching (MKT), pedagogical content knowledge (PCK), curriculum or pedagogy; it can focus on teaching for equity or for excellence; it can prioritise generalised or contextual knowledge; and so forth. It can also be influenced by certain perspectives on teaching or learning mathematics, such as constructivism, or more recently student-centred and self-regulated teaching. All in all, we need to be mindful of the underlying notions of desired teacher knowledge, learning and best practices when engaging the empirically based research literature.

COMPETING ORIENTATIONS IN TEACHER EDUCATION

Three orientations to the balance of content in teacher education have been suggested by Rusznyak and Bertram (2015). First, a general pedagogic knowledge orientation, where student teachers are either introduced to a range of general pedagogic theories and approaches to teaching, or to privileged approaches to teaching, independently of the subject taught. Second, an orientation addressing knowledge for teaching a specific subject, including content and pedagogic content knowledge. Selection and representation of content is central, and educational propositional knowledge is joined with experience to form a basis for student teachers' reasoned professional judgement in different contexts. The third orientation centres around the situatedness of teaching, where individual development of prospective teachers becomes the focus.

From this perspective, student teachers should be encouraged to construct personal theories and/or philosophies from their contextually-specific practical teaching experiences, usually through conscious self-reflection and experience of community engagement. (Rusznyak & Bertram, 2015, p. 36)

An earlier systematic review engages the preparation for student teachers to teach in urban and/or high-needs schools in the US (Anderson & Stillman, 2013), and hence is related to both the first generalist orientation, and the third orientation of Rusznyak and Bertram. They found a focus on the change of beliefs and attitudes, a tendency of a reductionist view on culture and context, and not much evidence base for the development of teaching practice. The authors highlight the need for longitudinal research to investigate the impact of the situatedness of learning in the field.

Within the third orientation, own reflections have become commonly valued in teacher education (cf. Adler, Ball, Krainer, Lin, & Novotna, 2005; Anderson & Stillman, 2013). This may be seen as teachers engaging in continued development and adaptation to context, and the perspective aligns with the idea of the desired teacher as a *reflective practitioner* (Schön, 1987). Such practices are considered well established (Adler & Davis, 2006; Ensor, 2001), but are also critiqued for their tendency to engage in self-evaluative approaches (Adler & Davis, 2006; Österling, forthcoming).

Within the second orientation of Rusznyak and Bertram, a US review of research on teacher education indicates that there existed more research within *mathematics* teacher education compared to other content areas of teacher education (Cochran-Smith & Zeichner, 2005). In 2008, a systematic review of research on mathematics teacher education revealed a large number of studies focusing researchers' own practice as teacher educators, including efforts to demonstrate that a particular program works (Adler et al., 2005).

What we miss from these reviews is an engagement with the notions of good mathematics teaching underlying the research in the field.

RESEARCH QUESTION

This paper is aiming to answer the following research question:

- What do existing, empirically based research studies on practicum in pre-service mathematics teacher education imply as desired teacher knowledge, dispositions, and practice and as desired student teacher learning?

THEORETICAL/CONCEPTUAL FRAMEWORK

Internationally, teacher education differs in the organisation and role of the practicum, as does the terminology used. In this paper, we use *practicum* to describe the phenomena of teacher education taking place in a school context in all its forms.

Our key concept is the *desired teacher*. This notion recognises the teacher as both a subject and as subjected to discourses of power which constitute categories such as ‘teacher’ and ‘good teacher’ (cf. Montecino & Valero, 2015). The idea of what the good teacher is, has varied over time as well as between cultures and nation states. Images of the *charismatic teacher*, that is, the teacher with the *disposition* to teach, have been around for a long time, and still prevail in popular culture (Connell, 2009). Connell also identified a *technical-professional* model, which shows in descriptions of teachers’ necessary “technical know-how” (cf. Winch, Oancea & Orchard, 2015).

As generative as these notions may be, they lack operationalisation. Instead, we have started by inductively categorising the knowledge, beliefs and attitudes, and in particular practices reflected as desirable in the included articles. Finally, we considered the views on desired student learning and supervision practices reflected in the articles.

For the categorisation of the desirable practices, we have drawn on the distinction between whether the *valued application* is of (a) *reasoned judgement* or (b) *preferred technique* (Christiansen et al., forthcoming; Rusznyak & Bertram, 2015). Strong arguments have been presented for the importance of professional judgement in teaching (Biesta, 2015; Shalem & Slonimsky, 2013; Winch et al., 2015). Within this perspective, “teaching is conceptualised as a complex principled practice requiring specialised disciplinary-based knowledge” (Rusznyak & Bertram, 2015, p. 35, drawing on Shalem & Slonimsky, 2013) and hence some degree of professional autonomy. The distinction between the two perspectives is one of the relations to specialised knowledge, where the ‘preferred technique’ does not explicitly refer to knowledge which may have informed the technique.

METHOD

In conducting the systematic review, we aimed at (i) applying a transparent and explicit search strategy, (ii) formulating criteria for inclusion and exclusion, and (iii) coding the included papers systematically, before (iv) constructing a synthesis.

Our first decision was to include only peer reviewed journal articles. Fourteen journals were searched in order to include both the majority of papers on practicum in mathematics teacher education and papers from different contexts, although only English language articles were included. We limited our search to articles published between

2001 and June 2017. The South African journal *Pythagoras* was however only electronically available to us from 2004.

We searched for the word “mathematics” together with one term for practicum at a time, in the abstracts of papers. We used the terms “practicum”, “field experience”, “student teaching”, “school based education”, “teaching practice” and “internship”.

Once the potentially relevant articles were selected, resulting in 107 papers, we checked that the articles indeed concerned practicum and mathematics teacher education; that they reported on empirical research; and that they engaged mathematics teaching. These were our inclusion criteria. We decided not to include single case studies (our exclusion criterion). This process resulted in a dataset of 51 articles.

Each paper constituted one unit of analysis. What the studies indicated as desired student teacher knowledge, dispositions, practice and learning, was often only partially explicit. At times, we had to infer what was privileged, from the foci of the articles. In doing so, we worked from the assumption that if a study argues for the relevance of more knowledge about a particular aspect of teacher education, this aspect is considered of value, unless explicit arguments to the contrary are presented. As an example, Cavanagh and Prescott set out to identify factors that facilitated or constrained student teachers becoming more reflective (2010). We inferred from this that the authors valued reflection in the practice of (student) teachers.

RESULTS

Using the distinction between desired practices using *reasoned judgement* and using *preferred techniques*, we found double the number of papers (24) in the former category as in the latter (12). Fourteen papers mentioned directed attention, noticing, and critical reflection as important teacher characteristics, and were categorised as favouring reasoned judgement practices. Articles which engaged scaffolding learners’ learning, student teachers moving from describing to using theory or theorising, balance obligations, an explorative approach in teaching, being able to analyse curriculum materials, or diversifying pedagogy were also seen as valuing reasoned judgement. A few papers were vaguer in their reference to teachers paying attention to learners or using reform pedagogy and thus may or may not reflect a focus on reasoned judgement.

Twelve papers that mentioned desired practices did not imply the use of reasoned judgement, but nonetheless mentioned *preferred techniques or approaches*. These papers advocated that teachers pay attention to learners or use reform pedagogy, or that teachers should use or teach problem solving or inquiry/investigations. High cognitive demands on learners and eliciting extended learner explanations was mentioned in one paper, and some papers stressing the use of connected representations.

The desired MKT or PCK of mathematics student teachers was explicitly engaged in seven of the papers, and three papers specifically mentioned the importance of content

knowledge, while several other studies addressed specific aspects of MKT such as being able to engage learner thinking, etc.

That beliefs or attitudes affect teaching wherefore student teachers should develop positive attitudes to mathematics and mathematics education, was an assumption informing the research of six papers. Also addressed were students' belief in his/her own ability to teach. Two studies foregrounded an explorative attitude or approach of students. An additional four articles used lesson studies as an approach, and could also be seen as foregrounding an explorative approach of student teachers.

When it came to desirable student learning, a collaborative, interactionist or dialogical model of teacher change was signposted fairly overtly in eight papers, half of which came from Scandinavia. Four papers focused on induction into the profession and/or developing teacher identity. Two papers specifically saw change of beliefs as crucial.

The supervision students were seen to benefit from encountering could be discerned in many papers as well. Balancing the reflective aspect or feedback with the evaluative aspect of the supervision was a perspective which we recognised in seven papers. Other views included that the supervision/teaching should be grounded in research or theory; should help connect content and pedagogy; should help student engage theory with practice; should focus on teacher effectiveness including classroom management; should focus on preparation; should provide different experiences for students depending on their attitudes to mathematics; should challenge students; or should engage content sequencing.

While many views relate, agreement on what qualities of the desired teacher should be foregrounded, and how to achieve this in teacher education, is obviously hard to come by!

CONCLUSION AND DISCUSSION

The sought after attributes of the mathematics student teacher of the 51 articles include: ability to reflect on teaching, content knowledge, MKT/PCK, positive beliefs and attitudes, and efficacy. Given our narrowing of the review to research on the practicum, it was to be expected that more papers engaged desired practices than engaged desired knowledge or beliefs/attitudes. A teacher who can employ reasoned judgement is held in esteem by many of the researchers, as reflected in the nature of the inquiries. Less frequently desired is a teacher who can employ specific techniques.

The learning outcomes for the student teachers were not always made explicit, but collaborative, dialogical or inductive perspectives were mentioned more frequently than other views on teacher learning. This does not imply that these models dominate - they may be made explicit more often than other approaches, or dominate in some contexts only. No clear picture of the supervision that student teachers should receive stood out, except for wanting summative and formative aspects to be balanced.

The outcomes from practicum discussed in the studies focused more frequently on student teachers' reflections, followed by their teaching, and thirdly attention to learner

thinking. Students' reflections were desired to be subject-oriented, use theory, be reasoned, engage PCK/MKT, be broad, be deep, be critical, be evidence-based, or to use theorizing or professional language – many of which reflect a preference for practices utilising reasoned judgement. Some of the outcomes concerning teaching were broad and vague: students being able to carry out instructional tasks, prepare for and justify their teaching, or have increased attention on aspects of teaching. Others were very specific: using reform mathematics, using code switching, using varied questioning, and using less whole class instruction.

The issues taken up in the research reported in the included paper implies a range of assumptions about what constitutes good teaching, and often also what facilitates the learning of student teachers. This of course reflects the normative character of education as a field.

Some studies may simply postulate the relevance or quality of a particular practice, cf.

... beginning teachers are encouraged to adopt a critical attitude toward their classroom practice by engaging in ongoing and focused reflection. This reflection involves constantly framing questions in response to classroom observations and experiences, seeking answers and re-framing new questions as they arise. Reflective practice is therefore widely regarded as one of the hallmarks of quality teaching. (Cavanagh & Prescott, 2010, p. 147)

The border between what is based on research or theory and the authors' own preferences may not always be clearly drawn, meaning that findings are highly relative to the assumptions of good teaching. Hence we see a need for a more coherent engagement with notions of desired teachers to make comparisons across studies more accessible. Furthermore, we would encourage a stronger critical engagement with the implications of assumptions about desired practices, but also about who can become the desired teacher (distributive rules).

Methodologically, surprisingly few papers utilised observations of student teacher practice teaching. Conclusions about what student teachers do or have learned cannot predominately be based on interviews or analysis of student writing, but must examine their actual practices as well. Studies combining more data sources would offer more nuanced perspectives.

Finally, we seek a more critical perspective on the path to the desired teacher, whichever that may be. We may ask: should the student teacher engage in the desired practice from the first practicum experience? If not, what may lead the students gradually towards this? Hugo & Wedekind define the 'imitation fallacy' to be:

... to assume that the end point aimed at by a developing education system should be imitated in its beginning. The mistake takes something like the following form - if a progressive end point is desired, then it has to be progressive from the beginning. (Hugo & Wedekind, 2013, abstract)

While they wrote this in a particular context, the question applies equally to other contexts. Is the best path to the desired teacher, whatever that may be, to imitate the desired practices from the beginning? And is the path the same for all students? We see

a strong need for more research that engages these issues, including questions of sequencing of content and pedagogical approaches in teacher education, and the accumulation of student learning.

References

(Due to space restrictions, only cited papers of the 51 reviewed papers are listed here. A full list is available upon request.)

- Adler, J., & Davis, Z. (2006). Opening another black box: Researching mathematics for teaching in mathematics education. *Journal for Research in Mathematics Education*, 37(4), 270-296.
- Adler, J., Ball, D., Krainer, K., Lin, F.-L., & Novotna, J. (2005). Reflections on an Emerging Field: Researching Mathematics Teacher Education. *Educational Studies in Mathematics*, 60(3), 359-381.
- Anderson, L. M., & Stillman, J. A. (2013). Student teaching's contribution to preservice teacher development: A review of research focused on the preparation of teachers for urban and high-needs contexts. *Review of Educational Research*, 83(1), 3-69.
- Biesta, G. (2015). How does a competent teacher become a good teacher? On judgement, wisdom and virtuosity in teaching and teacher education. In R. Heilbronn, & L. Foreman-Peck (Eds.), *Philosophical Perspectives on Teacher Education* (pp. 1-22). West Sussex: John Wiley & Sons.
- Cavanagh, M., & Prescott, A. (2010). The growth of reflective practice among three beginning secondary mathematics teachers. *Asia-Pacific Journal of Teacher Education*, 38(2), 147-159.
- Christiansen, I.M., Österling, L. & Skog, K. (forthcoming). *Images of the desired teacher in observation protocols*
- Cochran-Smith, M., & Zeichner, K. M. (2005). *Studying teacher education: The report of the AERA panel on research and teacher education*. Mahwah, NJ: Lawrence Erlbaum.
- Connell, R. (2009). Good teachers on dangerous ground: towards a new view of teacher quality and professionalism. *Critical Studies in Education*, 50(3), 213-229.
- Ensor, P. (2001). From Preservice Mathematics Teacher Education to Beginning Teaching: A Study in Recontextualizing. *Journal for Research in Mathematics Education*, 32(3), 296-320.
- Little, J., & Anderson, J. (2016). What factors support or inhibit secondary mathematics pre-service teachers' implementation of problem-solving tasks during professional experience? *Asia-Pacific Journal of Teacher Education*, 44(5), 504-521.
- Lunenberg, M., Dengerink, J., & Korthagen, F. (2014). *The Professional Teacher Educator Roles, Behaviour, and Professional Development of Teacher Educators*. Rotterdam: Sense Publishers.

- McDuffie, A. R. (2004). Mathematics teaching as a deliberate practice: An investigation of elementary pre-service teachers' reflective thinking during student teaching. *Journal of mathematics teacher education*, 7(1), 33–61.
- Montecino, A., & Valero, P. (2015). Product and Agent: Two faces of the mathematics teacher. In Mukhopadhyay, S. & Greer, B. (Eds.), *Proceedings of the Eighth International Mathematics Education and Society Conference* (pp. 794-806). Portland, Oregon.
- Rusznyak, L. & Bertram, C. (2015). Knowledge and judgement for assessing student teaching: A cross-institutional analysis of teaching practicum assessment. *Journal of Education*, 60, 31–61.
- Schön, D. A. (1987). *Educating the reflective practitioner: toward a new design for teaching and learning in the professions* (1. ed). San Francisco, Calif.: Jossey-Bass.
- Shalem, Y. (2014). What binds professional judgement? The case of teaching. In M. Young & J. Muller (Eds.), *Knowledge, expertise and the professions* (pp. 93-105). Oxon: Routledge.
- Shalem, Y., & Slonimsky, L. (2013). Practical knowledge of teaching practice: What counts. *Journal of Education*, 58, 67–86.
- Winch, C., Oancea, A., & Orchard, J. (2015). The contribution of educational research to teachers' professional learning: philosophical understandings. *Oxford Review of Education*, 41(2), 202–216.
- Österling, L. (forthcoming). *Confessions of Mathematics Student Teachers*.

THE NATURAL NUMBER BIAS IN ARITHMETIC OPERATIONS: THE CASE OF THE REPRESENTATIONAL FORM OF THE NUMBERS

Konstantinos P. Christou

University of Western Macedonia, Greece

In this study, it is tested the effect of the natural number bias (i.e. the tendency to ascribe characteristics of natural numbers to non-natural ones), to students' intuitions about the size and the representational form of the outcome of multiplication and division. A paper-and-pencil test was administered to 91 7th and 8th grade Greek students. The test included equalities between given and missing numbers (e.g., $4: _ = 7.6$) to be validated. The results showed a strong natural number bias effect both on students' anticipations about the size of the outcome of each operation (i.e., that multiplication always makes bigger and division makes smaller), and about the representational form of the outcome (i.e., that natural makes natural, and decimals make decimals). Educational implications are discussed.

INTRODUCTION

A fundamental part of mathematical literacy is shaping a clear image about students' difficulties with the number concept, especially when rational numbers are introduced. It has been noted that students tend to inappropriately apply natural number properties when non-natural numbers are involved, a tendency that leads to systematic mistakes and misconceptions and is well described by the "natural number bias" phenomenon (Ni & Zhou, 2005) (hereafter NNB). Among different approaches and explanations given about the origins of this bias, researchers seem to agree that cultural privilege, the type of notation, intuitions and early instruction support and perpetuate the construction of an initial conception for numbers grounded on natural numbers and the act of counting (Smith, Solomon, & Carey, 2005).

The tendency to imply this initial conception has multiple consequences that appear in the form of mistakes in different domains. For example, in the case of ordering rational numbers students erroneously think that longer decimals are larger, for example that 2.367 is larger than 2.6 (Hartnett & Gelman, 1998), or that the bigger the numerator and denominator of a fraction, the bigger the fraction (Nesher & Peled, 1986).

Previous studies in the field

Quite recently the research considering the NNB phenomenon has reached the area of arithmetic operations focusing on the size of the outcomes of each arithmetic operation. A series of studies are presenting findings that students tend to think that, regardless of the kinds of numbers involved, addition and multiplication between any two numbers always result in a larger number, as well as that subtraction and division

always produce smaller numbers than the minuend and the dividend respectively (Vamvakoussi, Van Dooren, & Verschaffel, 2013; Van Hoof, Vandewalle, Verschaffel, & Van Dooren, 2014).

This phenomenon, which often appear under the label “multiplication makes bigger” is well familiar to mathematics teachers. It has also appeared in the research literature already since the 80s when Fischbein and his colleagues (Fischbein, Deri, Nello, & Marino, 1985) interpreted those intuitions to lie in certain primitive, implicit models students hold for each operation; that is the model of addition as putting together, subtraction as taking away, multiplication as repeated addition, and division as equal sharing. From a NNB perspective we note that these assumed primitive models are compatible with—and based on—reasoning with natural numbers only (Vamvakoussi et al., 2013). That is because the result of addition or multiplication between two natural numbers is always a number bigger than the two initial numbers (unless 0 or 1 are involved). Similarly, the result of subtraction or division between two natural numbers is a number smaller than the minuend and the dividend, respectively. This is, however, not necessarily true for certain kinds of non-natural numbers, for which the output of operations depends on the numbers involved. For example, rational numbers smaller than 1 and also negative numbers falsify students’ expectations about the output of multiplication and division in the first case and addition and subtraction in the second case (e.g. $6 : 0.4$ is bigger than 6; 4×0.5 is smaller than 4).

The current state of the art considering the research on the NNB on arithmetic operations is that the NNB has a dual effect on reasoning about arithmetic operations between numbers and missing numbers (Christou, 2015). First, it privileges, and probably also shapes general rules about standard size of the output for each operation, such as that “multiplication always makes bigger,” which is in-line with the output from operations between natural numbers. Second, students think of unknown quantities as natural numbers. This latest aspect of the NNB is in-line with research findings that support a tendency on the part of the students to think of missing number symbols, such as literal symbols in algebraic expressions, to stand for natural numbers. For example, students appear to think that x in algebraic expression $2x$ stand only for natural numbers and thus the expression stand only for positive integers multiplicative of 2 (Christou & Vosniadou, 2012).

The present study

From a brief literature review presented above one may note that the research studies in this field have focused their interest in students’ intuitions considering only the size of the output of the arithmetic operations, that is whether the output of an operation is bigger or smaller than the operand numbers. The current study intends to provide evidence that there is also an issue of the kind of the effect of the operations. In other words, that due to NNB the students would tend to think that the output of the arithmetic operations on specific kinds of numbers should be a number of the same kind. For example, that the output of division between two decimal numbers should also be a decimal number (and not an integer, for example), or that the output of multiplication

between two fractions should also be a fraction. By kind of number it is meant the different representations of rational numbers, which may take the form of decimal or fraction. It will be made clearer on the following that students tend to think these different representations to stand for different “kinds of numbers” and this term is used in this way in what follows.

To the best of our knowledge there is no research that has focused on the kind of the output of arithmetic operations. However, the above-mentioned hypothesis stems from previous research findings on students’ understanding of the number concept. More specifically, students appear to have difficulties to see the set of rational numbers as a unified set of numbers (Kilpatrick, Swafford, & Findell, 2001) and they tend to treat decimals and fractions as if they were different kinds of numbers, rather than interchangeable representations of rational numbers (e.g. Khoury & Zazkis, 1994; O’Connor, 2001). In the same line, it would also be expected that in the context of arithmetic operations there would be differences between students’ responses depending on the kinds of numbers involved as operand or as outputs of an operation.

In order to test the effect of NNB on both the size and the kind (representational form) of the output of arithmetic operations, students will be asked to validate a series of operations between one given and one missing operand number, (e.g. $6 \times _ = 30$).

Research questions and predictions

Question 1 was whether, when judging the validity of the given equalities, students would be biased by the NNB. If so, it would be predicted that regarding the congruency/incongruency of both the size and the kind of the output of operations the students would make more mistakes on incongruent items (where an intuitive belief about both the size and the kind would lead to incorrect responses) than on congruent items; this would be expected to be present both in multiplication and in division (Prediction 1).

Question 2 was whether the NNB would appear to the same extent in both arithmetical operations (i.e., multiplication and division). A better performance would be expected on congruent than on incongruent items for both operations regarding again both the size and the kind congruency/incongruency (Prediction 1). Still, it would be expected that there might be differences in the strength of the natural number bias between the operations. One would expect higher accuracy in multiplication than in division in the size-congruent items (Prediction 2), but not necessarily in the size-incongruent items. That is because a different study in the past that had used the same kinds of items in primary school students showed that the students appeared more willing to accept that division can make the operand numbers bigger than to accept that multiplication can make them smaller (Christou, 2015). However, other studies on secondary students, with similar but not the same tasks with the previously mentioned study, showed the exact opposite result (Van Hoof, et al., 2014). No certain predictions could be made considering differences between operations as far as the kind congruency/incongruency is concerned based on previous studies.

METHOD

Participants

The participants were 91 students from a public school of Athens, Greece: 43 were 7th and 48 were 8th graders, 53 were girls.

Materials

The participants were addressed a questionnaire that included 46 equalities with operations between one given and one missing operand number, with an output also given (e.g. $6 \times _ = 30$). The tasks were equally shared between those involved multiplication and those involved division. Only natural and decimal numbers were involved in the equalities used in this study. Students were asked to decide about the validity of those relations by choosing one of two given alternatives: “it is possible” and “it is not possible,” without necessarily finding the specific missing number.

In order to test the hypotheses of the study, the designed tasks were combinations of congruency (congruent and incongruent) by means of size (bigger or smaller), kind (natural numbers or decimal numbers), and operation (multiplication and division). More specifically, Size-Congruent tasks are congruent by means of the size of the output of the operations, which means that the outputs are in-line with students’ intuitions about the size of the outputs of each operation such as that multiplication makes bigger and division makes smaller. Size-Incongruent tasks, on the other hand, present operations with the output size to be counter to students’ intuitions (i.e., multiplication would make smaller, e.g. $41 \times _ = 7$). Kind-Congruent tasks were those tasks that were congruent by means of the symbolic representation of the numbers involved in the operations, that is in-line with the intuitive belief that the outputs of the operations should be of the same symbolic representation with the operand numbers (i.e., that natural numbers appear both as operands and as outputs, decimal numbers as operands and as output, e.g., $6.3 \times \dots = 2.1$). Kind-Incongruent tasks presented operation outputs that were counter to this later intuitive belief, that is equalities with the given outputs being of different symbolic representations from the given operand numbers (i.e., natural numbers as operands and decimal numbers as outputs and vice versa, e.g., $4: \dots = 7.6$). There were also buffer tasks, that acted as distractors, in which the correct response was “it is not possible.”

Procedure

The students completed the tests in their classroom during their mathematics course with the presence of their teacher and the researcher. Students were told there was only one correct answer for each question and that they should choose one of the two given alternatives that best represents their opinion. They were also explicitly told that they could think with any kind of number they know.

RESULTS

Participants' responses were scored on a right/wrong basis. Analysis of variance of students' total score (52%) showed no main effect for gender [$F(1, 89) = 1.173, p = .282$]. Students from 8th grade performed better than 7th graders ($M = .48, SE = .03$ vs $M = .49, SD = .02$) but those differences were not statistically significant [$F(1, 89) = .103, p = .749$], which indicated that students are not necessarily getting better by age. Mean accuracy in each of the main categories of task were calculated and are presented in Figure. 1.

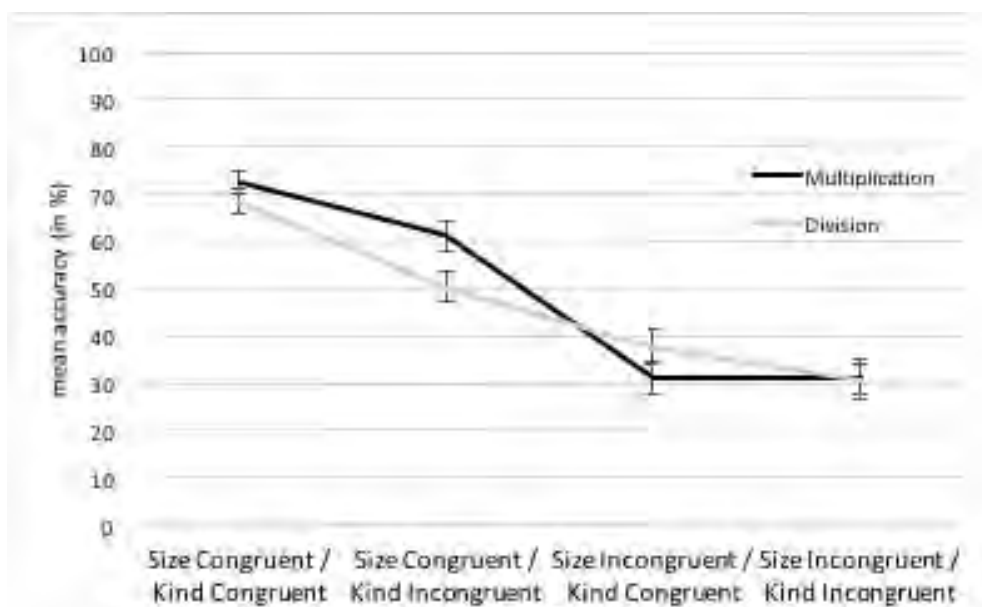


Figure 1: Mean accuracy per congruency and operation.
Error bars represent 95% CIs.

In order to answer the research Question 1, pairwise comparisons of the mean accuracies were conducted for the congruent and incongruent items within each operation category. The results showed a size-congruency effect because there appeared statistically higher accuracy in Size-Congruent than in Size-Incongruent items when both tasks were Kind-Congruent in multiplication $t(90)=0.807, p < .001$, and also in division $t(90)=6.623, p < .001$. There also appeared statistically higher accuracy in Size-Congruent than Size-Incongruent items when both tasks were Kind-Incongruent in multiplication $t(90)=7.301, p < .001$, as well as in division $t(90)=4.717, p < .001$.

Considering the effect of the representational form of the numbers that appear in the operations the results showed statistically higher accuracy in Kind-Congruent than in Kind-Incongruent items when both tasks were Size-Congruent for multiplication $t(90)=3.806, p < .001$, and also for division items $t(90)=6.605, p < .001$. In addition, there was statistically significant higher accuracy in Kind-Congruent than in Kind-Incongruent items when both tasks were Size-Incongruent in division $t(90)=2.445, p < .001$, but not in multiplication where students scored almost the same. The above results support Prediction 1.

To test research Question 2, pairwise comparisons of the mean accuracies were conducted between operations. The results showed that students performed higher in multiplication than in division, for size-congruent items that were also kind-congruent (Prediction 2), however these differences were not statistically significant $t(90) = 1.748, p = .084$. On the other hand, the students performed higher in the division than in the multiplication items when the outputs were in-line with their intuitions by means of kind but not by means of the size of the outputs of the operations. This means that the students were more willing to accept that division may result in bigger numbers than to accept that multiplication may result in smaller numbers, but these differences were again not statistically significant $t(90) = 1.491, p = .139$. When the size of the outputs of the operations were in-line with students' intuitions the students appeared more willing to accept that the outputs of division can be numbers with different representation form from the operand numbers than to accept this for the outputs of multiplication $t(90) = 3.202, p < .05$ (for Size-Congruent / Kind-Incongruent items). In other words, students tended to accept that division is more likely to produce outputs of different representation from than multiplication.

DISCUSSION

The results supported the main hypothesis of the study that in the domain of arithmetic operations there is a NNB which interferes with students' reasoning about the size and also about the representational form of the output of each operation.

More specifically, it appeared that the NNB affects and probably also shapes students' tendencies to intuitively associate each operation with specific output size, that is, that the output of multiplication is bigger than the operand numbers and the output of division is smaller than the dividend. This conclusion is supported by the finding that the participants performed significantly better in the tasks that were in-line with their intuitions (i.e., multiplication produces bigger numbers than the operands), compared with the size incongruent tasks that violated those intuitions. These findings are in-line and further support previous findings in the field (Christou, 2015; Fischbein, Deri, Nello, & Marino, 1985; Vamvakoussi et al., 2013; Van Hoof, et al., 2014).

The innovative finding of this study is that except of the size of the results, the NNB also appeared to affect students' anticipation of the kind of the result, meaning that, for the students, the number representation should be of the same kind between operand numbers and outputs. This conclusion was supported by the finding that the students performed significantly better to those items that were in-line with their intuition that the outputs of each operation should be numbers of the same kind as the operand numbers, than those items that violated this intuition. This latest finding further supports previous ones that students tend to treat decimals and fractions as if they were different kinds of numbers, rather than interchangeable representations of rational numbers (Khoury & Zazkis, 1994; Kilpatrick, Swafford, & Findell, 2001; O'Connor, 2001). This may be approached to be a by-product of students' tendency to use their initial conception of number which was organized around the concept of natural

number. In support of this claim it should be noted here that in the natural number set numbers have unique representation and operations between natural numbers result on natural number. Of course, this is not the case with division between natural numbers, however, before introduced to rational number arithmetic students have learned division as a combination of natural numbers using multiplication and addition between natural numbers (i.e. $\text{dividend} = \text{divisor} \times \text{quotient} + \text{remainder}$).

The results of this study have multiple educational implications considering ways to help the students excide the barriers of reasoning using their initial conception of numbers which is grounded on natural numbers. These findings could be applied in the design of learning environments that could help students dissolve the misconceptions imposed by the NNB and acquire a number concept closer to the mathematical concept of rational and real number. This could be accomplished by specifically designed interventions, which would motivate the students to change their intuitive beliefs about numbers and their properties, that reflects on their anticipations about the size and the kind of the output of each arithmetic operation. The tasks that were used in this study could also be used in this direction, as educational material that could create cognitive conflict falsifying students' incorrect intuitions. As a means students could become aware of their NNB which could be a first step in the process of changing it.

References

- Christou, K. P. (2015). Natural number bias in operations with missing numbers. *ZDM Mathematics Education*, 47(5), 747-758. doi: 10.1007/s11858-015-0675-6
- Christou, K. P., & Vosniadou, S. (2012). What kinds of numbers do students assign to literal symbols? Aspects of the transition from arithmetic to algebra. *Mathematical Thinking and Learning*, 14(1), 1-27. doi: 10.1080/10986065.2012.625074
- Fischbein, E., Deri, M., Nello, M., & Marino, M. (1985). The role of implicit models in solving problems in multiplication and division. *Journal of Research in Mathematics Education*, 16, 3-17. doi: 10.2307/748969
- Hartnett, P. M., & Gelman, R. (1998). Early understandings of number: Paths or barriers to the construction of new understandings? *Learning and Instruction*, 8(4), 341-374. doi: 10.1016/S0959-4752(97)00026-1
- Kilpatrick, J., Swafford, J., & Findell, B. (2001). *Adding it up. Helping children learn mathematics*. Washington, DC: National Academy Press.
- Khoury, H. A., & Zazkis, R. (1994). On fractions and non-standard representations: Pre-service teachers' concepts. *Educational Studies in Mathematics*, 27, 191-204.
- Nesher, P., & Peled, I. (1986). Shifts in reasoning. *Educational studies in mathematics*, 17(1), 67-79. doi: 10.1007/bf00302379
- Ni, Y. J., & Zhou, Y.-D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist*, 40(1), 27-52. doi: 10.1207/s15326985ep4001_3

- O'Connor, M. C. (2001). "Can any fraction be turned into a decimal?" A case study of a mathematical group discussion. *Educational Studies in Mathematics*, 46, 143-185.
- Smith, C. L., Solomon, G. E. A., & Carey, S. (2005). Never getting to zero: Elementary school students' understanding of the infinite divisibility of number and matter. *Cognitive Psychology*, 51, 101-140. doi: 10.1016/j.cogpsych.2005.03.001
- Vamvakoussi, X., Van Dooren, W., & Verschaffel, L. (2013). Educated adults are still affected by intuitions about the effect of arithmetical operations: evidence from a reaction-time study. *Educational Studies in Mathematics*, 82(2), 323-330. doi: 10.1007/s10649-012-9432-8
- Van Hoof, J., Vandewalle, J., Verschaffel, L., & Van Dooren, W. (2014). In search for the natural number bias in secondary school students' interpretation of the effect of arithmetical operations. *Learning and Instruction*, 30, 30-38. doi: 10.1016/j.learninstruc.2014.03.004

RE-THINKING ‘CONCRETE TO ABSTRACT’: TOWARDS THE USE OF SYMBOLICALLY STRUCTURED ENVIRONMENTS

Alf Coles¹ and Nathalie Sinclair²

¹University of Bristol and ²Simon Fraser University

In this theoretical report, we question the prevalent assumption that teaching and learning mathematics should entail a movement from the concrete to the abstract. Such a view leads to reported difficulties in students moving from manipulatives and models to more symbolic work – moves that many students never make, with all the implications this entails for life chances. We propose working in “symbolically structured environments” as an alternative way of conceptualising students’ direct engagement with the abstract and we exemplify one such environment, that involves early number learning.

INTRODUCTION

Both in most learning theories in mathematics education, and in intuitive approaches to pedagogy, there are widespread assumptions that the teaching and learning of mathematics should begin with the concrete and familiar – more abstract and symbolic knowledge arising later. This assumption is supported by the large number of manipulatives, metaphors and real-world connections that pervade the mathematics education landscape – in an attempt to offer entry points that are ‘concrete’ and meaningful to students. In this report, we aim to critique such assumptions and provide theoretical arguments and empirical examples that support a rethinking of mathematics teaching and learning; in doing so, we will survey some of the different ways in which the terms ‘concrete’ and ‘abstract’ have been used in the mathematics education literature.

CONCRETE AND ABSTRACT

A number of theorists posit a concrete to abstract developmental progression. This is evident in Piaget’s (1954) work, for example, where the “concrete operational stage” precedes the formal or abstract one. Bruner’s (1996) three stages of enactive, iconic and symbolic similarly privileges action on concrete objects over the more abstract manipulative of symbolic ones. Vygotsky’s (1974) approach, which is sometimes contrasted with that of Piaget because of the way it posits learners as moving from abstract to concrete, may seem to counter the prevailing assumptions. However, Vygotsky is deploying the words concrete and abstract in slightly different ways than Piaget or Bruner, attending more to the learner-object relation. For example, the concept of “honour” will be abstract at first for learners because they have no experiences of that concept; but it will become more concrete as they gain experiences related to war, promises, etc. This view of the concrete/abstract duo (that we see as a ‘relational’ one) is also put forward by Wilensky (1991), who argues that while the alphanumeric

inputs of programming languages such as Logo may seem abstract—being symbolic as they are—can also be seen as concrete for some children inasmuch as some children will have had experiences with these symbols that give them a direct and visible reference. Wilensky sidesteps the question of whether or not learning should begin in the concrete or the abstract, and instead advances the relational perspective that both sides, concrete/abstract, are context-bound and subjective (i.e., one is not necessarily harder for learners than the other) and further, that symbolic infrastructures may be suitable environments for learning mathematics.

The relational perspective on the concrete/abstract rarely, if ever, features in official curriculum or standards documents. Consider for example the well-known and influential counting principle of Gelman and colleagues (see Gelman & Meck, 1983), who propose that the earliest experiences around number should centrally involve counting objects—that is, transitive counting—and culminate in the ability to determine the number of object in a given set, that is, in successfully answering the “How many?” question. The assumption about the concrete may not at first be obvious, but that is because so much of early number learning is conceptualised as being cardinal in nature, in which the purpose of number is precisely to determine the numerosity of a set. In such a conception, it is natural to assume children should begin their number experiences by counting things, that is, by taking ‘things’ as metaphors for quantities. In this line of thinking, the very meaning of a number such as seventeen—for early number learners—is the existence of a set of seventeen objects.

A different conceptualisation of number can call into question this ‘natural’ assumption. For example, in a more ordinal conception of number, the meaning of seventeen derives from the fact that it follows sixteen and precedes eighteen, in which case its meaning is a relational one. One may also attend to the symbolic patterning of the numeral 17, in which case it is the metonymic nature of number that is being stressed (e.g., its link, as a symbol, to other number symbols), rather than the metaphoric one (e.g., its link to objects). Within a relational conception of number, intransitive counting (reciting the number sequence) may be seen as a proper starting point. Such an approach dispenses with physical objects to be counted, and thus may strike educators as being rather abstract. But in the Wilensky sense, its concreteness might be found in its connection to experiences like singing the number song. While Wilensky’s motive was to point to the potential for so-called abstract mathematical environments being motivating for children, through the connection that these children can make with such environments, we are *also* interested in how working within the ‘abstract’ may also be conceptually advantageous, a view that will challenge deepseated assumptions about what is meaningful and basic in mathematics learning.

In the next section, we trace problems that are reported around how students come to operate with abstract concepts and consider different historical perspectives on the use of manipulatives and their links to symbols. These differences suggest the move from concrete to abstract is not a developmental universal but an outcome of curriculum and pedagogical choices.

MANIPULATIVES AND SYMBOLS

One result of the assumption that learning proceeds from the concrete to the abstract has been the historical drive, for example seen in the 1980s in the USA, to use manipulatives in learning mathematics (Sowell, 1989). The assumption here has been that manipulatives provide a mechanism for students to engage in giving meaning to mathematical objects, which can then be mapped onto more standard, abstract notation. The idea that meaning is created through an individual, or group, grappling with manipulatives and metaphors of mathematical concepts, then generalising from particular experiences, is linked to the constructivist views of learning (Piaget, 1954) discussed above. We take a different perspective, rooted in our respective research backgrounds of enactivism (Maturana and Varela, 1987) and inclusive materialism (de Frietas and Sinclair, 2014). We view meaning as a feature of relationships, that reside neither in an individual, nor group, nor an object or tool. In every interaction between two organisms, both are changed; our relationships with the world are in constant flux, hence so is every meaning. As humans, how we come to experience the world is a result of our history of co-evolution with everything around us; meaning is an achievement of joint activity and we view knowing and doing as synonymous (Maturana and Varela, 1987). In the same way that every action is an interaction, everything we might interpret as an individual construction of knowledge can only be separated, *at the cost of insight*, from the wider relationships in which it arises.

Rittle-Johnson et al. (2015) have suggested that there has been an unquestioned (and un-tested) assumption, which we see driving manipulative use, that instruction in mathematics should proceed from the conceptual to the procedural. Manipulatives are then used with the aim of beginning instruction with conceptual understanding of processes, e.g., base-10 blocks to understand a subtraction algorithm (Fuson & Briars, 1990). We acknowledge that positive benefits have been reported in relation to manipulative based approaches to learning arithmetic (Mix et al., 2017).

However, as far back as 1997, Uttal et al. report on a series of experiments that suggest the use of manipulatives may set up a dual-representation system, one system being the manipulatives and the other being the symbols or operations intended to be represented. Students frequently become confident working within the manipulative system, but did not see the connection to the symbolic system. Furthermore, learners who are able to translate between systems are doing more and harder work than those able to operate purely within the symbol system (as mathematicians do). These critiques raise the question, whether the reported benefits of using manipulatives may be dependent on curriculum and pedagogical choices that do not benefit all learners.

In our own work on learning number we argue that offering low-attaining students concrete models of number may re-inforce the very way of thinking about number (as solely linked to objects) that students need to move away from to become successful in arithmetic (Sinclair & Coles, 2017). It appears a short step from assuming learning begins with the concrete, to concluding some children are not ‘ready’ for the abstract.

There have been two important strands of work (Davydov, 1990; Gattegno, 1974), in the case of number learning, that examine an alternative to starting instruction with a concrete phase before moving to the abstract. In Davydov's curriculum (Dougherty, 2008) students' first experiences with number are as a comparison of measurements (of length, area or volume). The natural numbers appear as a symbolisation for the number of times a unit measure fits into a second measure. Similarly, Gattegno's curriculum for early number begins with an exploration of relationships of lengths (greater than, less than) leading to the first numerals appearing, to symbolise the situation when one length fits an exact number of times into a second length.

In both the Davydov and Gattegno curriculum, concrete materials play an important part. However, in contrast to the kind of use of manipulatives critiqued by Uttal et al. (1997), there is a significant and previously unmarked difference in how the manipulatives and symbols are related. A typical example of using manipulatives (Fuson & Briars, 1990) introduces a direct and absolute relation between a symbol and object, e.g., in base-10 equipment, 'units' are single cubes, 'tens' are lines of 10 cubes, 'hundreds' are squares of 100 cubes. For Davydov and Gattegno, their manipulatives also have direct and absolute symbol references (Davydov gets students to create their own notation; Gattegno uses colour names for wooden rod lengths) but these are not the symbols that are important. The key mathematical symbols are the numerals. Numerals are introduced, in both curricula, as relations between objects (Coles, 2017). Hence the concrete objects are used as a context in which to make meaningful a set of symbols, but where the key symbols are abstract from the start, denoting actions on, or relations between objects. An immediate consequence is that the numeral symbols develop connections to each other, independent of objects, because a relationship (which is what the symbols symbolise) can be viewed from two sides, e.g., if one length is double another, then the second length is also half the first, so students come to see, if $2w=r$, then $w=\frac{1}{2}r$. Davydov and Gattegno both introduce fraction notation in the first year of their curricula; conceptual work (giving meaning to symbols) and procedural work (using symbol-manipulation rules) proceed together.

The reported successes of the Davydov and Gattegno approaches to learning number provide evidence that there is no universal pattern of learners reaching the abstract as a culmination of experiences that begin with concrete examples, and/or in which mathematical symbols start off having direct and absolute concrete referents. We propose that a common feature of these approaches is captured by the concept of a "symbolically structured environment", which we explore in the next section. Our definition of this concept comes towards the end of the report.

SYMBOLICALLY STRUCTURED ENVIRONMENTS

The notion of a "symbolically structure environment" (SSE) first arose for us in exploring the role of ritualisation in early mathematics learning. In the work of anthropologist Catherine Bell (1991), we found an approach to ritualization that eschewed the dichotomy between thought and action that characterises much of the work on

ritual in the mathematics education research, and instead proposed that ritualization “is embedded within the dynamics of the body defined within a symbolically structured environment. [...] It is designed to do what it does without bringing what it is doing across the threshold of discourse or systematic thinking” (p. 93). Although Bell was speaking of environments that one might find in religious ceremonies, we see a resonance to the mathematics classroom in the notion of a way “of acting that is designed and orchestrated to distinguish and privilege what is being done in comparison to other, usually more quotidian, activities” (p. 74).

We see the symbolically structured environments as having features in common with the notion of a microworld, first introduced by Papert to describe self-contained worlds where students can “learn to transfer habits of exploration from their personal lives to the formal domain of scientific construction” (1980, p. 177). For Papert, the microworld was an immersive environment in which students could develop mathematically fluency. It was not a manipulative that made mathematics concrete; it *was* mathematics, but restricted to an accessible and generative subdomain that was both formal and body-syntonic. These latter qualities connect closely with Bell’s emphasis on SSEs that enables a certain structured act of moving. In the case of Turtle Geometry, a microworld for exploring shape, the Logo commands provided structured ways of body-syntonically moving a turtle on the screen.

Another SSE that is not digital in nature is found in the Gattegno tens chart (Figure 1) which makes available the structure of our written number system, not via a cardinal linking of numerals and objects, but via a more ordinal or relational sense of making links among numerals themselves, for example, how the spoken number names form patterns, and how you can get from one number to the next (see Coles, 2014, for further possibilities for working with the chart).

1	2	3	4	5	6	7	8	9
10	20	30	40	50	60	70	80	90
100	200	300	400	500	600	700	800	900
1,000	2,000	3,000	4,000	5,000	6,000	7,000	8,000	9,000
10,000	20,000	30,000	40,000	50,000	60,000	70,000	80,000	90,000
100,000	200,000	300,000	400,000	500,000	600,000	700,000	800,000	900,000

Figure 1: An example of Gattegno’s whole number tens chart

The symbols used in both Turtle Geometry (such as “fd 10” or “rt 90”) and the Gattegno chart are meaningful from the start because they are used to perform actions or make distinctions. Entry into a SSE requires a constrained beginning, to establish the “rules” of the game (some of which are arbitrary (Hewitt, 1999)). Later on, students can have opportunities to instigate their own use of the symbols so that their activity is not *merely* repetitive or procedural. In these SSEs, student work is metonymical since

they act on objects that are not taken to represent mathematics, but on mathematics itself, as we illustrate in the next section, drawing on empirical work we have carried out using the tens chart. This data is offered in an illustrative manner, to point to possibilities and ground the theoretical ideas above.

LEARNING IN A SYMBOLICALLY STRUCTURED ENVIRONMENT

We have reported elsewhere (e.g., Sinclair and Coles, 2017) on student work using the tens chart and other SSEs. The project for which we have the most student data was one run by the first author, with 7-8 year old children in which the tens chart was used as an introduction to multiplication and division. Work with the class occurred in a fairly typical (in terms of prior attainment) rural primary school in the UK, as part of a project linked to the charity “5x5x5=creativity” who place artists (in this case, a ‘mathematician’) into schools to run projects. Having set up how to multiply and divide by 10 and 100 on the chart, emphasising the visual and gestural way this can be done, a challenge was proposed. The students had to choose a number on the chart, go on a journey multiplying or dividing by 10, 100 and to get back to where they started. Here was a constrained beginning, in which the students had to learn how to ‘play the game’ and use the symbols (for \times and \div) in the way that the first author modelled. Linked to our discussion of Davydov and Gattegno, the symbols we cared about in this activity (\times and \div) arose as standing for actions or relations on the chart (i.e., how you go ‘down’ and ‘up’ a row). The availability to students, of the symbolic structuring of the chart, then allowed scope for innovation. One student, on her fourth ‘journey’ wrote what we have copied in Figure 2.

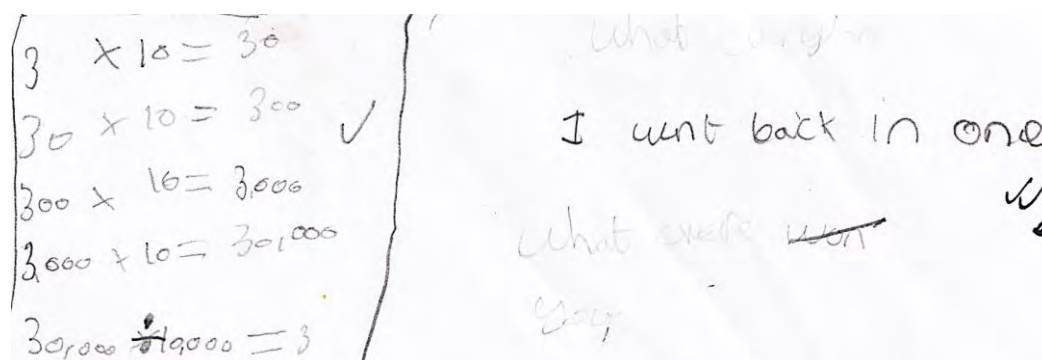


Figure 2: A student's ‘journey’ on the tens chart, “I went back in one”

The student had chosen to extend the pattern of multiplication and division by powers of 10 (only 10 and 100 had been modelled) to include division by 10,000, having set herself the challenge to “get back in one”. Many of the students showed evidence of similar creativity in setting themselves challenges and extending their symbol use, in a metonymic manner, beyond that to which they had been introduced. The student who wrote Figure 2 might not have been able to say “10,000” as a number and we realise this kind of metonymical way of working may concern some readers. However, we would like to highlight the important role that children’s bodies are playing in their interactions within this SSE (i.e., gestures on the chart). This bodily engage-

ment—which we see as an embodied extension with a tool—and the evident inventiveness of what children do and write, indicates that what we are advocating is far from a mechanical activity of meaningless symbol manipulation.

We are now in a position to offer a tentative characterization of a SSE as one in which: (a) symbols are offered to stand for actions or distinctions; (b) symbol use is governed by mathematical rules or constraints embedded in the structuring of environment; (c) symbols can be immediately linked to their inverse; (d) complexity can be constrained, while still engaging with a mathematically integral, whole environment; (e) novel symbolic moves can be made.

CONCLUSIONS

In the space of school mathematics, the focus on *manipulatives as concrete representations* seems to have eclipsed a more fundamental reason to incorporate tools into mathematics learning. As articulated by Piaget (1954), and since elaborated by many mathematics education researchers with interests in the bodily basis of understanding (see de Freitas & Sinclair, 2014), the goal of using manipulatives is neither to make a concept concrete nor to recover its Platonic meaning, but *to move*. That is, one's senses of number, shape, and so on “have more to do with *structured acts of moving* than with *acts of moving structures*” (Ng et al., *submitted*). From our perspective, the main purpose of a manipulative is not to re-present mathematical concepts, but to mould the learner's motions, in the process occasioning opportunities for them to expand and weave their repertoires of mathematically relevant structures.

With the notion of a SSE, we are attempting to provide an alternative kind of tool for mathematics learning that is both abstract and metonymical, but also accessible and meaningful. We argue that such tools—along with careful teacher support—may provide students with less cumbersome and indirect ways of developing mathematical fluency than through the kinds of manipulatives that insist on offering direct, concrete representations. This argument has been informed by theoretical insight and our experiences working with the SSEs described above. More empirical research will be required to further justify our claims and, more importantly, to better appreciate the ways in which our approach will involve significant changes in current teaching practices and curricular progressions.

References

- Bell, C. (1991). *Ritual theory, ritual practice*. New York: Oxford University Press.
- Bruner, J. (1966). *Toward a theory of instruction*. Cambridge, MA: Harvard University Press.
- Coles, A. (2014). Transitional devices. *For the learning of mathematics*, 34(2), 24-30.
- Coles, A. (2017). A relational view of mathematical concepts. In E. deFreitas, N. Sinclair, A. Coles (Eds.) *What is a mathematical concept?*. Cambridge: C.U.P., pp.205-222.

- Davydov, V. (1990). *Types of generalization in instruction: Logical and psychological problems in the structuring of school curricula*. Reston, VA: NCTM.
- de Freitas, E., & Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the mathematics classroom*. New York, NY: Cambridge University Press.
- Dougherty, B. (2008). Measure up: A quantitative view of early algebra. In Kaput, J. J., Carraher, D. W., & Blanton, M. L. (Eds.), *Algebra in the early grades*, (pp. 389–412). Mahwah, NJ: Erlbaum.
- Fuson, K. & Briars, D. (1990). Using a base-ten blocks learning/teaching approach for first and second grade place-value and multi-digit addition and subtraction. *Journal for Research in Mathematics Education*, 21, 180-206.
- Gattegno, C. (1974). *The common sense of teaching mathematics*. NY: Educational Sol'ns.
- Gelman, R., & Meck, E. (1983). Preschoolers' counting: Principles before skill. *Cognition*, 13(3), 343–359.
- Hewitt, D. (1999). Arbitrary and necessary: A way of viewing the mathematics curriculum. *For the Learning of Mathematics* 19(3), 2–9.
- Maturana, H., & Varela, F. (1987). *The tree of knowledge: the biological roots of human understanding*. Boston: Shambala.
- Mix, K., Smith, L., & Barterian, J. (2017). Grounding the symbols for place value: evidence from training and long-term exposure to base-10 models. *Journal of Cognition and Development*, 18(1), 129-151.
- Ng, O., Sinclair, N., & Davis, B. (submitted). Drawing off the page: How new 3D technologies provide insight into cognitive and pedagogical assumptions about mathematics.
- Piaget, J. (1954). *The construction of reality in the child*. (M. Cook, trans.). NY: Basic Books.
- Sinclair, N., & Coles, A. (2017). Returning to ordinality in early number sense: Neurological, technological & pedagogical considerations. In F. Ferrara, E. Faggiano, A. Montone (Eds.) *Innovation and Technologies in Mathematics Education*. Springer: Rotterdam, pp.39-58.
- Sowell, E. (1989). Effects of manipulative materials in mathematics instruction. *Journal for Research in Mathematics Education*, 20(5), 495-505.
- Uttal, D., Scudder, K., & DeLoache, J. (1997). Manipulatives as Symbols: A new perspective in the use of concrete objects to teach mathematics. *Journal of applied developmental psychology*, 18, 37-54.
- Vygotsky, L. (1978). *Mind in society: The development of higher psychological processes*. Cambridge, MA: Harvard University Press.
- Wilensky, U. (1991). Abstract meditations on the concrete and concrete implications for mathematics education. In I. Harel & S. Papert (Eds.), *Constructionism* (pp. 193-204). Norwood, NJ: Ablex Publishing Corporation.

TEACHER BELIEFS AND SUPPORT FOR ARGUMENTATION

AnnaMarie Conner¹ and Laura Marie Singletary²

¹University of Georgia, ²Lee University

Two teachers' beliefs were inferred from four interviews over the course of two years. Their support for argumentation in high school mathematics classes was recorded and analysed, resulting in differences in support. While the two teachers had similar beliefs about mathematics and proof, they differed in their beliefs about teaching mathematics. In particular, their belief about who was responsible for explanations in a mathematics class was most visible in their support for argumentation.

Collective argumentation in mathematics is important for student learning. Through the processes of argumentation, students and teachers can begin to understand the disciplinary underpinnings of mathematics and learn the kinds of reasoning that lead to proof. Additionally, there is agreement that teachers' beliefs relate to choices made in classroom practice, although this connection is influenced by contextual factors. This study begins to connect the classroom argumentation practices of teachers with their beliefs by addressing the following question: What aspects of two student teachers' beliefs are evident in their support for collective argumentation in mathematics?

FRAMEWORK AND RELATED LITERATURE

We define collective argumentation as a group arriving at a conclusion through consensus (Conner et al., 2014). Acknowledging the significance of the teacher's role in collective argumentation, Yackel (2002) called for more research into how teachers support collective argumentation. These results build on Conner et al.'s (2014) description of teacher support for argumentation, which is in turn built on Toulmin's (1958/2003) framework for arguments in many fields.

Toulmin (1958/2003) described a framework for deconstructing arguments in any field. According to Toulmin, all arguments have several components, of which the most central are claims, data, and warrants. A claim is the statement one is attempting to establish. Data are the information on which a claim is based. A warrant is the justification linking the data to the claim. Krummheuer's (1995) adaptation of Toulmin's model has been used in mathematics education research as an analytical tool for mathematical arguments (e.g. Conner et al., 2014; Rasmussen & Stephan, 2008). Toulmin diagrams, which pictorially represent relationships between an argument's components, allow researchers to investigate the characteristics of the arguments constructed by the collective in mathematics classrooms. Expanded Toulmin diagrams (Conner, 2008), which add colour to denote participants in the argumentation and include the teacher's actions in support of argumentation, allow further exploration of characteristics of arguments. In tandem with the teacher support for collective argu-

mentation framework (Conner et al., 2014), expanded Toulmin diagrams can be used to explore the ways teachers support collective argumentation.

The teacher support for collective argumentation (TSCA) framework (Conner et al., 2014) was developed from a study of two prospective teachers' orchestrations of mathematical discussions. The TSCA framework categorizes three ways teachers can support collective argumentation. A teacher can (1) directly contribute components of the argument (e.g. claim, data, warrant), (2) use questions to elicit components of the argument, or (3) use other supportive actions to encourage students in developing components of the argument, reinforce students' contributions, or bring students' attention to particular aspects of the argument (e.g. drawing representations, gestures, revoicing). The power of the TSCA framework is in providing the opportunity to see across the three types of teacher support and how they are working together to help students in the construction of arguments. This framework, when coupled with an examination of beliefs, allowed us to construct differentiated descriptions of support and the rationales for those supportive actions.

To study beliefs, we draw from Leatham's (2006) perspective of sensible systems of beliefs. This theoretical perspective suggests that in order for a belief to exist within a system it must make sense to the other surrounding beliefs within the system. Therefore, when a teacher's descriptions or actions seemed to contradict our inferences of her beliefs, we looked deeper to develop a better understanding of how a particular belief makes sense within a given system.

METHODS

Our findings are based on data collected in a study that explored prospective secondary mathematics teachers' beliefs about mathematics, teaching, and proof and how they supported collective argumentation during their student-teaching experiences. For this paper, we focused on the beliefs and practices of Ms. Carr and Ms. Bell. Data sources included video recordings and transcripts of a unit of instruction in each classroom and video recordings and transcripts of four interviews with each participant. For this paper, we focused on the first two days of classroom instruction for each student teacher, which captured the diversity of teaching practices observed in each unit.

To analyse the data, we used expanded Toulmin (1958/2003) diagrams to study how teachers support collective argumentation. We developed diagrams to represent each of the arguments and used the TSCA framework (Conner et al., 2014) to code the argument components and teacher support. We then reviewed counts of the coded data to look for common supportive actions and actions that were uncommon or possibly absent from the argumentation. This process allowed us to begin to identify patterns of support within individual arguments and across the dataset for each teacher.

To study beliefs, we analysed the interview data iteratively by first using broad codes, such as mathematics, proof, and teaching. We then created a more refined set of descriptive codes to describe the ways they talked about mathematics, proof, and

teaching. We constructed thematic narratives to characterize the beliefs we inferred and document the evidence we had to support our claims in sensible ways.

We created concept maps to investigate connections between our findings for teacher support and beliefs. We looked for disconfirming evidence and considered alternative explanations for the connections observed. We constructed narratives to refine our interpretations, which required us to search for evidence to make reasonable claims about relationships perceived among beliefs and practice.

RESULTS

Episode of Argumentation in Ms. Carr's Class

Ms. Carr taught geometry to tenth and eleventh grade students (ages 15-17) in a rural high school. In the focal class period, Ms. Carr introduced the concept of congruence, proved triangles congruent by the definition of congruence, and introduced triangle congruence theorems. The excerpt of an argument in Figure 1 is from Ms. Carr and her students establishing the congruence of a pair of corresponding angles in the given diagram (see green data in Figure 1). After this excerpt, the class went on to prove the triangles congruent. The transcript of this episode is found below. Due to space limitations, we include only a part of the beginning of the argument, in which the class establishes the congruence of two labelled angles.

- Ms. Carr: By the definition of congruent angles, since P and N have the same measure, we know that what? If the angles have the same measure, we can say that they are what? I know this seems like a silly question, but it is that magic word again.
- Students: {Congruent.}
- Ms. Carr: If the angles have the same measure, then we can say that they are congruent by the definition of congruent angles.

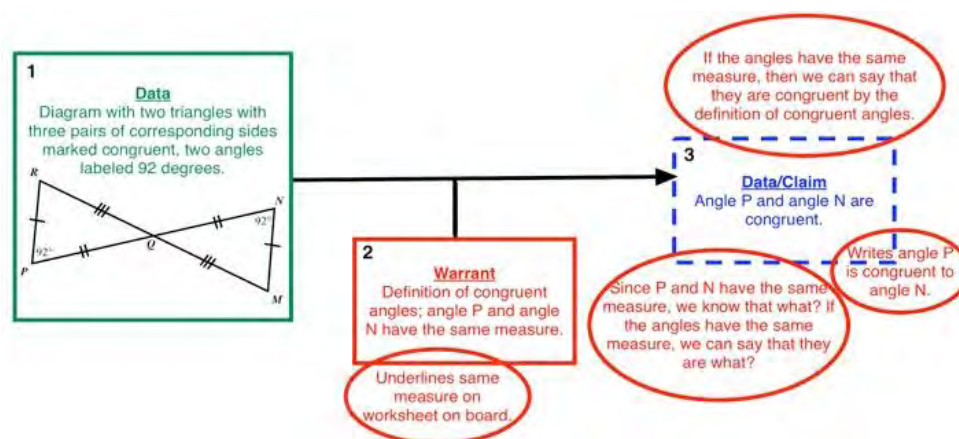


Figure 1: Diagram of Episode of Argumentation in Ms. Carr's Class

Characteristics of Ms. Carr's Support for Argumentation

Arguments in Ms. Carr's class were characterized by teacher-contributed components, and components of arguments not contributed by Ms. Carr were surrounded by ques-

tions and other supportive actions. She contributed to almost one-third of the data and claims and over half of the explicit warrants. Notice that she contributed the warrant in Figure 1 by saying, “By the definition of congruent angles, since P and N have the same measure.” Patterns of contributions of warrants and their surrounding support was important to understand her teaching because warrants are crucial to understanding the reasoning and explanations contributed within a class.

Most of Ms. Carr’s questions requested a factual answer; she asked students to identify a part of a figure or a term more than any other kind of request. Most of her questions that elicited warrants were classified as justification questions in the elaboration category. For example, when the class was discussing a theorem earlier in this class, Ms. Carr asked, “Why did you say it had to be true?” Her students gave a warrant, “The other angles are congruent,” after which she validated their statement, saying “Good” and restated it. When students contributed warrants during whole class discussions, most were both prompted by a question and accompanied by other supportive actions.

Two kinds of other supportive actions were commonly found with student-provided components: repeating and informing. These usually involved Ms. Carr restating a student’s contribution or expanding a student’s statement. In the argument in Figure 1, students contributed the claim that the angles are “congruent” (data/claim labelled 3 in Figure 1). Ms. Carr expanded their contribution by saying, “If the angles have the same measure, then we can say that they are congruent by the definition of congruent angles” and repeated their contribution by writing $\angle P \cong \angle N$ on the board.

Ms. Carr’s Beliefs

Ms. Carr began her first interview by sharing that she has “always really loved math.” She reiterated her love for mathematics in each interview. She described mathematics as objective, applicable, consistent, logical, and rewarding. Ms. Carr defined proof in mathematics as the process used to show something is true. When discussing proof, she thought it was important for students to be able to “coherently put thoughts together to make a reasonable argument.” Ms. Carr consistently preferred arguments that, in her words, “explained why” instead of just “explaining how.”

Ms. Carr’s beliefs about teaching involved imparting knowledge. She said, “I think that being able to convey what you’re trying to teach in different ways, so that everybody gets it, is really important.” After student teaching, Ms. Carr continued to describe teaching as imparting knowledge to students.

Teaching is like, you have this whole amount of knowledge that to you is organized and makes sense, and you have to take all that and just like pick little pieces out of it and try to like one day at a time impart that on somebody else.

She continued by saying her favourite part of student teaching was her ability to explain something well so that her students understand. Ms. Carr consistently described teaching as explaining and emphasized imparting knowledge to students. From statements across the interviews, we inferred that Ms. Carr believed that she was responsible for the explanations in the classroom.

Beliefs Visible in Ms. Carr's Support for Argumentation

Ms. Carr emphasized explanation in her beliefs about teaching, mathematics, and proof. In each interview, she said she loved mathematics, and she usually followed this saying she enjoyed explaining mathematics. She shared, "it's really exciting to love math and then explain it to somebody and have them get it." We found her descriptions of proof highlighted the importance of the explanatory role of proof in school mathematics. We inferred her beliefs about explanation were some of her core beliefs based on her statements in interviews and her observed support for argumentation.

Ms. Carr's beliefs about who was responsible for explanations in class provide insight into her support for argumentation. Even when students provided explanations with their warrants, she surrounded warrants with questions and other supportive actions. We believe she did this so the explanations were clear to other students. She believed she was responsible to impart knowledge by giving explanations, resulting in student contributions completely surrounded by teacher support.

One could conceive of an emphasis on explanation resulting in a completely different pattern in supportive actions, one in which students were expected to provide a majority of the warrants and were asked questions that encouraged them to explain their thinking. However, we hypothesize that one aspect of a teacher's beliefs would have to be different for this to be the case: This scenario requires a belief that *students* should be responsible for the explanations. Instead, with an emphasis on teacherprovided explanations, we see Ms. Carr contributing over half of the explicit warrants in her teaching and emphasizing the importance of the teacher's explanations.

Episode of Argumentation in Ms. Bell's Class

Ms. Bell taught a mathematics class for ninth grade students (ages 14-15) in the same rural high school. In the focal class periods, her students grappled with angle measures and angle sums in n -sided polygons. The argument shown in Figure 2 took place while students were working in small groups. Each group worked to find a formula for the sum of the interior angles of an n -sided polygon from measurements they had made on polygons on a worksheet (and on the board). Ms. Bell stopped to talk with a group of students about their progress.

- Martin: It goes up by 180 each time.
- Adam: Each one, yeah, and then, uh--
- Ms. Bell: So if it goes up by the same amount each time, what kind of function is it?
- Karin: Linear.
- Ms. Bell: Linear.
- Martin: So it's something--
- Ms. Bell: It's a linear function.
- Karin: I think the slope is going to be 180, because if you look at the table, it's going up by one and then by 180.

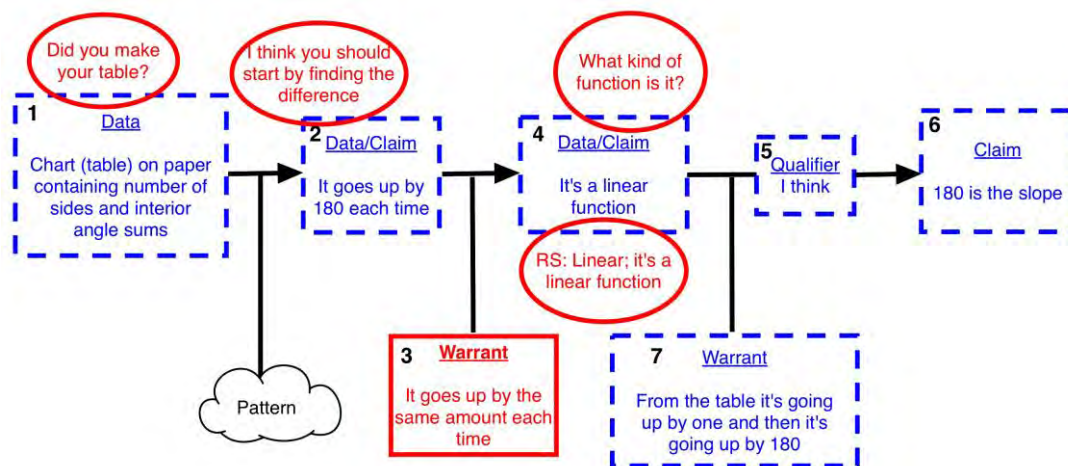


Figure 2: Diagram of Episode of Argumentation in Ms. Bell's Class

Characteristics of Ms. Bell's Support for Argumentation

Arguments in Ms. Bell's class were characterized by student-contributed components, and she prompted or otherwise supported some but not all components of arguments (see Figure 2). Ms. Bell contributed to less than one-fourth of the components of arguments in the focal class periods. She contributed approximately the same proportion of warrants as other components. This is significant in that it means she expected her students to contribute all components of arguments, including warrants.

Ms. Bell asked a wide variety of questions. She most frequently asked her students to make a conjecture, describe a method, or recall mathematical information. Two-thirds of the arguments during the focal class periods contained questions involving justification, explanation, interpretation, or conjecture. For instance, in the episode above, Ms. Bell said, "What kind of function is it?" to prompt Karin to interpret her observations. Ms. Bell did not always prompt students' contributions of warrants; Ms. Bell's students contributed 20 out of the 69 warrants without prompting. For instance, Karin said, "I think the slope is going to be 180, because if you look at the table, it's going up by one and then by 180." Karin offered the claim labelled 6 in Figure 2 and the warrant labelled 7 in the same sentence, without prompting from Ms. Bell.

Ms. Bell also exhibited a range of other supportive actions, many of which were categorized as repeating. These actions included displaying mathematical ideas on the board or restating what a student said, such as her restatement of Karin in Figure 2. Sometimes Ms. Bell asked a question to prompt a component and also repeated or otherwise supported the components (109 times in the two class periods). These actions appear to be related to a desire to clarify what was said.

Ms. Bell's Beliefs

Ms. Bell enjoyed many things about mathematics, including its logic and universality. In her first interview, she said, "I'm a very logical person, and I like the logic and the pureness behind it...it's just like a universal language." She described mathematics as "very pure and simple and elegant." She explained that mathematics was simple in the

same way that she found it to be elegant and pure because it is not clouded by opinion. Ms. Bell strongly believed students should understand where mathematical ideas come from, and this belief relates to her definition of proving as “trying to make an argument as to why something’s true...and build on that to form a conclusion.” She believed teachers and students should prove, saying students “should be able to explain why something is true, and investigate mathematically why that’s true.”

Ms. Bell had strong beliefs about the roles of the student and the teacher in a mathematics classroom. A teacher should “guide students towards finding their own understanding.” She intended for students in her classes to “ask as many questions as they want” and “actually come up with their own ideas.” Her picture of a classroom included group work and individual work as well as whole class discussions, and she saw herself guiding students toward “conjectures of what they think is true about a certain concept that they are investigating, and then..., as a class, prove that it is correct.” She intended to engage her students in exploring mathematical ideas, help to clarify their ideas, and ensure they could develop explanations.

Beliefs Visible in Ms. Bell’s Support for Argumentation

Ms. Bell’s belief that students should understand where mathematical ideas came from was visible in her emphasis on asking questions and particularly in the kinds of questions she asked. Ms. Bell believed explanation was important, particularly in the connections that students should make. Her focus on explanation, though, included a belief that students should actively engage with mathematics, participate, and figure out the mathematics, with some assistance from the teacher.

Ms. Bell’s emphasis on student engagement is clear in her few direct contributions and in her students’ contributions of warrants even when she did not explicitly prompt them. The norm of providing explanations or reasons for claims by giving warrants is evidence of her belief that the teacher should be a guide and everyone (including herself) should participate.

Ms. Bell’s focus on the teacher as a guide, one of whose responsibilities was to clarify ideas for students, can best be seen in her other supportive actions. She engaged in a range of supportive actions, but most of these came from the repeating category, in which she either restated what a student contributed or wrote that contribution on the board where it would be visible to others in the class.

CONCLUSION AND IMPLICATIONS

Ms. Carr and Ms. Bell were two teachers who had similar beliefs about mathematics, proof, and even the teaching of mathematics. However, their beliefs differed in a significant way. While Ms. Carr and Ms. Bell both believed that explanation was important in mathematics, Ms. Carr believed that she was responsible for the explanations, and Ms. Bell believed that explanations should be constructed and provided by her students. This difference in beliefs was visible in their different support for argumentation. Ms. Carr surrounded her students’ contributions with questions and other

supportive actions and provided many direct contributions. Ms. Bell asked a wide variety of questions and also engaged in other supportive actions, but many parts of arguments were contributed without her prompting, and she made few direct contributions.

There are many studies that examine teachers' beliefs about mathematics and teaching mathematics. These studies often report attempts to change teachers' beliefs about mathematics. This study suggests that more attention should be paid to teachers' beliefs about teaching mathematics, particularly to issues of authority and teacher and student roles in the classroom. It can be argued that Ms. Bell's and Ms. Carr's beliefs about mathematics and proof were productive in that they liked mathematics, appreciated its beauty and logic, and valued proof particularly in its explanatory role. However, different beliefs about who was responsible for explanations coincided with completely different support for collective argumentation in their classes.

Acknowledgments

This paper is based on work supported by University of Georgia Research Foundation under Grant FRG772 and the National Science Foundation under Grant 0227586. Opinions, findings, and conclusions do not necessarily reflect views of these agencies.

References

- Conner, A. (2008). Expanded Toulmin diagrams: A tool for investigating complex activity in classrooms. In O. Figueras, J. L. Cortina, S. Alatorre, T. Rojano, & A. Sepulveda (Eds.), *Proc. of the Joint Meeting of the Int. Group for the Psychology of Mathematics Education 32 and the North Am. Chap. of the Int. Group for the Psychology of Mathematics Education XXX* (Vol. 2, pp. 361-368). Morelia, Mexico: Cinvestav-UMSNH.
- Conner, A., Singletary, L. M., Smith, R. C., Wagner, P. A., & Francisco, R. T. (2014). Teacher support for collective argumentation: A framework for examining how teachers support students' engagement in mathematical activities. *Educational Studies in Mathematics*, 86(3), 401–429. doi:10.1007/s10649-014-9532-8.
- Krummheuer, G. (1995). The ethnography of argumentation. In P. Cobb & H. Bauersfeld (Eds.), *The emergence of mathematical meaning: Interaction in classroom cultures* (pp. 229–269). Hillsdale, NJ: Erlbaum.
- Leatham, K. (2006). Viewing mathematics teachers' beliefs as sensible systems. *Journal of Mathematics Teacher Education*, 9(1), 91-102.
- Rasmussen, C. L., & Stephan, M. (2008). A methodology for documenting collective activity. In A. Kelly, R. Lesh & J. Baek (Eds.), *Handbook of design research methods in education: Innovations in science, technology, engineering, and mathematics teaching and learning*. New York: Routledge.
- Toulmin, S. E. (2003). *The uses of argument* (updated ed.). New York, NY: Cambridge University Press. (Original work published 1958).
- Yackel, E. (2002). What we can learn from analyzing the teacher's role in collective argumentation. *Journal of Mathematical Behavior*, 21, 423-440.

FINANCIAL LITERACY: PRACTICING MONEY MANAGEMENT AS YOUR FUTURE-SELF

Mary B. Connolly and Cynthia Nicol

University of British Columbia, Canada

This paper explores middle school students' experiences of financial literacy, specifically money management. Thirty-seven students aged 13-14 from a range of socio-economic backgrounds participated in individual audio-recorded interviews on a budget-making task for a 25-year-old future self. Transcribed interviews were analyzed for students' perspectives on the budgeting activity and mathematical challenges they encountered. Results indicate that although familiar with the term 'budget' only 15% were initially able to describe a budgeting process. Nonetheless, all students developed a budget and almost all reported the activity deepened personal understandings of money management. The study provides insights to the developing literature on youth conceptions and experiences of financial literacy.

INTRODUCTION

The global community recognizes financial literacy as an essential life skill young people should be introduced to at school before they become “active financial consumers” (p. 30); this “early as possible” approach is seen as a more effective strategy than “attempting remedial actions in adulthood” (OECD, 2017, p. 45). Financial challenges for young people today are more complex than in previous generations. Reduced social safety nets and longer life expectancies increase risks and underscore the importance of financial literacy (OECD, 2014 p. 27). With this in mind, the OECD (2017) has modified its (adult) definition of financial literacy for young people to incorporate the ability to meet these future challenges by not just applying knowledge and skills but by analyzing, planning, and solving problems in a variety of situations (p. 49). Their definition of financial literacy for the 2015 PISA financial literacy assessment of 15 year-olds follows:

Financial literacy is knowledge and understanding of financial concepts and risks, and the skills, motivation and confidence to apply such knowledge and understanding in order to make effective decisions across a range of financial contexts, to improve the financial well-being of individuals and society, and to enable participation in economic life. (OECD, 2017, p. 50)

This new emphasis on analysis, planning and problem-solving across a wide range of financial contexts could be considered a “second generation” literacy definition and has implications for curriculum developers, educators and policy makers around the type of educational programs and pedagogies that may be most effective in equipping young people for the future. As countries around the world develop national financial

literacy strategies and define standards for youth, educators and curriculum developers are challenged to offer meaningful learning experiences that increase their students' financial literacy.

Financial education initiatives prove disappointing as many studies show little impact on students' financial knowledge. For example, young people in Canada scored no differently on surveys of those who did or did not take a high school financial course (BCSC, 2011) with similar findings in the U.S. with the Jump\$tart biennial surveys (Mandell, 2008). Recent meta-analyses of hundreds of financial education studies from around the world also showed virtually no changes in financial behavior (Fernandes, Lynch, & Netemeyer, 2014; Miller, Reichelstein, Salas, & Zia, 2015).

Studies have established that people with lower financial knowledge are more likely to hold consumer debt, and less likely to develop and maintain a budget and have savings for emergencies and retirement (see review in Lusardi & Mitchell, 2014). The ability to budget effectively, while requiring relatively basic number operation skills, has proven to be a challenge for many adults.

Budgeting and money management were found to be the most frequently covered topics, included in more than 85% of programs, in a review of over 90 adult financial education programs and 150 websites (Vitt et al., 2001). Saving and spending and budgets are also topics included in most financial literacy education standards and curriculum guidelines for young people around the world (see World Bank study of budget literacy practices from 34 countries by Masud, Pfeil, Agarwal, & Gonzalez, 2017). The topic's importance is recognized: Canadian middle school teachers ranked 'creating a budget' as the most valuable financial concept for their students to learn (Connolly & Nicol, 2015) and in a U.S. survey by financial services firm T. Rowe Price (2013), parents felt 'living within your means' was by far the most important advice they could give their 8-14 year-old children (p.19). Introducing concepts that provide students with a solid grounding in the fundamentals of money management is also seen as a strategy for contributing to a more equitable society by leveling the playing field, as financial knowledge has been shown to be strongly linked to a family's economic standing (Mandell, 2008; Pinto, 2013).

With the need for ongoing educational efforts for adults around such an important topic, and its recognized importance within youth financial literacy standards and among educators and parents, a better understanding of the learning experiences of young people around money management seems vital. To date, there is little research on the pedagogical practices of developing middle school students' money management skills that would provide a research-based conceptual foundation for the development of effective materials and practices (Connolly & Nicol, 2015 and Sawatzki, 2014 are exceptions).

This study provides foundational research with a sociocultural lens that contributes to a baseline understanding of student's conceptions and experiences with the process of decision-making around money management. Consistent with the OECD's emphasis

on analysis, planning and problem-solving for youth financial education, money management in this study context describes the ongoing process of budgeting: goal-setting and making spending and saving decisions that involve planning and problem-solving skills.

Curriculum designers and educators, many of them K-12 mathematics teachers, are tasked with helping young people develop and practice making money decisions. In this paper, we examine a middle school situated learning activity around budgeting to better understand the learner's perspective: the conceptions and experiences they bring, the concepts they find difficult to understand, and some of the math challenges they experience.

METHODS

As part of a broader phenomenography dissertation study of middle school conceptions of money management, this paper discusses the experiences of students during an activity where they are tasked with developing goals and a personal budget as their 25-year-old future-selves. "Tasks" are used in education research with the intention to "elicit in subjects estimates of their existing knowledge, growth in knowledge and also their representations" (p. 579), and ways of reasoning; tasks offer "insights into the creative activity of students in constructing new knowledge" (p. 581) as they engage in the problem-solving activities (Maher & Sigely, 2014). A key feature of phenomenography is that the unit of analysis is the experience of the individual student with the phenomenon, in this case the budgeting process, rather than the nature of the phenomenon itself from the researcher's perspective as with phenomenology (Giorgi, 1999; Marton, 1986).

The Participants

The participants in this study were 37 middle school students (ages 13-14) who had completed eighth grade, and were living in Vancouver, Canada. The boys (n=14) and girls (n=23) came from 10 different public and private high schools with a wide range of socioeconomic backgrounds. Many of the students had completed some financial education through an optional Business 8 module. The sampling was purposeful to provide a rich range of variations in backgrounds and perspectives. Students volunteered for the study and were provided a \$10 honorarium.

Data Collection

Semi-structured, one-on-one interviews (50-75 min) were conducted with each participant during which they were asked open-ended questions designed to reveal their personal experiences with money, their conceptions of the budgeting process, in addition to completing a situated-learning task that involves students creating a personal budget for a 25-year-old future-self. The interviews were audio-recorded and transcribed verbatim. Goal setting and budgeting worksheets and any mathematics hand calculations were scanned as part of the interview transcript documents.

Data Analysis

The phenomenographic analysis of the interview transcripts is currently in-process, where an outcome space will be created with categories of descriptions that represent the variation in the ways students experience or conceptualize money management. In this paper, we are exploring some of the dominant themes and perspectives that have emerged to-date from over 900 pages of participant interview transcripts. The themes discussed in this paper will focus on student perspectives that relate to the budgeting activity, including some mathematics challenges they encountered.

RESULTS AND DISCUSSION

In advance of the budgeting activity, students were asked if they had heard the term budget, and what they knew about it. There was a wide range of understandings, about 25% of the students were unfamiliar with the term, or thought it meant the cost of an item. Over half of the students described a budget as a limit or fixed amount that you could spend "...what I know about budget is that you basically just have a certain amount of money that day or a certain amount of money in your bank account or something." About 15% of students understood it to be more of a plan: "Budgeting is basically when you plan out how much money you're going to spend and how much you are going to save and you try to stick to that plan when you are buying stuff" and "...I think of like a spreadsheet, showing what you spent your money on and what you should spend your money on."

Students were then asked to suggest how a young adult, who was starting to get a regular paycheck, decides what to do with their money. About 45% of students described a few purchases they would suggest such as furniture or a vehicle or rent, and another 40% suggested covering needs first and then wants: "First pay for their necessities like food, where they live, rent, and then...just buy whatever" and another, "kind of break it up and see, 'Okay, I have this much to spend on...going out for dinner, and that much to spend on groceries and stuff,' so like kind of be careful or know your spending limit, I guess." A few students (15%) were able to describe a budgeting process: "They could probably set like a weekly budget...and then try to stay under that, and then divide it up into certain sections...like rent and then groceries and then leftover money." About 18% mentioned saving for emergency situations without prompting: "spend as little as you can until something big comes up because if you spend however you like, maybe something big comes up and then you don't have the money to pay for it, so then you're kind of stuck."

Activity: Budgeting for your future 25-year-old self

Students were asked to consider their future life goals. About one-third of students quickly jotted down items such as "buy a car and a house," 40% provided additional goals such as travel and specific purchases, while about 25% provided more detailed visions of their careers, family and life goals including contributing financially to their family and charities. These goals informed the allocation of their income in the second phase of the activity.

Students were given net incomes that represented a typical wage for a skilled 25 year-old employee in this geographic area, and provided a worksheet filled with blank boxes, each representing a \$100 bill, in which to allocate their income. They were also provided a list of possible, though not complete, expenses to assist them with estimates: rent for various sized apartments, utilities, cell phone and television choices, used car expenses, transit pass, vacation options, and house/condo down-payment savings. With these scaffolds, students were generally able to complete a first pass of their budget in around 5-10 minutes though most needed some prompting to allocate income for additional expenses such as clothes, entertainment, and savings for some of the goals they had created earlier.

It was during this phase that some challenges and misconceptions arose. For example, students might suggest funds for clothes or entertainment or vacations be taken from a generic “savings” amount, which then quickly became over-subscribed. By suggesting they designate their savings to specific expenses and goals, they gained insight into the true costs, and what they could afford. We reviewed their specific goals, such as traveling or purchasing a home and how those might be accomplished. Most students initially abandoned travel goals as they could not find funds for the \$900 expense in their budgets. When asked how they might accomplish it, say one year into the future, one student exclaimed “Oh! You could save-up!” excited and surprised for this discovery. The concept of breaking down a large expense into smaller monthly amounts was new to most students.

Estimating the monthly amount to save for large items was challenging for many students, many were not fluent with twelve times-tables and estimated with a large number, then a smaller one, then something in-between. Many also struggled with number calculations. In Figures 1 and 2, the student attempted to compute \$170 per month for a year, with an incorrect answer on both tries.

Figure 1:
$$\begin{array}{r} 170 \\ \times 12 \\ \hline 340 \\ 1700 \\ \hline 1840 \end{array}$$

Figure 2:
$$\begin{array}{r} 17 \\ \times 12 \\ \hline 34 \\ 170 \\ \hline 1760 \end{array}$$

Figures 1 and 2

Figure 3:
$$\begin{array}{r} 400 \\ \times 12 \\ \hline 800 \\ 4000 \\ \hline 4800 \end{array}$$

Figure 3

Figure 4:
$$\begin{array}{r} 250 \\ \div 12 \\ \hline 00 \end{array}$$

Figure 4

In Figure 3, the student, while getting the answer correct, put considerable effort into \$400 per month for one year. Nearly a third of students had similar challenges, using pencil and paper for calculations others did in their head. Another abandoned a calculation because she had “forgotten how to do this” when multiplying \$250 by twelve (Figure 4). Using her cell phone calculator, another confused addition with multiplication while attempting to determine an expense. The worksheet with its \$100 boxes

simplified computing the budget total, staying within one's net income, and visually determining available and allocated funds. While only basic numeracy (arithmetic) skills were required, many students had challenges that slowed down the completion of the exercise and required guidance from the researcher.

The final phase was to determine how a sudden job loss might affect their budget and students were told to assume they would lose one month's income a year from now before finding a new job. Most initially suggested using all available savings to pay their expenses and a few suggested borrowing from the bank, their friends or their parents. We worked on determining their monthly "run-rates" which could be calculated by adding up their required expenses and many students initially underestimated their monthly expenses and were surprised to learn how much they would need. They frequently went on to adjust their expenses by adding a roommate or giving up a car for a monthly transit pass. Students were asked to estimate the monthly savings they could put in an emergency fund for this purpose.

Virtually all students were very pleased with their budgets and felt they reflected a future they would look forward to. Several expressed relief that the future "wasn't so scary now" and another was disappointed to find she could not afford to pay for herself and her mother to travel the world.

Post-Activity: Reflection

Following the 30-minute activity, over 85% of students said they found the budgeting exercise useful, meaningful, and age appropriate for a grade eight student. The activity extended students' earlier understandings of budgets, with 70% able to describe specific benefits budgeting could bring to their current or future lives:

Financial decision-making: "I really liked this [budgeting]...because any idea I had [for my future] I always thought 'oh, it is too much' or 'I'm spending too much'...but now that I actually wrote down everything that I needed, it seems it is pretty good. Like it's doable." Another example: "if I had [a budget], I would basically look at it a lot. Whenever I think I'm not going to have enough money for something, I could just look at this to see what I can do or what I can change to make it so that I have enough..." and "it would also help me plan ahead for the future, and it helps me understand what's good and not good, and it's just a really useful exercise."

Improving planning and organization: "If [your plan] is in your mind, it's kind of jumbled, but if you write it down on paper, it's organized. You know what to do. It's like you can see a visual representation of everything, where it's going." And "[the budget was useful] because I might actually...do something like this if I'm thinking about my future...to help me get organized" and "all the bills are laid out here...you can see what area you can use to spend and what area you don't need to spend on."

Creating savings plans: "Kids my age would probably want to learn more about budgeting and budgets...how to spend your money [and] what you can do to save your money for specific things that you really want to do in the future with your family and friends." And "I learned that I could save money over time...save like \$50 every

month, and then over time it would get bigger and bigger.”

Making lifestyle decisions: “I might need more money...I probably would have a part-time [second] job.” And “I noticed that it is cheaper to share the cost with [a roommate] because you are not spending as much...” “With the vacation if you just set aside like, \$100 a month and you need \$900, you'll still get [there]...you can still go on your vacation.”

Other observations included relevance to their current situation: “I’m kind of doing it [budgeting] right now...sometimes I make my own lunch and then save some money for Friday so I can spend with my friends” and recognition of obstacles to budgeting: “it would help a lot, but some people just don't want to sit down and plan everything out...but I think it would be a good thing to do.”

While the budget activity was developed to learn about student’s experiences with money management, the task itself became a learning space. Almost all students reported deepened understanding of their pre-activity definitions of budgeting – no familiarity (25%) or a limited fixed amount of money (almost 60%) or a basic plan (15%) – through completion of the “future-self” budgeting activity. Students were able to see its applicability and benefit to their lives and found the activity rewarding and worthwhile, as well as age appropriate and relevant.

CONCLUSIONS

The ongoing challenges adults face around consumer debt and financial decision-making are being addressed through financial educational programs that focus extensively on budgeting and money-management topics (Vitt et al., 2001). These challenges are a global concern and the World Bank has begun assembling research around “budget literacy” practices for youth and adults from 34 countries around the world with a goal of improving economic outcomes (Masud et al., 2017). The OECD (2017) recently modified its definition of financial literacy for young people in order to emphasize the importance of not only financial knowledge and skills but also the ability to use them to analyze, plan and problem-solve in a variety of situations.

In this study, we contribute to a much-needed baseline understanding of student’s conceptions and experiences around money management and budgeting that would provide a research-based conceptual foundation for the development of effective materials and practices. Through a task-based one-on-one interview, that itself became a learning space, students revealed their understandings, misconceptions and challenges while working through a personal budget. Following the activity, their reflections revealed complex thinking about financial concepts: Budgeting helps with financial decision-making, improves planning and organization, helps in the creation of savings plans and in making lifestyle decisions. Some students recognized direct applicability to their lives and could describe planning obstacles. The insights provided by the students are invaluable for educators, curriculum developers and policy makers looking to advance financial literacy education.

Acknowledgements

This research was supported by the Social Sciences and Humanities Research Council of Canada, the University of British Columbia, and the Vancouver School District.

References

- BCSC. (2011). *National report card on youth financial literacy*, (pp. 1–72). Vancouver, Canada: BC Securities Commission.
- Connolly, M. B., & Nicol, C. (2015). Students and financial literacy: What do middle school students know? What do teachers want them to know? In K. Beswick, T. Muir, & J. Wells (Eds.), *Proceedings of PME 39*, (Vol. 2, pp. 185–192). Hobart, Australia: PME.
- Fernandes, D., Lynch, J. G., Jr., & Netemeyer, R. G. (2014). Financial literacy, financial education, and downstream financial behaviors. *Management Science*, 60(8), 1861–1883.
- Giorgi, A. (1999). A phenomenological perspective on some phenomenographic results on learning. *Journal of Phenomenological Psychology*, 30(2), 68.
- Lusardi, A., & Mitchell, O. S. (2014). The economic importance of financial literacy: Theory and evidence. *Journal of Economic Literature*, 52(1), 5–44.
- Maher, C. A., & Sigely, R. (2014). Task-based interviews in mathematics education. In S. Lerman (Ed.), *Encyclopedia of Mathematics Education*. Dordrecht: Springer Netherlands.
- Mandell, L. (2008). *The financial literacy of young American adults* (pp. 1–258). Washington: Jump\$tart Coalition for Personal Financial Literacy.
- Marton, F. (1986). Phenomenography: A research approach to investigating different understandings of reality. *Journal of Thought*, 21(3), 28–49.
- Masud, H., Pfeil, H., Agarwal, S., & Gonzalez Briseno, A. (2017). *International practices to promote budget literacy: Key findings and lessons learned* (pp. 1–233). Washington, DC: The World Bank.
- Miller, M., Reichelstein, J., Salas, C., & Zia, B. (2015). Can you help someone become financially capable? A meta-analysis of the literature. *The World Bank Research Observer*, 30(2), 1–27.
- OECD. (2014). *PISA 2012 results: Students and money. Financial literacy skills for the 21st Century* (Vol. VI, pp. 1–202). PISA: OECD Publishing.
- OECD. (2017). *PISA 2015 results: Students' financial literacy* (Vol. IV, pp. 1–270). PISA: OECD Publishing.
- Pinto, L. E. (2013). When politics trump evidence: Financial literacy education narratives following the global financial crisis. *Journal of Education Policy*, 28(1), 95–120.
- Sawatzki, C. (2014, September). *Connecting social and mathematical thinking: Using financial dilemmas to explore children's financial decision-making*. Monash University.
- T. Rowe Price. (2013), 5th Annual parents, kids & money survey. March 2013 (pp. 1–49). Retrieved from <https://corporate.troweprice.com/Money-Confident-Kids>
- Vitt, L. A., Anderson, C., Kent, J., Lyter, D. M., Siegenthaler, J. K., & Ward, J. (2001). *Personal finance and the rush to competence* (pp. 1–234). Fannie Mae Foundation: ISFS.

BOUNDARY CROSSING IN DESIGN BASED RESEARCH – LESSONS LEARNED FROM TAGGING DIDACTIC METADATA

Jason Cooper and Shai Olsher

University of Haifa, Israel

Teachers are coming to take a crucial role in designing the curriculum they teach, relying to varying extents on learning-resources that they gather. The internet, though rich in resources, does not support didactically-sensitive searching. To address this, we are developing a pair of tools, one for tagging didactic aspects of learning resources, and one for searching based on tagged metadata. Employing a design-based-research approach, we search for a set of metadata categories that will support changes in teachers' practices, yet will be comprehensible to teachers and useful in their current practices. We describe the "boundary" that this research has exposed between the communities of teachers and researchers, and the mutual learning that took place through boundary-crossing.

INTRODUCTION

Designing innovative tools for teachers poses a challenge; designers often aim to support and encourage transformative practices, yet if tools do not fit in with familiar goals and practices that teachers currently engage in, it is unlikely that they will be adopted. Design-based methodologies, which involve cycles of development and research, aim to address such challenges. Yet there are no clear-cut rules that determine how findings from research should influence re-design. If, for example, teachers are not using tools in accordance with the designers' goals, should the design be changed to help achieve the goals, should the goals be changed, or perhaps both?

In this article we describe the development of a tool for communal tagging of learning resources – assigning didactic metadata to tasks – which can then be used by individuals to search for tasks in a didactically-informed manner. The question that guided our design-based research was: What aspects of learning resources should be tagged to support teachers' needs as co-designers of the curriculum that they enact? To answer this question, it is necessary to assign meaning to the terms *teachers' needs* and *didactically-informed*. We recognize that teachers' perspectives, influenced by instructional practices of supplementing conventional textbooks, may be different from the perspectives of researchers concerned with the coherence of a curriculum that is substantially co-designed by teachers. Our goal is not to reconcile these conflicting perspectives. Rather, we aim to develop tools that will support the emergence of new practices that draw productively on both perspectives. In the research reported herein we aim to gain a deeper understanding of teachers' and researchers' perspectives on instructional design, in order to guide the re-designing of our tools. We report on an

experiment conducted with 7 practicing high-school teachers. Our focus is on a 90-minute discussion between these teachers and the first author that followed a tagging experiment. The contribution of this work is twofold; A. Research findings on teachers' interactions with curricula and teaching resources suggest how the availability of particular categories of tagged metadata can influence curricular decisions; B. Research methods suggest a model for co-designing tools for teachers, that stresses the need for a deep insight into teachers' practices and perspectives.

LITERATURE REVIEW AND THEORETICAL FRAMEWORK

Growing expectations that teachers integrate technology in their instruction, along with the emergence of “dynamic” e-textbooks (Pepin, Gueudet, Yerushalmy, Trouche, & Chazan, 2015), are placing teachers in the role of co-designers of the curriculum that they enact (Remillard, 2009). To design curricular sequences skilfully, teachers need to be sensitive to aspects of curricular coherence, such as mathematical correctness, epistemological stance to mathematical topics, sequencing that avoids gaps in the mathematical progression, consistent handling of mathematical objects, and consistency with national curricula (Gueudet, Pepin, & Trouche, 2013). Furthermore, this coherence-of-design needs to be maintained in the curriculum that is eventually enacted in classrooms (what Gueudet et al. call coherence-in-use, *ibid.*). Currently there is a lack of tools to support such notions of coherence, both in-design and in-use.

Development of such tools for teachers, and in particular the design-based research that we are conducting on teachers' interactions with these tools, exposes a “boundary” between mathematics education researchers and teachers, defined by Akkerman & Bakker as “a sociocultural difference leading to discontinuity in action or interaction” (2011, p. 133). Drawing on theories and methodologies of boundary research, we view a tagging tool, along with its categories of metadata, as a *boundary-object* (Star, 1989), allowing teachers and researchers to collaborate in spite of differences in perspectives; teachers may achieve one notion of coherence, using a tool that was designed with a somewhat different notion of coherence in mind. Such a boundary object can support learning – on the part of teachers and researchers – in the sense of gaining new understandings, developing identities, or changing practices (Akkerman & Bakker, 2011). Both teachers and researchers can learn about each other's perspectives through *boundary-crossing* – interacting with the goals and perspectives of others. Cycles of design/research may afford opportunities for such boundary-crossing. In this article we focus on one such cycle – an experiment based on an early design of the tool, and implications of the experiment for re-design. Accordingly, our research question is:

- What are the teachers' perspectives on metadata for curricular design, and how do they differ from the researchers'?

METHODOLOGY

The research reported herein was conducted with 7 experienced high school teachers of mathematics enrolled in a 2-year M.Ed. program, throughout a one-semester course on

uses of technology in teaching. The categories of metadata that were available for tagging were based on a preliminary review of relevant literature. The stages of the experiment were: 1. Teachers tagged one chapter (20 tasks) from an e-textbook on quadratic functions (Yerushalmy, Shternberg, & Katriel); 2. The tagging tool and its categories of metadata, along with other issues and dilemmas, were discussed, first in an online forum, and then in a 90-minute discussion conducted with the first author; 3. Teachers co-created a shared collection of 63 tagged teaching/learning resources on the topic of trigonometric equations; 4. Teachers used novel representational tools to construct didactically-informed learning sequences based on the tagged metadata. In the present article we focus on the discussion (stage #2). Data was analysed and presented as follows: First, we articulated principles that guided our choice of metadata categories for the initial design (section 4). Next, we selected parts of the discussion that highlighted teachers' perspectives, and compared them with the researchers' perspective to reveal differences and conflicts. We then analysed these differences, searching for themes (section 5). Finally, we address what was learned, in discussing implications of our findings for re-designing the tool (section 6).

BACKGROUND: WHAT METADATA IS RELEVANT FOR TAGGING?

The design of the tagging tool that was used in the experiment was based on a review of pertinent literature, in alignment with our goal of supporting the emergence of new curricular practices among participating teachers. We briefly describe the main metadata categories, and their rationale.

Curricular coverage: We follow Schwartz et al. (1995) in our conceptualization of the mathematics curriculum: Categories of mathematical and general *skills* (e.g. modelling, manipulating, inferring), enacted in four main mathematical domains (number/quantity, shape/space, pattern/function, chance/data), involving a variety of operations on and with mathematical objects (e.g. numbers, functions, shapes). A *balanced* curriculum should cover all relevant combinations of skills, objects and operations. Accordingly, we tag mathematical domain and object, and the extent to which each skill and operation features in a resource. These categories support a modular approach to curricular design; tagging the mathematical nature of the task, and avoiding categories such as grade and difficulty levels, imply that tasks can be used in many different contexts.

Mathematical expressivity and curricular specificity of a task (Sinclair & Jakiw 2005) refer to the richness of mathematical ideas, representations, and approaches on one hand, which often comes at the expense of the ease with which a task fits a specific curriculum on the other hand.

Representational modality of mathematical objects: Yerushalmy (2006) has demonstrated the importance of linked multi-modal representations of mathematical objects (e.g. functions) in interactive learning resources. Each modality (verbal, numeric, symbolic, graphic) is tagged separately.

Resource usage: We attend to two central aspects of enacting resources that are expected to be relevant for didactically-informed searching: curricular “role” (e.g. opening a topic, practice, homework, assessment, enrichment), which is relevant for sequencing tasks, and “class arrangement” (e.g. whole class, individual, pairs, groups).

FINDINGS – TEACHERS’ PERSPECTIVES ON TAGGING METADATA

In this section we describe the main themes of the teachers’ perspective on tagging learning resources that emerged from our analysis of the discussion (summarized in Table 1), and relate them to the researchers’ perspective. Teachers’ perspectives were substantiated through discussion with the first author, justifying their significance.

Teachers are pressed for time – implications for tagging and searching

Tim explained how teachers’ chronic lack of time can influence their approach to searching for resources: “I’ll never know which tasks I didn’t find. I’m not concerned about this. I won’t review 30 [tasks]. I’ll search until I find the first that works for me and go with it, even if in retrospect the next five are better”. Many of the teachers’ statements can be traced to this pressure their work exerts on them. Mike suggested that 4 categories of metadata are as much as teachers will be prepared to tag or to search by. Jim stressed the importance of a 1-bit value (good/bad) for efficiently filtering out inappropriate resources.

Wisdom of crowds

Many teachers expressed a tendency to rely on aspects of wisdom of crowds, assuming that resources would be tagged by many users, and that search tools would aggregate the results. The “quality” of a resource was considered an important aspect (“what good is it if I have a task that apparently answers all my pedagogical needs, but it’s simply *not good*”, Tim). The researcher expressed a possible concern regarding implications of quality-based tagging, since the taggers and the searcher may not share perspectives on quality. In response, two possible modes of use emerged. In one, teachers would first filter based on didactic metadata, and then use an aggregated measure of “quality” to narrow down the options. Tim explained: “If enough people agreed that it was good, I’d be more inclined to use it, less skeptical”. Amy recognized in tagging an opportunity for professional development: “You [the researcher] have a coherent view of teaching. You know what you want. But sometimes teachers can’t express what they’re looking for. While looking at the tasks that others liked, they can form an idea of what they want”.

Objective vs. subjective tagging

Though it was agreed that tagging, as a human endeavor, is inherently subjective, some categories of metadata were considered more objective (e.g. mathematical topic, representations) than others. Objective categories were considered most reliable as a basis for searching, since the tagger and the searcher are likely to agree on how a task should be tagged. Nevertheless, many teachers recognized affordances in tagging contextual data, which is inherently subjective. It was suggested that such data should be tagged

after a resource had been tried out in a particular classroom context. Suggested categories were the grade level in which the task had been used, its level of difficulty, how interesting it had been for the students, and how well it supported classroom heterogeneity. Some even suggested that there should be a free-text category for taggers to provide details on how they had used the resource, and tips on how to make the most of it. Amy explained how such contextual information could be relevant in the context of searching for resources: “Free-text is a powerful tool. I hear that Tim disliked the task for three reasons, and I decide if they’re relevant for me”. Amy was describing a mechanism by which a searcher can become familiar with a tagger by “hearing” his voice through free-text tagging, and can thus decide which aspects of this tagger’s metadata are subjectively relevant.

	Researchers’ perspective	Teachers’ perspective
Role of tasks	Modular – can be used in various instructional contexts	Linear – associated with a point in the curriculum
Salient task properties	Didactic attributes	Quality, classroom usability
Extent of tagging	Should be comprehensive	Should be efficient
Individuality of task selection	Varied needs in accordance with approaches to teaching	Reliance on wisdom of crowds
Nature of metadata	Objective, searchable	Idiosyncratic, reflecting personal experience
Roles of metadata	Support informed search for learning resources	(Additionally) Extend curricular repertoire

Table 1: Main differences between researchers’ and teachers’ perspectives

WHAT WAS LEARNED - IMPLICATIONS FOR RE-DESIGN

In this section we discuss implications of what we have learned for redesigning the tagging tool, noting some changes that we have already introduced, and their rationale.

Supporting the emergence of a hybrid practice for searching and selecting tasks

Learning through boundary-crossing often results in the emergence of hybrid practices (Akkerman & Bakker, 2011), combining aspects from both sides of the boundary. Currently, teachers’ methods for choosing resources rely on search engines such as google, that cannot narrow down their results by didactic relevance, but do rank them. This implies a 2-phase process: searching by keywords, and selecting from among the highest-ranking results. In discussing how didactic metadata would be used for choosing tasks, teachers envisioned a new practice that retains the second phase of the familiar practice. Metadata was seen as a way to narrow down the search, eliminating

irrelevant items, yet the final phase – selecting from among the search results – was based on personal preferences that cannot always be articulated in terms of didactic metadata. With this in mind, filtering on a range of grade levels, though inconsistent with the researchers’ modular approach to learning resources, may nevertheless be an effective means of reducing the number of relevant resources that need to be reviewed.

Importance of instructional context for tagging and selecting tasks

While the researchers tended to discourage tagging contextual data, viewing it as inconsistent with a modular approach to curriculum design, teachers tended to embrace contextuality. An attempt to reconcile these perspectives suggests that tagging can be a two-phase activity – the first based on “objective” aspects of resources, and the second based on context-dependent aspects of their enactment. Contextual metadata poses a challenge for a search tool: how to support teachers’ reliance on contextual data to achieve coherence-in-use, while at the same time encouraging them to make use of “objective” metadata for achieving coherence of design. A possible reconciliation is to rely on a 2-phase search process, where filtering is conducted on objective categories, and selection from among results may rely on contextual metadata. Currently, contextual metadata appears as a separate section of metadata categories, under the title “Resource Usage”, and can be used for filtering. Following the discussion, we decided to add three of the categories that were suggested by teachers: 1. Task duration; 2. Supports heterogeneous class; 3. “I liked this task”. Our reasons for endorsing these categories were: Duration resonates with the framework of balanced assessment, since a balanced curriculum should blend short-straightforward and long-complex tasks. Support for heterogeneity resonates with the notion of mathematical expressivity / curricular specificity, in addressing complementary concerns about curricular structure; where Sinclair & Jakiw (2005) were concerned with coherence of design, this new category expresses a parallel concern for coherence in use. The decision to support a “like” category is discussed in a separate section.

Sharing meanings within communities

Many of the teachers were concerned that the exact meanings of the metadata categories are not always self-evident (validity), and that different taggers might assign different values to the same resource (reliability). Professional development may help teachers conform to the designers’ curricular discourse. A more symmetrical approach, that does not privilege the designers’ discourse, is to view tagging as a community endeavour. Real communities (teachers in a particular school) or virtual communities (teachers whose tagging I choose to follow) may develop shared meanings for keywords, and come to agree on tagging norms through joint work. This suggests that tools should support the emergence of such communities.

The importance of “quality”

Activities that on paper appear promising, can nevertheless “fail” in class, for reasons that are difficult to anticipate. Knowing this, teachers were concerned with aspects of quality that transcend categories of didactic metadata. Currently, teachers concerned

with quality rely on wisdom of crowds (“likes”, ranking), on the reputation of proven sources (developers, repositories), or on the recommendation of a trusted peer. Some of these may need to be integrated into the tagging tool. We have taken a step in this direction, and have added a “Like” checkbox. Furthermore, it is possible to filter a search based on the identity of the tagger. Combined with “like”, this supports the notion of following recommendations of individual taggers. Moving from “wisdom of trusted individuals” to “wisdom of crowds” requires careful deliberation on the representation of multiple-taggings for a single resource.

CONCLUSIONS

We have described a cycle of our design-based research – an experiment conducted with an early design of a tagging tool, to elicit teachers’ perspectives on metadata for curricular design – and have discussed the implications of our findings for redesigning the tool. We propose to view this experiment as a case of boundary-crossing, using the tagging tool as a boundary-object that encourages the parties – teachers and researchers – to learn through reflection, which Akkerman and Bakker (2011) describe as “coming to realize and explicate differences between [perspectives of communities] and thus to learn something new about their own and others’ practices” (p. 144-145). We have shown how teachers’ perspectives are sometimes in conflict with researchers’ perspective that guided the development of the tool, and have demonstrated how such conflicts can lead to insights on the parties’ perspectives, and to potentially productive reconciliations. Reflecting upon a discussion on categories of metadata led to three kinds of insight: 1. Some suggestions for new categories (duration, support for heterogeneity) were found, upon reflection, to be consistent with the researchers’ perspectives; 2. Explication of teachers’ perspectives suggested *modes of use* that can support the co-existence of categories from different perspectives (e.g. objective didactic categories alongside subjective categories of quality in a 2-phase search procedure, or organizing tagging around communities where meanings are shared); 3. Teachers reflecting on contextual categories suggested unexpected *roles* for a tagging tool (e.g. as a means for teachers to expand their curricular discourse). Furthermore, many of these implications for re-design enhance the tagging tool’s role as a boundary-object, in supporting perspectives of both researchers and teachers.

Ultimately, assessment of categories of metadata as a boundary object needs to be extended to the context of didactically-informed searching for resources.

Acknowledgements

The research was supported by the Ministry of Science, Technology and Space, grant 3-12946. The authors wish to thank Michal Yerushalmy for her guidance and advice.

References

- Akkerman, S. F., & Bakker, A. (2011). Boundary crossing and boundary objects. *Review of Educational Research*, 81(2), 132-169.

- Gueudet, G., Pepin, B., & Trouche, L. (2013). Collective work with resources: an essential dimension for teacher documentation. *ZDM*, 45(7), 1003-1016.
<https://doi.org/10.1007/s11858-013-0527-1>
- Pepin, B., Gueudet, G., Yerushalmy, M., Trouche, L., & Chazan, D. (2015). e-textbooks in/for teaching and learning mathematics: A disruptive and potentially transformative educational technology. *Handbook of International Research in Mathematics Education. Third edition.*, 636-661.
- Remillard, J., (2009). Considering What We Know About the Relationship Between Teachers and Curriculum Materials. In Remillard, J., Herbel-Eisenmann, B. A., & Lloyd, G. M. (Eds.). (2009). *Mathematics teachers at work: connecting curriculum materials and classroom instruction* (pp. 85-92). New York: Routledge.
- Schwartz, J. L., Kenney, J., Ilias, S., Kelly, K., Sienkiewicz, T., Sivan, Y., . . . Yerushalmy, M. (1995, September). *BA - Assessing Mathematical Understanding and Skills Effectively* (AMUSE). Retrieved from Balanced Assessment:
<http://hgse.balancedassessment.org/amuse.html>
- Sinclair, N., & Jakiw, N. (2005). Understanding and projecting ICT trends in mathematics education. In S. Johnston-Wilder and D. Pimm (eds), *Teaching Secondary Mathematics with ICT* (pp. 235-251). Maidenhead: Open University Press.
- Star, S. L. (1989). The structure of ill-structured solutions: Boundary objects and heterogeneous distributed problem solving. In L. Gasser & M. Huhns (Eds.), *Distributed artificial intelligence* (pp. 37-54). San Mateo, CA: Morgan Kaufmann.
- Yerushalmy, M. (2006). Slower algebra students meet faster tools: Solving algebra word problems with graphing software. *Journal for Research in Mathematics Education*. 37(5), 356-387.
- Yerushalmy, M., Shternberg, B., & Katriel, H. *Products of Linear Functions*. Accessed Jan. 1, 2018, from VisualMATH - Functions and Algebra:
http://visualmath.haifa.ac.il/en/quadratic/products_of_linear_functions

JOURDAIN AND DIENES EFFECTS REVISITED – PLAYING TIC TAC TOE OR LEARNING NON-EUCLIDEAN GEOMETRY?

Jason Cooper and Alon Pinto

Weizmann Institute of Science

Research mathematicians often play a central role in determining educational policies, yet the relevance of their mathematical expertise may be (and indeed often is) questioned. A mathematician who developed a game based on non-Euclidean geometry, and a high school teacher, participated in a group discussion on the game, and were interviewed separately to elicit their perspectives on enriching advanced-track students through inquiry. Though their perspectives appeared in many ways incompatible or incommensurable, we suggest ways in which many of their conflicting concerns can be reconciled. In this we are proposing a model of cooperation among communities, where mathematicians' and teachers' contribution to mathematics education is mediated by mathematics education researchers.

INTRODUCTION, THEORETICAL FRAMEWORK

The founding fathers of mathematics education were research mathematicians such as Hans Freudenthal and Felix Klein. Yet, as mathematics education established itself as a separate academic discipline, drawing on theories and methodologies from the social sciences and psychology, the role of research mathematicians is no longer obvious (Fried, 2014). In fact, there have been many conflicts between the communities of mathematicians and mathematics educators, including the debate over educational reforms known as the Math Wars (e.g., Klein et al., 1999). Based on our professional interactions with teachers and with mathematicians, reported in previous research (e.g., Cooper & Karsenty, 2016; Pinto & Cooper, 2017), we see a great potential in supporting collaboration between these communities for promoting various aspects of mathematics education, including curricular reform, development of teaching resources, and teacher education. Achieving such collaboration is challenging, since the views on teaching and learning mathematics prevalent in these communities are not only different but in some ways incommensurable (Sfard, 1998). We take a Commognitive approach (Sfard, 2008), viewing communities' perspectives as discourse – acceptable modes of communication within professional communities. Sfard (1998) claims that the very notion of what constitutes mathematics is in a flux, and calls for dialogue among stakeholders to clarify their positions. Thus, in order to develop productive models of cooperation, it is first necessary to gain a deeper understanding of the communities' mathematical and pedagogical discourse.

The nature of this challenge is demonstrated in the example of Dienes, a research mathematician who designed “blocks” for the teaching of position systems of writing

numbers, and proposed a theory of the ‘psychodynamic process’ of teaching and learning mathematics. Brousseau’s criticism (1997) of Dienes’s approach highlights two related didactic pitfalls: 1. Declaring that children playing with these blocks have learned something about the decimal system is a case of the *Jourdain effect* – giving a scientific name to a trivial activity; 2. Underestimating the need for a teacher’s mediation in artifact-based learning is a case of the *Dienes effect* – expressing a belief in an infallible artificial genesis of mathematical knowledge.

Henri, a research mathematician who developed a version of Tic Tac Toe that models a geometry of affine spaces over finite fields, may appear to have fallen in both these traps, if he believes that children can learn about non-Euclidean geometries through playing a game that models the mathematics. Nevertheless, we were quite taken with the game, and felt that it might have pedagogical merit. We decided to investigate this by engaging mathematicians and teachers in discussions about the game. In this we aim to address the following question: *What are mathematicians’ and teachers’ perspectives on enriching students through game-based inquiry?*

METHODOLOGY

Our investigation of mathematicians’ and teachers’ perspectives focuses on two representatives of these communities – Henri – the mathematician who designed the game, and Abby – an experienced advanced-track high school teacher who holds a Ph.D. in Mathematics Education. We held a 2-hour discussion on the game with the participation of 2 research mathematicians (including Henri), 3 experienced secondary mathematics teachers (including Abby), and 3 researchers of mathematics education (including the authors). We conducted semi-structured interviews with Abby and Henri in order to further elicit their perspectives: 30-minute interviews with Abby before and after the discussion, and a 60-minute interview with Henri after the discussion. Though Henri and Abby are representatives of their communities, their discourse necessarily has individual aspects. The presence of additional representatives in the discussion allows our analysis to go beyond individual aspects, however in the present account we focus on the discourse of these two representatives, acknowledging that additional work will be required in order to generalize our findings. The discussion and the interviews were fully transcribed by the authors.

Our analysis is presented in three parts. We begin with summaries of Henri’s and Abby’s perspectives, based on all three data sources (group discussion and three interviews). Accuracy, credibility and validity were addressed through memberchecking – the members reviewed these sections, and changes were introduced to achieve accounts that are faithful to the members’ perspectives, while remaining faithful to the data. Next, we contrast the two perspectives, guided by our underlying theoretical framework of Commognition (Sfard, 2008), which relies on four aspects of discourse – keywords (e.g., *learning*, *concrete*), visual mediators (e.g., the game screen), repeating routines (e.g., patterns of teaching mathematics), and commonly endorsed narratives (e.g., “a wall is not painted in one coat”). In particular, we focus on differences in the

ways that Henri and Abby make use of common or similar keywords, on differences in the role they attribute to the visual mediation of the game, on differences in the teaching routines that are common in university and in high school, and on conflicting narratives regarding mathematics learning.

THE GAME

The idea for the game originated in popular talks that Henri gave on SET – a game consisting of 81 cards with different shapes, colors, and fills, in which players compete to form sets of three cards that satisfy certain conditions. In his talk, Henri attempted to present the game as a model of $A^4(F_3)$ – the 4-dimensional affine space over \mathbb{Z}_3 (Polster, 1998). However, his attempts were only partially successful. Though the audience soon recognized that any two cards can be uniquely extended to a set, they did not necessarily appreciate why this property creates a “geometry” of points (cards) and lines (valid



Figure 1: all three cards agree on color (green) and on count (3), and disagree shape and on fill.



Figure 2

sets, see Figure 1), where any two points define a unique line. Seeking a more “concrete” realization of an affine space, Henri developed a prototype mobile-phone version of Tic Tac Toe – PenTacToe – played on a 5×5 cyclic grid, where winning lines can have any slope. Figure 2 shows a possible board after four moves, where both players have each nearly completed a line. This board is schematically represented in Figure 3a. The game offers four line-preserving transformations that can be used to make winning situations more perceptually salient – vertical and horizontal affine translation, achieved by dragging up/down or left/right, and vertical and horizontal shearing transformations, achieved by clicking green buttons. Figure 3b shows the result of horizontal cyclic dragging to better see the O line, while Figure 3c shows the result of vertical shearing to change the slope of the X line. The game can be played against a friend, or with the computer taking the role of the opponent – in competition or in learning modes.

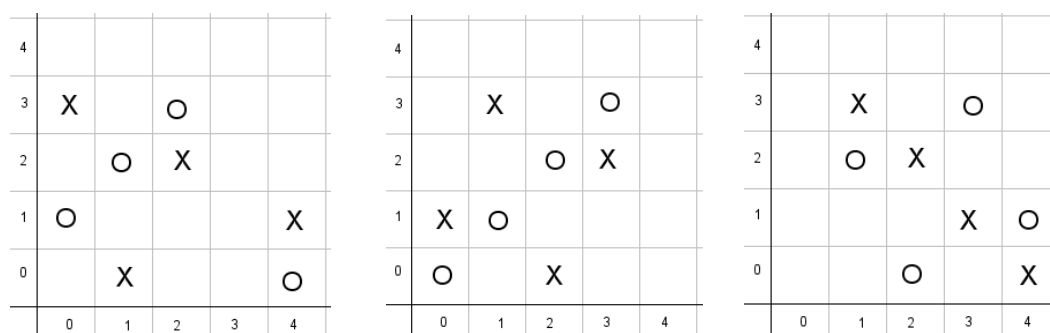


Figure 3: In the leftmost board, X has four squares on the line $y = 2x + 3$ and O has four squares on the line $y = x + 1$. The middle board is the result of dragging horizontally, to show O's line more clearly ($y = x$). The rightmost board is the result of a horizontal shearing transformation, showing X's line more clearly ($y = -x - 1$).

THE MATHEMATICIAN'S PERSPECTIVE

While Henri referred to the game as a minor side-project, he became quite enthusiastic about it after receiving positive feedback from colleagues and mathematics educators. He acknowledged that he had not yet given much thought to what children could gain from playing the game, or to the nature of mediation required to achieve such goals. He explicated his perspective in real time in response to questions and critique during the discussion, substantiating his claims on the basis of his experiences as a learner and an instructor, and his orientations towards mathematics. Though he had not designed the game as an in-school activity, at the end of the group discussion he was confident that the game could provide a worthwhile learning activity, even in the school context.

Student-centered concerns: motivation and engagement

Henri considered it crucial that the game engage children, regardless of whether they recognize the underlying mathematics. This influenced his design; for example, he chose a 5×5 grid, since with a smaller grid (3×3) the first player can easily win, whereas with a larger one (e.g., 7×7) it is quite easy for both players to force a tie. Henri did not consider students' motivation to learn a critical issue, emphasizing that the game was designed as enrichment for children who take an interest in mathematics.

Educator-centered concerns: The game's affordances and limitations

Initially, Henri maintained that playing the game is tantamount to doing mathematics (forming lines in a discrete geometry), even if the players are unaware of it. He believed that even without mediation, abstract mathematical ideas would 'percolate' at some intuitive level. During the group discussion he softened this assertion, noting that while the game could help bring abstract mathematical ideas 'closer' to children, they are not likely to learn these ideas without mediation. As learning objectives, Henri identified non-curricular topics such as modular arithmetic and analytical geometry over finite fields. He suggested sketching graphs of linear functions over a finite field and investigating their properties alongside the game, arguing that in this setting, linear algebra is accessible to most secondary students since it is concrete – it can be seen and played with. However, from Henri's perspective, such learning objectives were secondary. Viewing the learning of mathematics as an iterative process of abstraction and concretization, and saw great value in providing students with experiences of reconceptualization. The analogy between the game and the familiar Tic Tac Toe, and between the properties of valid configurations in the game and lines in the Euclidean plane, can inspire children to abstract and reconceptualize the concept of 'line'.

THE TEACHER'S PERSPECTIVE

Abby was quite critical of the game when interviewed before the group discussion, stating that it was "not her cup of tea". She substantiated her responses to our questions, before and after the group discussion, based on familiarity with her students and with what motivates them, on her teaching experience, on her theoretical approach to teaching and learning, and on her orientations towards mathematics.

Student-centered concerns: motivation and engagement

Generally speaking, Abby sees her students as ambitious and goal oriented, aiming to do well on their matriculation exams. They have little patience for enrichment activity that is not directly related to the content on which they will eventually be assessed.

Educator-centered concerns: The game's affordances and limitations

Though she did eventually consider the game to be engaging, Abby did not see affordances for the kind of learning she values. She does not favor teaching university-level topics that have little relevance for the high-school mathematics curriculum, and even if she could be convinced to teach non-Euclidean geometry, the time she would need for mediating finite fields and discrete geometry would be prohibitive in her institutional context. Furthermore, in order to mediate this knowledge responsibly, she would need to re-learn the university mathematics that she had learned many years ago. She does not believe that her students would be able to make sense of the underlying mathematics simply by interacting with the game. Abby does value opportunities to engage her students in mathematical investigation, however this does not require her to move outside the curricular content that her students are already familiar with. On the contrary, she has a vested interest in the content that she has taught, and cherishes opportunities to present it as a powerful tool for investigating interesting problems, not just for solving exercises. She suggested that rather than exploring transformations of affine functions over finite fields, students would be better off stretching the curriculum borders by further exploring functions they are familiar with, such as trigonometric functions. Following the discussion held with Henri, when asked if she could see what Henri sees in the game, Abby did agree that there is pedagogical merit in extending students' conception of line. However, she felt that Henri, from his university perspective, was underestimating the gap between what she called a concrete, continuous line and its extension to something discrete, which for students does not yet justify the name line. She felt that a larger grid might have provided a better bridge between continuous and discrete lines.

DISCONTINUITIES AND INCOMMENSURABILITIES

Though there are significant differences in Henri's and Abby's discourse, we first note some similarities: They both value extra-curricular enrichment and object to acceleration of talented students toward higher grade-level content, and they both believe that such enrichment should engage students in mathematical inquiry. In this section we contrast some major differences in their discourse, and discuss which aspects of their discourse are incompatible, in the sense that they have conflicting practical implications, and which are incommensurable, in the sense that it may not be possible – empirically or theoretically – to decide which to privilege over the other.

Nature of valued mathematical activity

For Henri, mathematics is about ideas and theories that are built through abstraction of familiar mathematical objects. The immediate aim of the game is to engage children in

the process of abstraction: taking the familiar notions of *geometry*, *point*, *line*, and developing a more abstract notion through engagement with a new and different model – 2-dimensional affine geometry, itself realized concretely in a game. Henri believes that students can be encouraged to develop competency in abstraction by providing them with the opportunities to practice it, which are rare in school. In contrast, Abby sees mathematics primarily as a problem-solving activity, where the curricular content that she teaches provides tools for engaging students in interesting and meaningful problems. Henri's and Abby's perspectives are incompatible in this respect, since Henri's entails activity with mathematical objects outside the curricular content.

Image of the learner

Affordances for learning that Henri and Abby see depend on their image of learners. Abby, an experienced high school teacher, arguably has a much more realistic image in mind than Henri, who in lieu of pedagogical experience, may be tacitly seeing a young version of himself as a potential learner – curious, talented, and highly motivated. As Abby pointed out in her interview, only a tiny minority of talented students are future-mathematicians, which questions the validity of Henri's pedagogical perspectives on the game, its implementation and its affordances. Conversely, Abby's estimation that only future-mathematicians will be interested in the mathematics underlying the game might also be skewed. The two discourses are incompatible in this respect, yet they are not incommensurable; it should be possible to empirically decide what motivates students, and what they can or cannot learn, with or without mediation.

Pedagogical roles of educators, theories of learning

Henri, in designing a game that models a mathematical object – $A^4(F_5)$, was not driven by explicit didactic goals, but rather by an agenda, not fully articulated, to foster a certain kind of learning. Accordingly, his main role was one of designer, and he had not planned any structured activity, though he did design a variety of playing modes – with the computer (learning or challenge) or with a friend. Though he does not have a clearly-articulated theory of learning, he appears to take a Piagetian approach in creating opportunities for learners to “rediscover” a piece of mathematics through individual interactions. Yet he values students' engagement with the game, even if they do not explicate any new mathematics, quoting a metaphor: “a wall is not painted with a single coat of paint”. In contrast, Abby, who holds a Ph.D. in mathematics education, explicitly holds a discursive Vygotskian theory of learning (Sfard, 2008), that is generally critical of the idea that students can construct new mathematical objects on their own. She clearly articulated the didactic affordance of the game in her view, as a vehicle for promoting an inquiry attitude to learning mathematics, yet for this inquiry to be productive, the teacher has a crucial role in mediating new mathematical ideas. Henri, in contrast, believes that playing the game could “bring the children closer to the mathematics”. Henri's and Abby's perspectives on learning are incommensurable, and to a large extent incompatible; however, the pedagogical roles they envision are not incompatible – Henri could explicate measurable long-term learning goals for his

game, even if he does not require that they be achieved in the short run, and he is not likely to object to teacher mediation, even if he does not consider it crucial.

What mathematical objects are *concrete*?

Both Henri and Abby felt that inquiry should build on *concrete* prior knowledge, but they used this word differently. Abby felt that students' prior meanings for "line" are *perceptually* concrete; "the concept of line is particular, concrete, lines are continuous, it has very closed properties, [students] can't see the extensions... The collection of discrete points [in the game] do not yet warrant the title *line*". For Henri, in contrast, the game provides an opportunity to connect lines in the game to a concrete *algebraic representation* of analytical lines $y = ax + b$. The two notions of concreteness are neither incompatible nor incommensurable; it should be possible to design models of new mathematical objects that are both perceptually and mathematically concrete.

	Henri	Abby
Nature of math	Abstracting familiar objects	Problem solving
Image of learner	Tacit, based on self	Based on experience
Pedagogical role	Design of game	Articulate learning goals Mediate learning
Theory of learning	Piagetian, cumulative	Vygotskian
Concrete means:	Mathematically familiar	Perceptually familiar

Table 1: Henri's and Abby's discourse on game-based inquiry

RAPPROCHEMENT

We suggest possible avenues for rapprochement – actions that can productively draw on Henri and Abby's discourse, satisfying what is crucial for them both.

Extending Henri's pedagogical discourse of students

Abby suggested a rapprochement: "one lesson, 45 minutes, let him gain some insights from sitting in, on my lesson or on someone else's, and then let him re-think the role of this game in the classroom". However, this may be a pedagogical version of the Dienes effect, in underestimating the need for mediating pedagogical knowledge in helping Henri gain a deep understanding of students, teaching and classrooms. Furthermore, it is possible that though Abby's image of students is grounded in pedagogical experience, it is also somewhat self-fulfilling. A more symmetric rapprochement may be achieved by encouraging Henri to participate in the design and execution of educational research on students' interaction with his game. Mathematics education researchers could play a central role in this avenue of rapprochement.

Designing mediations

Regarding mediation, three routes for rapprochement seem possible; in one, the role of designer and teacher are split – teachers can design classroom activities around tools

designed by mathematicians; in another, Henri could be encouraged to design interactions, much as he did for his lecture on SET; a third possibility, which to us appears the most promising, is for mathematicians and teachers to co-design activities that would be both mathematically and pedagogically appropriate.

IN CONCLUSION

Our point of departure in this paper was a mathematical game, designed by a research mathematician, that appeared to epitomize two well-known didactical pitfalls – the Jourdain and the Dienes effects. An experienced teacher concurred, and raised many more concerns regarding affordances of the game. Having discussed the game with mathematicians, teachers, and mathematics education researchers, and after carefully analyzing and contrasting the mathematician's and the teacher's perspectives, we have outlined how productive rapprochement may be possible, and have suggested a role for mathematics education researchers in achieving such rapprochement. Though Abby herself is a qualified researcher in this field, it appears that her stake as a teacher may make it difficult for her to take on this role herself.

References

- Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Kluwer Academic Publishers, Dordrecht.
- Cooper, J., & Karsenty, R. (2016). Can teachers and mathematicians communicate productively? The case of division with remainder. *Journal of Mathematics Teacher Education*. Retrieved Dec. 15, 2017, from <https://doi.org/10.1007/s10857-016-9358-7>
- Fried, M. N. (2014). Mathematics and mathematics education: Searching for common ground. In M. N. Fried, & T. Dreyfus (Eds.), *Mathematics and Mathematics Education: Searching for Common Ground* (pp. 3-22). New York: Springer, Advances in Mathematics Education series.
- Klein, D., Askey, R., Milgram, R. J., Wu, H.-H., Scharlemann, M., & Tsang, B. (1999). *An open letter to Richard Riley, United States Secretary of Education*. Retrieved Dec. 15, 2017, from California State University Northridge: <http://www.csun.edu/~vcmth00m/riley.html>
- Pinto, A., & Cooper, J. (2017). In the Pursuit of Relevance—Mathematicians Designing Tasks for Elementary School Teachers. *International Journal of Research in Undergraduate Mathematics Education*, 3(2), 311–337.
- Polster, B. (1998). A geometrical picture book. Springer.
- Sfard A. (1998) The Many Faces of Mathematics: Do Mathematicians and Researchers in Mathematics Education Speak about the Same Thing?. In: Sierpinska A., Kilpatrick J. (eds) *Mathematics Education as a Research Domain: A Search for Identity*. Glaukom, vol 4. Springer, Dordrecht.
- Sfard, A. (2008). Thinking as communicating: Human development, the growth of discourses, and mathematizing. Cambridge: Cambridge University Press.

FROM ‘FROWNS AND GROANS’ TO ‘ASTONISHMENT AND DELIGHT’: SEEKING INDICATORS OF A MATHEMATICS TEACHERS IDENTITY

Cosette Crisan and Melissa Rodd

UCL Institute of Education, University College London, UK

This paper reports on a research project based on designing and teaching in-service courses for Non-Specialist Teachers of Mathematics (NSTM). An NSTM is a school teacher who qualified to teach in a subject other than mathematics, yet teaches mathematics in secondary school (11-16 year old students). While the overall aim of our research was to describe what constitutes a trajectory towards a mathematics teacher identity for a NSTM, in this paper we explain how we sought indicators of a mathematics teacher identity. We do so by first describing how we adapted Wenger’s notion of identity, then advanced our ‘Modes of Belonging’ Mathematics Teacher Identity framework. After that we exemplify how we used our framework to locate indicators of mathematics teacher identity in the data from a narrative of NSTMs working on a particular piece of mathematics.

INTRODUCTION

In England, the shortage of mathematics teachers is well-recognised with the demand far outstripping supply. The latest available statistics on teacher supply gathered by the Department for Education revealed that “79.8 per cent of mathematics lessons taught to students in year groups 7-13 were taught by teachers with a relevant qualification; a decrease from 82.7 per cent in 2013” and “75.8 per cent of teachers of mathematics to year groups 7-13 held a relevant post A level qualification (down from 77.6 per cent in 2013)” (Ross 2015, p. 13). The crisis in teacher supply means that subjects like mathematics have to be covered by teachers who are not specialists in these subjects.

The current on-going need for specialist mathematics teachers is not unique to England, but well recognised at international level, too. In Eire, for example a national survey found that 48% of teachers of mathematics at post-primary schools were not mathematics qualified, while in Germany, research on ‘fachfremd’ (meaning ‘non-specialist’ in German) teachers of mathematics, includes Bosse’s (2014) findings that these teachers enjoyed teaching mathematics even though they viewed mathematics as if it was the mathematics of elementary school and they had had little professional development in mathematics teaching. In the United States, the NSTMs teachers are referred to as teaching ‘out-of-field’ (e.g., Ingersoll & Curran 2004), while in Australia, Hobbs (2013) found that teachers who were ‘teaching across specialisations’ (TAS) experienced discontinuities which can impact negatively on their confidence and efficiency as a teacher of the new subject.

At the first Teaching Across Specialisation (TAS) Collective convened in August 2014, presentations from countries across the world indicated the wide spread of the TAS phenomenon and a call for “Research is needed to establish the key features of effective professional development that leads to such transformation in identity and practice for out-of-field teachers.” (Hobbs & Törner 2014, p. 46) was launched.

OUR IN-SERVICE COURSE DESIGN

The design principle of our in-service mathematics courses for NSTMs was informed both by our research that showed that learning to teach a subject without a background in either content or teaching approaches requires focused re-training (Crisan and Rodd 2014) and also an appreciation of the fact that it can actually be quite difficult to teach out-of-field (du Plessis, Carroll & Gillies 2015; Hobbs 2013).

There are two key practices for teachers of mathematics: engaging in (school) mathematics and being a teacher. In the case of our NSTMs, the latter is an established practice, while the former is the practice they are developing. On our in-service mathematics courses, we brokered these two key practices, enabling connections between them, through explicit teaching of school mathematics content within discourses familiar to school teachers.

Our view that effective secondary mathematics teaching is founded on sound subject knowledge, together with a thorough, interconnected, knowledge of the curriculum and sympathetic understandings of students’ needs and interests informed the design of our in-service courses. In these courses, there was emphasis on revisiting and teaching school mathematics. This served not only to develop the NSTMs’ technical fluency of some of the more challenging topics taught at different levels of school education, but also to promote modes of mathematical enquiry such as generalisation, abstraction, reasoning and proof. We also emphasised precision in mathematical language, as well as recognition of conceptual structures within mathematics. Discussions of pedagogical nature, such as common students’ misconceptions, multiple representations of a concept, or different teaching approaches, were integral to course delivery.

OUR RESEARCH INTEREST

A prompt for our research came from the NSTMs themselves. One of our NSTMs (trained to teach humanities), who was applying for a promoted mathematics teacher post, told us that she cried when she saw simultaneous equations and, when that topic came up, always asked a colleague to teach it for her. On one hand, this teacher wanted to be thought of as an expert mathematics teacher, while on the other hand, she was not able either to fluently solve problems on this standard topic in the mathematics curriculum within our class or to contemplate teaching the topic to her students in school.

Such a disjunction confirmed our thinking that issues of identity were relevant to our work with NSTs. We became particularly interested in how to make sense of our NSTMs’ mathematics teacher identities formation and development over the duration of the course. The research presented in this paper is part of a larger research study roo-

ted in our teaching of four cohorts of NSTMs over the past four years in London, UK with an overall aim of answering our research question ‘What constitutes a trajectory towards a mathematics teacher identity for a NSTMs on an in-service course?’.

However, in order to answer this research questions, we first needed to be able to recognise indicators of a mathematics teacher identity and in this paper we offer an insight into how we engaged with theory and our data in seeking such indicators. We thus proceed to firstly explain how we adapted Wenger’s notion of identity to mathematics teacher identity, then we put forward a framework accounting for the three interlinked ‘Modes of Belonging: engagement, imagination and alignment’ (Wenger, 1998, p. 174) in order to make sense of identity formation in the two key practices of our NSTMs: learning mathematics and being a teacher and lastly, we illustrate how we explicitly sought indicators of a mathematics teacher identity through our engagement with the framework and data from a narrative of NSTMs working on a particular piece of mathematics.

ENGAGEMENT WITH WENGER’S PERSPECTIVE ON IDENTITY

While a variety of frameworks have been employed by researchers to describe teachers’ identity development in mathematics teacher in-service courses (e.g., Boaler 2001; Fennema & Nelson 1997), Graven & Lerman (2003) argued that Wenger’s (1998) social practice perspective of learning is a suitable framework to use to analyse the process of becoming a teacher of mathematics.

Hence we engaged with Wenger’s “Social ecology of identity” (Wenger 1998, p. 190) and adapted it and operationalised it as an analytic tool in the following way: the general illustrative examples in the table on page 190 (*ibid.*) were replaced by mathematics education-specific examples of indicators of aspects of identity, by drawing on our own teaching experiences at secondary school level and expertise in research informed teaching of prospective and practicing teachers. In this way, Wenger’s notion of identity was adapted to mathematics teacher identity by interpreting the three interlinked “Modes of Belonging: engagement, imagination and alignment” (Wenger 1998, p. 174) in the two key practices of doing mathematics (Identification with school mathematics) and being a teacher (Negotiability in mathematics teaching) as indicated in Table 1 below.

OUR ‘MODES OF BELONGING’ MATHEMATICS TEACHER IDENTITY FRAMEWORK

In our study, *Identification with school mathematics* refers to how the NSTMs constructed identities as learners of mathematics during our in-service course. Identification through engagement, imagination, and alignment refers to how the NSTMs invested themselves in learning about and doing school mathematics topics, how they constructed images about how students learn mathematics and how their views converged towards an increasing connection with how the mathematics teaching community views mathematics as a practice.

Negotiability in mathematics teaching through engagement, imagination, and alignment refers to how the NSTMs negotiated their ways in the mathematics teaching community, how the NSTMs constructed images of themselves as potential specialist mathematics teachers and how their views converged towards an increasing connection with the mathematics teaching community.

MATHEMATICS TEACHER IDENTITY				
Identification with (school) mathematics		Negotiability in mathematics teaching		
Identities of participation	Identities of non-participation	MODE	Identities of participation	Identities of non-participation
e.g. Enjoy thinking about the mathematics to be taught.	e.g. Avoid mathematical activity.	Engagement	e.g. Do in-service courses; facilitate students' presenting partial proofs which are discussed	e.g. Rely on text book or on downloaded powerpoint resources.
e.g. Find new ideas in standard topics.	e.g. Act as if there was only 'one correct method'; avoid thinking about alternative approaches.	Imagination	e.g. Share ideas, applications, etc. about mathematics with students; imagine self as a mathematics teacher.	e.g. When being asked by a student 'why are we doing this?' reply 'you need it for exam'.
e.g. Want to understand why, expect proof, work detail.	e.g. Routinely get answers to mathematics problems from internet/elsewhere; make errors.	Alignment	e.g. Discuss, with students, what progression they have made in mathematics.	e.g. Only show methods in exam mark scheme; want certification of maths specialism without engagement.

Table 1: 'Modes of Belonging' Mathematics Teacher Identity framework

SEEKING INDICATORS OF A MATHEMATICS TEACHER IDENTITY

Data

Throughout the delivery of the four year-long in-service courses we collected biographical data: routes into teaching; subject specialism of their teacher training; teaching

experience: of mathematics, if any, or of their subject specialism; mathematics-related material (written diagnostic assessment of mathematics subject knowledge and capacity to diagnose students' errors/misconceptions; collection of on-going mathematical work); and written reflections (done during and at the end of their course and essay assignments) from all participating teachers as an integral part of their respective course. We also conducted interviews and carried out school observations specifically for this research.

Data analysis

In the following we first explain how we interpreted and hence allocated data from a narrative related to a particular piece of mathematics as indicators of *Identities of participation* in both *Identification with (school) mathematics* and *Negotiability in mathematics teaching* in the table above.

Identification with (school mathematics): Identities of participation-Engagement: One of the activity we designed was intended to give the NSTMs opportunities to investigate number patterns in Pascal's triangle, at the same time facilitating for opportunities to identify for themselves patterns with which they were already familiar. In each cohort there were expressions of astonishment that there was so much mathematical content represented in 'Pascal's triangle', for instance: "how did he ['Pascal'] manage to fit it all in such a simple format?" (Lech, session discussion).

When looking at the mathematics within the Pascal triangle, the teachers were amazed to discover 'in the triangle' many mathematics topics they had previously studied. "It's all in there!" exclaimed one participants in disbelief.

The teachers experienced joy and surprise at noticing connections between different topics, starting to see mathematics in a new light, more than just a set body of independent knowledge and skills, clearly expressed by one other participant: "I actually quite like that. I couldn't grasp it and I can only just touch it – but I really like the fact that it's connected in different ways and we talk about...for example, Pascal's triangles here, there and then!".

Identification with (school mathematics): Identities of participation-Alignment: More advanced mathematics topics, such as binomial coefficients and combinatorial identities were also introduced using Pascal's triangle. However, when the identity

$$\sum_{i=0}^k \binom{n+i}{i} = \binom{n+k+1}{k} \quad n, k \in \mathbb{N}, n > k$$
 was projected on the board, NSTMs' responses expressed non-participation–mathematics-engagement through their **frowns and groans** and comments of 'frightening', 'scary', 'illegible'.

Our role as tutors on the course included helping the participants overcome the negative affect, and so we introduced the hockey sticks visually (as shown in the diagram) and encouraged the participants to investigate hockey sticks of different sizes.

1							
	1						
		1					
			1				
				1			
					1		
						1	
							1

The NSTMs then described the sticks numerically (e.g., $1 + 4 + 10 + 20 + 35 = 70$) and noticed a pattern emerging.

The hockey sticks were then represented using the binomial coefficient notation:

$$\binom{3}{0} + \binom{4}{1} + \binom{5}{2} + \binom{6}{3} + \binom{7}{4} = \binom{8}{4}.$$

Table 2: Pascal's triangle and the Hockey Stick Theorem

The NSTMs were then able to write down a generalization of the hockey sticks patterns, thus describing the Hockey Stick Theorem with the very expression that 'scared' them when shown earlier, namely $\sum_{i=0}^k \binom{n+i}{i} = \binom{n+k+1}{k}$. The NSTMs were **aston-**

ished to have arrived at this concise representation of the identity themselves and expressed **delight** at being able to 'see' the identity in all these formats!

Identification with (school mathematics): Identities of participation–Imagination: Being able to make sense of an abstract mathematical expression, as above, contributed to NSTMs' identity of belonging to the mathematics community through their participation in doing mathematics and alignment with the mathematics that specialist teachers know and do. For example, when yet another emergence of Pascal's triangle got the whole class excited, we classified this as the NSTMs participating mathematically by noticing connections between different mathematical topics, and also as an instance of "joy and satisfaction in undertaking mathematical practices" (Grootenboer & Zvenberger 2008, p. 246) that helped to create a positive group atmosphere, important in building a community of practice within the class of NSTMs and helping, through development of positive affect, the NSTMs participate in other communities of mathematics.

DISCUSSION

In this paper we described the first stage in our data analysis, namely that of explicitly seeking indicators of identity as conceptualised from interpreting Wenger's structure and adapted to mathematics teacher identity by us.

Data from the mathematical episode has been classified in terms of indicating participation in the three different 'Modes of Belonging' – engagement, imagination and alignment' as a result of engaging with the framework tool we put forward in Table 1. Having provided examples of how data were allocated to some of the cells in the Table 1, during the conference presentation we will engage further with our framework and we will exemplify how data collected from one NSTM participant was analysed and a narrative produced, once allocated to the table. In our research project, a similar exercise was applied to all the data we collected from our NSTMs, and this enabled us to

produce a narrative and describe our NSTMs positioning on (different) trajectories towards a mathematics teachers identity.

By considering practices central to being a secondary mathematics teacher, namely *Identification with school mathematics* and *Negotiability in mathematics teaching*, we have offered a way of thinking about mathematics teacher development. In Wenger's terms, our NSTMs were newcomers to the mathematics teaching community and as such they negotiated their trajectories towards becoming a mathematics teachers in their own ways and their individual 'Table 1's looked different from each other's and were different at different points in time. Using our framework at different points in an in-service course provided a way to evidence how mathematics teacher identities emerged and developed.

Graven (2005) points to identity transformation seldom being the focus of in-service courses and the researcher proposes that identity interacts with teachers learning and thus should be a focus of the design and provision of any in-service training. As such, our contribution to knowledge is in drawing attention to how mathematical knowledge is realised within non-specialist teachers' mathematics teacher identity and in developing understandings of non-specialist teachers' experience on an in-service course.

References

- Bosse, M. (2014). The practice of out-of-field teaching in mathematics classrooms. In L. Hobbs & G. Törner (Eds.), *Taking an international perspective on out-of-field teaching* (Proceedings and Agenda for Research and Action, 1st TAS Collective Symposium, pp. 33–34). Porto, Portugal, 30–31 August 2014.
- Crisan, C., & Rodd, M. (2014). Talking the talk...but walking the walk? How do non-specialist mathematics teachers come to see themselves as mathematics teachers?. In L. Hobbs & G. Törner (Eds.), *Taking an international perspective on out-of-field teaching* (Proceedings and Agenda for Research and Action, 1st TAS Collective Symposium, pp. 33–34). Porto, Portugal, 30–31 August 2014.
- Du Plessis, A., Carroll, A. & Gillies, R. (2014). The Meaning out-of-Field Teaching Has for Educational Leadership. *International Journal of Leadership in Education: Theory and Practice*, 20(1), 87–112.
- Graven, M. (2005). Mathematics teacher retention and the role of Identity: Sam's story. *Pythagoras*, 61, 2–10.
- Grootenboer, P. J., & Zevenbergen, R. (2008). Identity as a lens to understand learning mathematics: Developing a model. In M. Goos, R. Brown & K. Makar (Eds.), *Navigating currents and charting directions* (Proceedings of the 31st annual conference of the Mathematics Education Research Group of Australasia, Brisbane, Vol. 1, pp. 243–250). Brisbane: MERGA.
- Hobbs, L. (2013). Teaching 'out-of-field' as a boundary crossing event: factors shaping teacher identity. *International Journal of Science and Mathematics Education*, 11, 271–297.

- Hobbs, L. (2014). An agenda for Research and Action. In L. Hobbs & G. Turner, In L. Hobbs & G. Törner (Eds.) *Taking an international perspective on out-of-field teaching* (Proceedings and Agenda for Research and Action, 1st TAS Collective Symposium, pp. 38–48). Porto, Portugal.
- Ingersoll, R. M., & Curran, B. K. (2004). *Out-of-field teaching: The great obstacle to meeting the “Highly Qualified” teacher challenge*. NGA Centre for Best Practice. Issue Brief. Retrieved on 31 August 2015 from <http://www.gse.upenn.edu/pdf/rmi/Out-of-Field.pdf>.
- Roos, H., & Palmér, H. (2015). Communities of practice: exploring the diverse use of a theory. In K. Krainer and N. Vondrova (Eds.), *Proceedings of 9th Congress of European Research in Mathematics Education* (pp. 162–172). Prague, Czech Republic: Charles University in Prague, Faculty of Education and ERME.
- Wenger, E. (1998). *Communities of practice: Learning, meaning and identity*. New York: Cambridge.

LINKING THEORY AND PRACTICE: PROSPECTIVE TEACHERS CREATING FICTIONAL CLASSROOM DISCUSSIONS

Annalisa Cusi* and Francesca Morselli**

*University of Torino, Italy, **University of Genoa, Italy

In this contribution we discuss a course for prospective primary school teachers aimed at providing them with theoretical tools that could support their future work. Specifically, the course is based on a theory-informed design of tasks for pupils and subsequent creation of fictional classroom discussions focused on the same tasks. Our research questions concern prospective teachers' reflections on the activities in which they are involved and on the ways in which these activities could promote their professional development. By means of a qualitative analysis, we highlight categories of prospective teachers' reflections and interpret these results in terms of levels of awareness.

INTRODUCTION AND BACKGROUND

Research in mathematics teacher education stresses on the need for teachers' reflection about their own practice (Mason, 1998; Jaworski, 2004). Jaworski (2004) introduces a dialogical model for teacher professional development that involves teachers' critical reflection about their practice and the sharing of these reflections between teachers and researchers within communities of inquiry. The need for teachers' reflection about their own practice and an emphasis on the role of awareness in teaching is also highlighted by Mason (1998, 2008), who stresses on the importance of leading teachers to become aware “not simply of the fact of different ways of intervening, but of the fact of subtle sensitivities that guide or determine choices between types and timings of interventions” (2008, p. 49). According to him, this requires teacher educators to promote teachers' shifts of attention toward constructs, theories, and practices that can inform and guide their future choices.

This last claim suggests to focus on the complex interplay between theory and practice within teacher education programmes. With the aim of directing teachers' attention in a way that could foster their development of awareness about their own practice, Cusi and Malara (2016) introduce a methodology of working with in-service teachers that involves the use of specific theoretical tools for both the design of classroom activities (tasks and methodology) and the a-posteriori analysis of teaching-learning processes. This methodology can be outlined through this structure: (1) *sharing and study of theoretical tools*; (2) use of the theoretical tools to analyse classroom activities and excerpts of classroom discussions conducted during teaching experiments (*analysis of the practice of other teachers*); (3) use of the theoretical tools for the design of class-

room activities and implementation of these activities in the teachers' classes (*phase of planning/action*); (4) use of the theoretical tools to analyse classroom discussions conducted by the teachers in their classes (*analysis of teachers' own practice*); (5) *shared reflections* between teachers and researchers.

In the case of courses for prospective teachers, who do not work at school, the participants cannot implement in their classes the activities they created and, consequently, cannot analyse their own practice and share these reflections (phases 3-4-5 in the methodology). For this reason, it is necessary to introduce activities for prospective teachers that enable them to imagine what could happen in the class when the designed activities are implemented. We think that introducing activities with a narrative component could represent a way to fulfil this objective.

Narrative is a promising methodology of study, widely used for the investigation of beliefs and identity (Kaasila, 2007). Organizing facts in a narrative form encompasses finding/attributing temporal and causal connections, thus turning a sequence of events in a coherent whole in which each part contributes to the global meaning (Bruner 2003). This leads the narrator, and those who read or listen a narrative, to become aware of more or less implicit relations. Hence, narrative can constitute an excellent starting point to foster self-awareness and reflection. Different kinds of narrative activity may be proposed, such as narration of previous experiences as students (Dettori & Morselli, 2010), narration of lived experiences as teachers (Zaslavsky, Chapman & Leikin, 2003), reconstruction of classroom episodes that were previously seen on videos (Dolk & den Hertog, 2008). In Chapman (2005) prospective teachers write stories of both actual teaching (from their schooldays or the running practicum), and ideal, best practice teaching. Trainees are given theoretical means to analyse their narratives so that this approach can become part of their professional competence and also used in future work.

In the present contribution we confine ourselves to a special case of narrative activity, namely the creation of fictional classroom discussions. An example may be found in Lloyd (2006), who uses prospective teachers' fictional stories as a means to investigate their emerging identities. Lloyd concludes highlighting the importance of teacher education activities that “*focus intently on the roles of teacher and students in student-centred classrooms and on relationships between the treatment of mathematical subject-matter and students' learning*” (p. 81) and suggesting that further research should address the use of story creation tasks in prospective teacher education. The author also suggests promoting teachers' analysis of their own and other teachers' stories. Our study can be set in this research development, for the explicit intention of turning story writing into an educational activity, and for the focus on the role of the teacher.

More recently, Zazkis, Sinclair and Liljedahl (2009a, 2009b) advocate the use of lesson play, as a way to improve the traditional lesson plan by inserting zooms “*on one specific aspect of a lesson – interaction with students in general and with students' emerging conceptions in particular*” (p. 43, 2009a). The authors claim that writing a

virtual dialogue between teacher and students leads the prospective teacher to focus on the process of the teaching, on the mathematical language used for communicating, on the various forms of mathematical reasoning that might emerge in the classroom and on the possible models of pupils' conceptual schemes. The planning, implementation and evaluation of teacher education activities grounded on lesson play is discussed in Zazkis, Sinclair and Liljedahl (2009b), where the task design and prospective teachers' professional development is analysed referring to Mason (1998)'s three levels of awareness: *awareness-in-action*, that is the ability to act in the moment (for instance, a teacher is able to pose a question or correct a mistake, but is not able to justify his action); *awareness-in-discipline*, that is awareness of awareness-in-action (in comparison to the previous example, the teacher is able not only to take instructional choices, but also to justify them); *awareness-in-council*, that is awareness of awareness-in-discipline (the teacher is also aware of what is necessary to develop awareness-in-discipline).

A COURSE FOR PROSPECTIVE TEACHERS FOCUSED ON THE CREATION OF FICTIONAL CLASSROOM DISCUSSIONS

In tune with the perspective outlined in the previous paragraph, in the last years we have designed a course for prospective elementary school teachers based on an adaptation of the methodology introduced by Cusi and Malara (2016).

The prospective teachers participating in this course are attending their first year of degree courses (5 years totally). Prospective teachers are involved in different training activities within schools, but only starting from the second year. So, when they participate to our course, they do not have any experience in teaching. For this reason, besides the activities aimed at the *sharing and study of theoretical tools* (namely tools useful to frame the classroom activities and to analyse the roles played by the teacher during classroom discussions) and those focused on the *analysis of the practice of other teachers*, we involve prospective teachers in activities during which they are asked to *design tasks for pupils and create fictional classroom discussions* focused on the same tasks. An important requirement is to explicitly refer to theoretical tools for both the design of tasks and the creation of fictional discussions. The course also encompasses moments aimed at the *sharing and comparison between the different tasks and fictional classroom discussions* created by prospective teachers.

RESEARCH QUESTIONS AND RESEARCH METHODOLOGY

As teacher educators, we want to investigate the effectiveness of the activities presented in the previous paragraph in promoting professional development.

In this paper, we focus on prospective teachers' perception of the activities in which they are involved during the course. Specifically, we address the following research questions: (a) *What kind of reflections do they propose on the activity of creation of fictional classroom discussions?* (b) *What kind of reflections do they propose on the role played by theoretical tools?*

In order to investigate these aspects, at the end of the course, we asked to prospective teachers to write some reflections on the course, focusing on: (1) strengths/weaknesses of the activities faced during the course; (2) difficulties met when facing the activities; (3) usefulness of the theoretical tools when working on the activities; (4) reflections on their future work of teachers.

We collected 67 prospective teachers' written reflections and performed a qualitative analysis of data (Patton, 1990), aimed at identifying categories of reflection. Each researcher analysed data separately. Afterwards, emerged categories were compared and discussed so as to reach a shared system of categories.

ANALYSIS

In our analysis we identified categories of reflection on three specific foci: the activity of creating fictional discussions, the role of the theoretical tools, the course as an educational project. In the subsequent part we discuss such categories of reflection. The categories are presented on a list and after exemplified by means of representative excerpts from data. For each excerpt the pseudonym for the author appears at the end of the sentence.

In their reflections prospective teachers adopted the perspective of *students* who have to carry out a task assigned by their educator, or the perspective of *future teachers* who look at the assigned activity as support for their future practice. Hence, for each category, we consider the perspective of the narrating subject (student vs future teacher).

Focus 1: reflections on the activity of creating fictional classroom discussions

Prospective teachers report that, when creating fictional discussions, they experienced some *difficulties* that can be organized into the following categories: a) imagining pupils' interventions (not having at disposal any direct field experience); b) creating efficient interventions of the teacher, on the basis of the theoretical tools that were presented in the course; c) creating effective classroom interactions, alternating pupils' and teacher's interventions.

- a. It was difficult because I never worked with pupils in the classroom context, then imagining how pupils could answer to some questions was not easy – Lor
- b. Creating classroom discussions was quite hard. Imagining pupils' answers was nice, but finding the good inputs from the teacher not so much; the teacher was supposed to help pupils, but not too much, not to give the answer but lead the pupils to find it. ... when writing down the teacher's intervention we also had to think about the strategy we wanted to be activated - Pi
- c. I found difficult to create a quite complete discussion between teacher and pupils: understanding which answers the pupils could give to the teacher's answers and which input the teacher could give to pupils – Ba

The aforementioned categories may be related to the perspective of a *student* who has to carry out a given task (creating a fictional discussion, possibly referring to the theoretical tools that were presented in the course). The subsequent categories refer to

reflections on the way the activity of creating fictional discussions may *support* the future teaching practice. Adopting the perspective of *future teachers*, they report that the creating classroom discussions helped them in: a) identify themselves with both the pupils and the teacher; b) understanding the complex work of a teacher; c) doing “mental experiments” of their future work of discussion planning and managing; d) becoming aware of those competences that must be improved to become a good teacher.

- a. One of the strength of the activity was the fact that we had to adopt different points of view; we had to figure out the possible difficulties of a pupil (hence, we had to think as a pupil); it was very useful to create a suitable activity and a related discussion (hence, we had to think as a teacher) – Esp
- b. Imagining the conversations between pupils and teacher I could reflect on the fact that such roles, once you are in class, must be played in an active way, and teacher’s interventions must be done in a goal-oriented way. What is difficult is finding a balance between a conversation that develops by its own, where the teacher gets cues from pupils’ answers, and a goal-oriented conversation, with well crafted interventions aimed at eliciting specific issues and promoting learning- Cal
- c. This activity makes you hypothesize what could happen in the class and think about how to guide the discussion. This is important, if you see the activity as an experiment you do in your head, and not as a rigid mental scheme. Carrying out this activity is a work on thinking ability and, moreover, creating the conversation one can see how different kinds of questions that are attributed to the teacher lead to different scenarios of discussion – Za
- d. I had the opportunity to reflect on classroom discussion as a way to check understanding; [...] hence, as a hint for the future I think I’ll try to improve my argumentation and discussion competences so as to to use them in class - Mar.

Focus 2: reflections on the role of the theoretical tools

In general, prospective teachers describe the theoretical tools as useful. Those who adopt the perspective of *students* recognize that theoretical tools may help in carrying out the assigned task. We identified the following categories of *support* provided by the theoretical tools: a) Theoretical tools as a way to remedy to the lack of field experience; b) Theoretical tools as a source of inspiration for planning activities; c) Theoretical tools as a support for the creation of fictional discussions.

- a. Studying theory was very useful! Being at the beginning of the academic studies, theory allowed to remedy to my little field experience – Fon
- b. Creating a totally different activity would have been very complex because it is very difficult to structure a complete mathematical activity without the necessary theoretical framework and the good examples to understand - Fer
- c. I found very useful the roles of the teacher that were presented during the course, because they helped us to understand what were the roles of the teacher who asked specific questions - Pi

The prospective teachers who adopt the perspective of *future teachers* value theoretical tools as useful *supports* for their future practice, since they: a) provide guidelines for the teacher; b) help understanding the complex work of the teacher and reflect on her responsibility.

- a. Thanks to the course I could look at mathematics in a different way and from a different perspective; taking into account the process more than the final result, making thinking visible, collaborating, using a mathematical language... these are all principles that I will keep with me and sue in my future classes – Esp
- b. I found very useful the focus on the roles of the teacher, because it made me reflect on the multiple facets of the work of a teacher. [...] This means that, when in class, we'll have to think about the direction we are taking. This reminds us the teacher has a great power in her hands - Za

Focus 3: reflections on the course as an educational process

Prospective teachers propose deep reflections on the whole *educational process* in which they were involved thanks to the course, highlighting their development of awareness about the educational project that frames the course. These reflections can be subdivided into the following main categories: a) reflections that highlight the different focus of the course and the corresponding educational aims; b) reflections that highlight how the course has fostered prospective teachers' shifts of attention (Mason 1998, 2008); c) reflections that stress that the activities carried out during the course contributed in the changing of prospective teachers' vision of mathematics; d) reflections that identify the activities of the course as initial steps in a wider educational process in which prospective teachers are involved.

- a. A further strength of the course was the possibility, for us, to take the challenge of developing and adopting two different perspectives. I am referring to the perspective of the pupils (when we solved problems) and the perspective of the teacher (when we analysed the different activities). Since I do not have any field experience in school, this enabled me to actually “enter” in my future work and to overcome my point of view of a student, putting myself in a professional perspective. - Ca
- b. I particularly appraised the activities during which we had to reflect on didactical aspects related to the mathematical problems proposed to us. This made us reflect on important issues for our future work, such as the analysis of the teachers' actions, formative assessment etc. - Bi
- c. The course succeeded in rekindling the interest; it made us, future teachers, feel the protagonists of the solution of the proposed activities, as if we were mathematicians at work. – Po
- d. It could seem simple, but actually it requires dedication and a metacognitive reflection, the capacity of “diving” in a situation and bringing out the best. These activities were a sort of training: at first everything is new, then little by little you get confident [...] and you get trained, as an athlete. The road is long, but these activities were a good and useful starting point. – Pe

The course made people who are at their first academic year, with no field experience in school, acquire a first mindset; it gave them a first grid through which to look at mathematics lessons. I have no experience in working in the classes, but this course made me feel more prepared for the second year field experience. I feel I found the coordinates. - Za

CONCLUSIONS

The aim of this contribution was to present and discuss a prospective teacher education course aimed at promoting the link between theoretical tools and teaching practice by means of specifically crafted activities, namely the theory-informed design of tasks and the subsequent creation of fictional classroom discussions. Creation of fictional discussions is in tune with Zazkis et al. (2009a, 2009b)'s idea of lesson play; moreover, during our course prospective teachers are explicitly encouraged to make a link between theoretical tools and the planning of tasks and the creation of classroom discussions. Our research questions concerned prospective teachers' perceptions of the course and of the activity of creating fictional classroom discussions, with a special focus on their comprehension and appreciation of theoretical tools. To this aim, we performed a qualitative analysis of prospective teachers' final written reflections on the course. We identified categories of reflection, which concern the specific activities, the theoretical tools and the general organization of the course. Creating fictional classroom discussions was valued as a sort of "mental experiment", where prospective teachers could sketch teacher's interventions, imagine pupils' interventions, craft powerful interactions between the different actors of the discussion. Interestingly, prospective teachers reported a general appreciation of the theoretical tools, that were seen as a support in carrying out the assigned tasks, but also as relevant guidelines for their future practice as teachers. This suggests that the activity of creation of fictional classroom discussions could foster a change of perspective, from university students to future teachers.

Referring to Mason (1998)'s levels of awareness, we may identify different levels of awareness in prospective teachers' reflections. Those who reflect on how they carried out the assigned tasks (planning an activity, creating a fictional discussion) without any reference to the theoretical tools at disposal, show an awareness-in-action. Those who, in their reflections, connect their planning and creative process to the theoretical tools, thus using theory as a real support, show also awareness-in-discipline. Finally, those who get the sense of the whole educational project of the course, thus linking activity, theoretical tools and their professional development, demonstrate also awareness-in-council.

As a further step of our research, we plan to analyse fictional discussions created by prospective teachers, as a mean to study their professional development throughout the course. Moreover, we plan to explore the potentiality of a side activity that was carried out during the course, namely the comparison of fictional discussions created by different prospective teachers.

References

- Bruner, J. (2003). *Making Stories*. Harvard University Press, Cambridge, MA.
- Chapman, O. (2005). Stories of practice: a tool in preservice secondary teachers education. *Proceedings of the 15th ICMI Study 'The professional education and development of teachers of mathematics'*. Águas De Lindóia / San Paolo, Brazil. (electronic version).
- Cusi, A., & Malara, N.A. (2016). The Intertwining of Theory and Practice: Influences on Ways of Teaching and Teachers' Education. In L. English, & D. Kirshner (Eds.), *Handbook of International Research in Mathematics Education 3rd Edition* (504–522). Taylor & Francis.
- Dettori, G. & Morselli, F. (2010). Eliciting beliefs with a narrative activity in mathematics teacher education. In F. Furinghetti & F. Morselli (Eds.), *Proceedings of the Conference MAVI 15: Ongoing research on beliefs in mathematics education*, pp. 89–100.
- Dolk, M., & Den Hertog, J. (2008). Narratives in teacher education, *Interactive Learning Environments*, 16(3), pp. 215–229.
- Jaworski, B. (2004). Grappling with complexity: Co-learning in inquiry communities in mathematics teaching development. In M.J. Hoines & A.B. Fuglestad (Eds.), *Proceedings of PME 28* (Vol.1, pp. 17–36). Bergen, Norway.
- Kaasila, R. (2007). Using narrative inquiry for investigating the becoming of a mathematics teacher. *ZDM –International Journal of Mathematics Education*, 39 (3), 205–213.
- Lloyd, G.M. (2006). Preservice teachers' stories of mathematics classrooms: explorations of practice through fictional accounts. *Educational Studies in Mathematics*, 63, 57–87.
- Mason, J. (1998). Enabling teachers to be real teachers: Necessary levels of awareness and structure of attention. *Journal of Mathematics Teacher Education*, 1, 243–267.
- Mason, J. (2008). Being mathematical with and in front of learners. In B. Jaworski & T. Wood (Eds.), *International handbook of mathematics teachers education: Volume 4*. (p. 31–55). Rotterdam: Sense Publishers.
- Patton, M.Q. (1990). *Qualitative evaluation and research methods, 2nd edition*. Sage Publications.
- Zaslavsky, O., Chapman, O., & Leikin, R. (2003). Professional Development in Mathematics Education: Trends and Tasks. In A. J. Bishop, M. A. Clements, C. Keitel, J. Kilpatrick, & F. K. S. Leung (Eds.), *Second International Handbook of Mathematics Education* (pp. 877–917). Dordrecht, the Netherlands: Kluwer.
- Zazkis, R., Liljedahl, P., & Sinclair, N. (2009a). Lesson Plays: Planning Teaching versus Teaching Planning. *For the Learning of Mathematics*, 29(1), 40–47.
- Zazkis, R., Liljedahl, P., & Sinclair, N. (2009b). Lesson Play – A vehicle for multiple shifts of attention in teaching. In Davis, B. & Lerman, S. (Eds.), *Mathematical Action & Structures of Noticing: Studies inspired by John Mason*, pp.165–177. Sense Publishers.

ELEMENTARY STUDENTS' CONDITIONAL REASONING SKILLS IN MATHEMATICAL AND EVERYDAY CONTEXTS

Anastasia Datsogianni, Stefan Ufer, and Beate Sodian

University of Munich (LMU), Germany

Reasoning about conditional “if..then” statements is a central component of logical reasoning. Given the early start of the development of everyday conditional reasoning skills and complaints about secondary students’ failure to correctly interpret mathematical conditionals, it is a desiderate to describe the conditional reasoning in mathematics already at younger ages. We report on a study that explored if it is feasible to survey conditional reasoning skills in everyday contexts and mathematics with primary school students. Questionnaire data from 55 Cypriot primary school students show that the applied instrument is accessible to students, and reflect central predictions of Mental Model Theories of conditional reasoning for differences between the two contexts. We discuss first implications for research and instruction.

INTRODUCTION

Logical reasoning is considered one core foundation of critical thinking, analytical thinking and problem solving skills (Liu, Ludu & Holton, 2015) and one key component of advanced thinking amidst human species (Markovits & Barrouillet, 2002). One of the main aspects of logical reasoning is conditional reasoning, i.e. reasoning with if-then statements. Research from developmental psychology has identified corresponding skills in young students and some clear developmental patterns, but also a great deal of variation in performance due to various external factors. Moreover, a link between conditional reasoning and mathematics has been found only in the case of late adolescence and adults (Attridge & Inglis, 2013; Inglis & Simpson, 2009). Our knowledge about elementary students’ conditional reasoning within mathematics is still weak, despite claims in the literature about the importance of logical reasoning for mathematics learning and success (e.g., Morsanyi & Szűcs, 2014; Nunes et al., 2007). Given that current theories describe conditional reasoning as a process that involves domain-specific knowledge, it is an open question to which extent students can transfer their existing logical reasoning skills to contexts that involve mathematical concepts. The main goal of this exploratory study is to investigate if it is feasible to survey elementary school students’ skills to reason with conditionals that involve mathematical concepts.

ACCOUNTS OF CONDITIONAL REASONING

Conditional reasoning tasks usually consist of a conditional rule of the form “if p then q ” as a major premise, and a minor premise. Four different minor premises differentiate four possible logical forms of inference: p is true (Modus Ponens, MP), p is false

(Denial of the Antecedent, DA), *q is true* (Affirmation of the Consequent, AC), and *q is false* (Modus Tollens, MT). Our work is based on the traditional interpretation of conditionals: *p* is sufficient, but not necessary for *q* (Evans & Over, 2004). Depending on the logical form (minor premise), different inferences can be made: Definite conclusions can be drawn for MP (“*q is true*”) and MT (“*p is false*”), while AC resp. DA do not allow for definite conclusions about *p* resp. *q*. Across studies on reasoning in everyday contexts, the expected correct answer is provided more frequently for MP than for MT and AC, and more frequently for MT and AC than for DA (Schroyens et al., 2001).

Two classes of theories are currently discussed to describe conditional reasoning: Mental Logic Theories (MLTs) assume that making inferences is based on formal mental rules (Schroyens et al., 2001). These theories primarily focus on the logical structure of the task. According to Mental Model Theories (MMTs), inferences are drawn by constructing mental models, that encode information about the meaning of the conditional (Johnson-Laird & Byrne, 2002). Both theories would support that it is easier to reason with inferences that are embedded in a familiar context than in an unfamiliar context. From the perspective of MLTs, however, these differences should be independent of the logical form. Contrary, MMTs would predict specific effects of knowledge about the context for different logical forms.

In the sequel, we will focus on the MMT account of conditional reasoning. According to this, individuals initially represent conditionals by the conjunctive model that describes a case in which *p and q* are valid, and check if this model is consistent with the minor premise. For MP, this is the case, and the conclusion *q is true* can be drawn. If the model is not consistent with the minor premise, further alternative models have to be constructed. Based on working-memory considerations, Barrouillet & Lecas (1999) propose that individuals’ treatment of conditionals evolves from a conjunctive-like interpretation (only *p and q*), to a biconditional (*p and q; not-p and not-q*), and then a conditional interpretation (*p and q; not-p and not-q; not-p and q*). Changes from a conjunctive interpretation to a biconditional interpretation of implication statements would thus be reflected in increased solution rates for MT, further changes towards a conditional interpretation would be reflected in increased solution rates for DA and AC.

Primary Students’ Conditional Reasoning in Everyday Contexts

Research in developmental psychology has shown that even very young children possess basic abilities in at least some forms of conditional reasoning when tasks are presented in a familiar everyday context (e.g., Markovits & Thomson, 2008). However, there is a great deal of variation in performance due to external factors such as the type of instructions (Saelen, Markovits & Klein, 2009), the context in which the inferences are drawn and evaluated (familiar premises, false premises, or abstract conditional premises; Markovits & Lortie-Forgues, 2011), or how strongly the antecedent and the consequent are associated (Markovits et al., 1998).

The reasoners' ability to generate alternative models (beyond p and q) for the given conditional is considered a crucial prerequisite to draw valid inferences. This is particularly important to infer the uncertainty of AC and DA inferences, where two models are consistent with the minor premise that lead to different conclusions (Markovits & Lortie Forgues, 2011), but also for MT where the initial model is not sufficient to draw inferences. A range of studies illustrate that this generation of alternatives is not a generic process of recalling what *might be possible given the premises*, but involves recalling concrete instances from semantic memory of *what is possible in the given context* (Markovits, 2014). For example, a study of Janveau-Brennan and Markovits (1999) shows that performance in conditional reasoning tasks was related to performance on a task that required students to generate alternative causes for the conclusion of the conditional, besides the given prerequisite. Thus, reasoning with conditionals, that involve mathematical concepts, will plausibly vary with available knowledge about these concepts.

Conditional Reasoning with in Mathematical Contexts

Problems about students' deductive reasoning in mathematics are often reported in the context of proof (e.g., Stylianides & Stylianides, 2007), often focusing on the difference between an implication and its converse (Küchemann & Hoyles, 2002). However, conditional reasoning is not restricted to proofs. For example, the application of simple mathematical conditionals during problem solving (e.g., "*If I arrange three rows of four squares each, I need 12 squares.*") can involve conditional reasoning processes in all logical forms. We use the term *conditional reasoning in mathematical contexts* to describe reasoning about such conditionals that describe mathematical structures. MMTs of conditional reasoning propose that, even though similar processes might be involved and similar strategies (cf. Barrouillet & Lecas, 1999) may be applied for conditional reasoning in mathematical and everyday contexts, the availability of knowledge about the underlying concepts will modulate reasoning in mathematical contexts. In particular, it can be expected that increasing concept knowledge will support conditional reasoning skills when alternative models are necessary, i.e. for AC, DA, and MT tasks.

Logical and conditional reasoning are considered to be closely linked with mathematics learning (e.g., Morsanyi & Szűcs, 2014), and it is plausible that conditional reasoning is involved when students discuss about mathematical concepts, their properties and relations in the classroom. However, in light of the tension between early conditional reasoning skills in everyday contexts (Markovits & Thomson, 2008) and frequent claims about students' fallacies when dealing with conditionals in mathematics (e.g., Küchemann & Hoyles, 2002), it is an open question how conditional reasoning in the two contexts is related.

GOALS AND METHODS OF THE STUDY

The goal of the current study was primarily to study if it is feasible to survey elementary students' conditional reasoning skills in everyday and mathematical contexts.

Moreover, we aimed to compare grade 2 and grade 4 students' conditional reasoning skills in everyday and mathematical contexts because of differences in mathematical knowledge. The study serves as a feasibility study for more detailed investigations in the future and, in particular, focused the following questions:

- (1) How do students' conditional reasoning skills in mathematics and everyday contexts develop during elementary school age?
- (2) In which way are students' conditional reasoning skills dependent on the required logical form of inference in each context?
- (3) Do students' conditional skills develop similarly for the four logical forms of inference similarly in both contexts, or are there specific differences?

A total of 55 elementary students (4th graders $n=13$: $M=9.5$ years, 6th graders $n=42$: $M=11.5$ years) from a public school of Cyprus participated in a cross-sectional survey study. A questionnaire was constructed with four conditional reasoning tasks within one common cover story. Two of the tasks focused on plausible conditionals in everyday situations, such as having fever, and two of the tasks focused on conditionals, which described multiplicative or additive structures in the context of the cover story. For example, one task in the mathematical context concerned dwarf houses. It was explained that these always consist of several rows, all with the same number of quadratic rooms of the same size. The major rule in this task was "*If a dwarf's house has exactly 2 rows of 4 rooms each, then it has 12 windows*". All four tasks contained one item for each logical form (MP, MT, DA, AC). The verbal structure of the tasks, items, and answer alternatives closely followed existing studies on conditional reasoning (e.g., Markovits & Lortie-Forgues, 2011). Tasks and items were arranged in a fixed, random order. The first author administered questionnaire in 40 minute classroom sessions. While the administrator presented the items to the class using a computer presentation, the students had booklets to provide their answers. For each item, students were presented with the major and the minor premise for the respective logical form as well as one possible conclusion. They were asked to indicate if it can be inferred from the provided premises that the conclusion is true for sure, if it can be inferred that the conclusion is false for sure, or if it cannot be inferred for sure, if the conclusion is true or false from the available information. No reasons for the answers were requested to gather students' intuitive responses.

Students' answers for each item were coded as correct or false according to the interpretation of conditionals described above. The data was analysed using Generalized Linear Mixed Models (GLMM), a generalization of logistic regression, that allows to analyze the data on item level, but still take into account dependencies between answers provided by each student. Apart from the questionnaire study, the items were also used in one-to-one-interviews with 16 second and 8 fourth graders.

RESULTS

Regarding the feasibility of the measurement, the group test sessions as well as the interviews showed that also children in grade 2 understood the tasks from both con-

texts. Out of 880 possible answers in the questionnaire survey (55 participants x 4 tasks x 4 items), participants provided 870 answers. Overall, 69.9% of the answers were correct, illustrating early conditional reasoning skills under specific conditions. The most frequent wrong answers were that no conclusion can be taken for MP (14.7% of all MP items) and MT (20.3%), that the proposed conclusion is correct for AC (33.3%), or that it is incorrect for DA (26.4%).

Regarding question (1), we expected higher solution rates for the everyday context than in the mathematical context, because it would be harder to generate mental models for the mathematical situations. Moreover, we expected this difference to be more pronounced in grade 4 than in grade 6 because we thought that students in grade 6 would have acquired more connected knowledge of addition and multiplication, allowing them to generate alternative mental models more easily. A GLMM analysis with independent variables *grade level* and *context* showed that students provided more correct answers in the everyday context than in the mathematical context in grade 4, while performance was similar in grade 6 (fig. 1a).

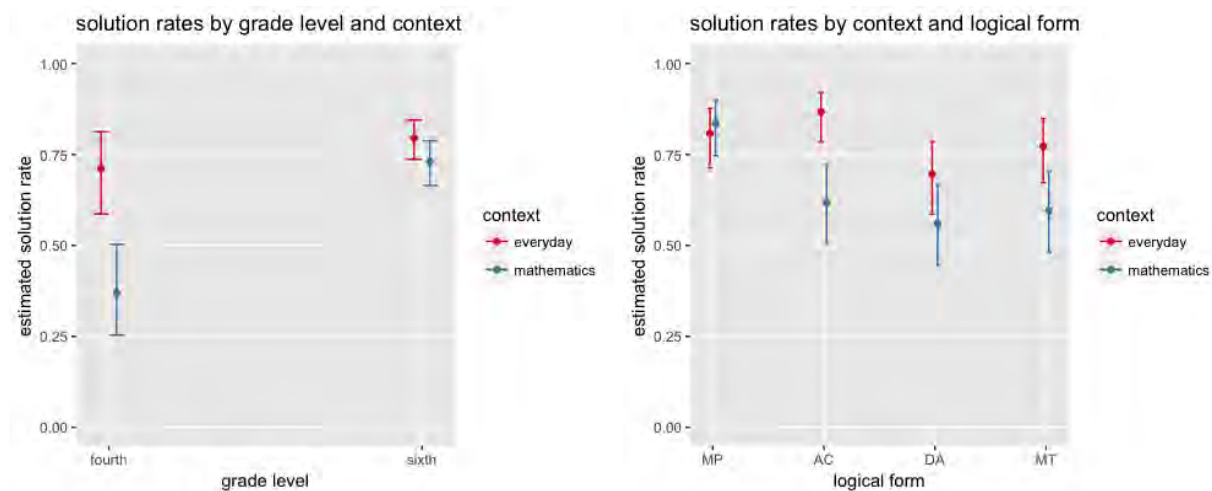


Fig. 1a and b: estimated solution rates and 95% confidence intervals by grade level and context (1a, left) resp. by context and logical form (1b, right)

Regarding question (2), we expected only small differences between the four logical forms in the everyday context, because early reasoning skills have been observed in the past for plausible everyday conditionals. However, we expected that in particular those forms of reasoning would be harder in the mathematical context, which involve construction of alternative models (MT, AC, DA). A GLMM analysis with independent variables *inference type* and *context* showed no significant differences between the solution rates for the four logical forms in the everyday context (fig. 1b). In the mathematical context, however, AC, DA and MT each turned out to be harder than MP (fig. 1b). The largest difference between the two contexts was found for AC, which requires generating a mental model that satisfies the consequent and does not satisfy the antecedent, the last stage of the Barrouillet & Lecas (1999) model.

Regarding the development for each inference type, question (3), we estimated

GLMMs for each context separately, with independent variables *grade level* and *logical form*. For the everyday context, no significant differences between grade levels or between logical forms occurred (fig. 2a). For the mathematical context, MP and AC items were solved significantly better by sixth graders than by fourth graders (fig. 2b). This indicates that, although a certain amount of students seems to be able to interpret implications as strictly conditional statements, a transfer to mathematical contexts did not seem to occur necessarily, unless sufficient knowledge of the involved concepts is attained. However, MT tasks, which require to build up only one additional model (*not-p* and *not-q*) seemed to be unaffected by these differences.

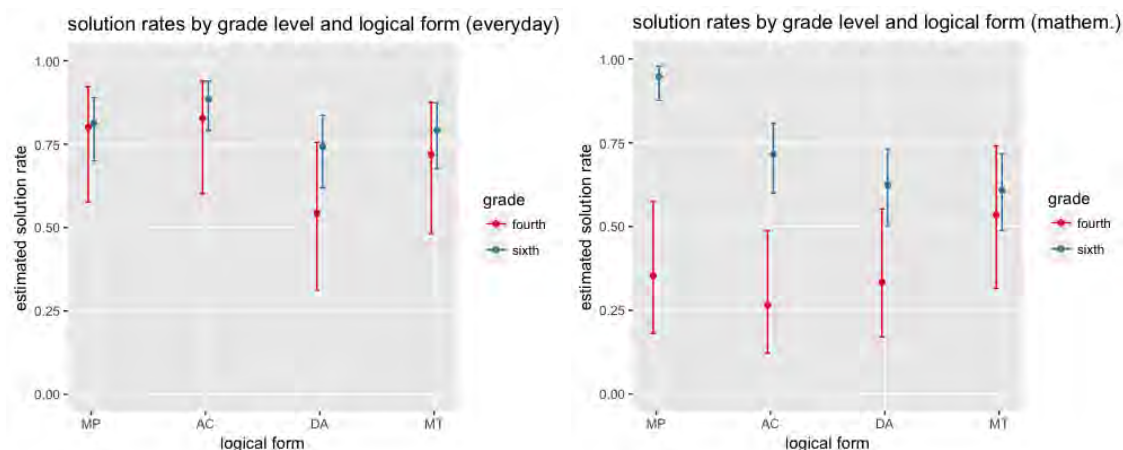


Fig. 2a and b: estimated solution rates and 95% confidence intervals by grade level and logical form (2a, left: everyday context; 2b, right: mathematical context)

DISCUSSION

The first goal of this study was to investigate if it is feasible to survey elementary school students' conditional reasoning skills in both, everyday and mathematical contexts. Results from interviews and one-to-one interviews indicate that the developed instrument is accessible to elementary school students. Moreover, the questionnaire survey results replicate early conditional reasoning skills reported in studies from developmental psychology (Markovits & Thompson, 2008). The second motivation for our study was to explore to what extent students could apply these skills when dealing with mathematical concepts in the classroom and beyond.

We assumed that grade 6 students' knowledge of addition and multiplication – the concepts underlying the tasks in the mathematical context – would be substantially higher than that of grade 4 students. Even though we could not report data on students' mathematics skills in this short report, the results are consistent with this assumption and MMTs: Only for the mathematical context did grade 6 students outperform their peers from grade 4, but results were comparable for the everyday context (1). This underpins the role of specific knowledge for conditional reasoning in mathematical contexts.

Moreover, our exploratory results indicate that the differences between the two con-

texts are not only due to task understanding: While the four logical forms were of similar difficulty for the everyday context, the logical forms requiring to analyse alternative models (AC, AD, MT) were significantly harder than MP in the mathematical context (2).

Finally, results indicate that the 6th graders deeper knowledge of mathematical concepts does not show equally for all logical forms, but primarily for MP and AC items (3). This is rather unexpected, since – following the model of Barrouillet & Lecas (1999) – an increase in AC and DA should occur only after a conditional interpretation of the conditional (and thus also MT) has been mastered. At the current state of research, reasons can only be hypothesized. For example, DA and MT involve negations in the problem formulation, but MP and AC not. However, if negations themselves were the problem, similar effects would be observable also in the everyday context. Further research will be necessary to describe these effects in more detail, in particular focusing on the role of mathematical knowledge.

Even though our study shows that it is possible to measure elementary students' conditional reasoning skills in both contexts, and obtain plausible results in many aspects, the further results of our study have to be considered carefully. Most importantly, the small sample size – even though tackled statistically by analysing on item level – forbids taking far-reaching conclusions apart from backing up the feasibility of the measurement and raising hypotheses for further research. Moreover, future research will have to collect really longitudinal data and include explicit measures of mathematical knowledge to support the interpretations.

However, the results obtained with the new instrument are consistent with theoretical expectations in some aspects that indicate educational implications – even though further research must back up these implications with further empirical data. Most centrally, the results illustrate that students are not necessarily able to transfer conditional reasoning skills from everyday contexts (Markovits & Thomson, 2008) to contexts involving mathematical concepts. Most likely the main reason for this is that knowledge of these concepts is necessary to analyse the necessary mental models when drawing inferences. If we agree that valid reasoning with mathematical concepts is a goal of classroom instruction, this implies the necessity to practice simple deductions on all logical forms when dealing with mathematical concepts. Moreover, it will be necessary to discuss why certain conclusions can or cannot be drawn – particularly for the more complex logical forms such as AC, DA, and MT (cf. Schroyens et al., 2001) – with a focus on the models underlying these inferences.

References

- Attridge, N., & Inglis, M. (2013). Advanced mathematical study and the development of conditional reasoning skills. *PloS one*, 8(7), e69399.
- Barrouillet, P., & Lecas, J. F. (1999). Mental models in conditional reasoning and working memory. *Thinking & Reasoning*, 5(4), 289–302.

- Evans, J. St. B. T., & Over, D. E. (2004). *If*. Oxford, England: Oxford University Press.
- Inglis, M., & Simpson, A. (2009). Conditional inference and advanced mathematical study: Further evidence. *Educational Studies in Mathematics*, 72(2), 185–198.
- Janveau-Brennan, G., & Markovits, H. (1999). The development of reasoning with causal conditionals. *Developmental Psychology*, 35(4), 904–911.
- Johnson-Laird, P. N., & Byrne, R. M. (2002). Conditionals: a theory of meaning, pragmatics, and inference. *Psychological review*, 109(4), 646.
- Küchemann, D., & Hoyles, C. (2002). Students' understanding of a logical implication and its converse. In A. Cockburn (Ed.): *Proc. 26th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 3, pp. 3–241). Norwich, UK: PME.
- Liu, H., Ludu, M., & Holton, D. (2015). Can K-12 Math Teachers Train Students to Make Valid Logical Reasoning? In X. Ge, D. Ifenthaler, & M. Spector (Eds.): *Emerging Technologies for STEAM Education* (pp. 331–353). Heidelberg: Springer.
- Markovits, H. (2014). Conditional Reasoning and Semantic Memory Retrieval. In A. Feeney & V. Thompson (Eds.): *Reasoning as Memory*, 53 – 70.
- Markovits, H., & Barrouillet, P. (2002). The development of conditional reasoning: A mental model account. *Developmental Review*, 22(1), 5–36.
- Markovits, H., Fleury, M. L., Quinn, S., & Venet, M. (1998). The development of conditional reasoning and the structure of semantic memory. *Child development*, 69(3), 742–755.
- Markovits, H., & Lortie-Forgues, H. (2011). Conditional reasoning with false premises facilitates the transition between familiar and abstract reasoning. *Child Development*, 82(2), 646–660.
- Markovits, H., & Thompson, V. (2008). Different developmental patterns of simple deductive and probabilistic inferential reasoning. *Memory & Cognition*, 36(6), 1066–1078.
- Morsanyi, K., & Szűcs, D. (2014). The link between mathematics and logical reasoning. In S. Chinn (Ed.): *The Routledge International Handbook of Dyscalculia and Mathematical Learning Difficulties*, 101–124.
- Nunes, T., Bryant, P., Evans, D., Bell, D., Gardner, S., Gardner, A., & Carraher, J. (2007). The contribution of logical reasoning to the learning of mathematics in primary school. *British Journal of Developmental Psychology*, 25(1), 147–166.
- Saelen, C., Markovits, H., & Klein, O. (2009). The effects of instructions on reasoning with stereotypical premises. *Psychologica belgica*, 48(4).
- Schroyens, W. J., Schaeken, W., & d'Ydewalle, G. (2001). The processing of negations in conditional reasoning: A meta-analytic case study in mental model and/or mental logic theory. *Thinking & Reasoning*, 7(2), 121–172.
- Stylianides, A. J., & Stylianides, G. J. (2007). The mental models theory of deductive reasoning: Implications for proof instruction. In D. Pitta-Pantazi & G. Philippou (Eds.): *Proceedings of the 5th CERME Conference* (pp. 665–674), Larnaca, Cyprus: CERME.

THE FIRST-TIME PHENOMENON: SUCCESSFUL STUDENTS' MATHEMATICAL CRISIS IN SECONDARY-TERTIARY TRANSITION

Pietro Di Martino* and Francesca Gregorio**

*Università di Pisa, Italy & **HEP Vaud, Lausanne, Switzerland

The huge difficulties related to the transition from secondary to tertiary mathematics are documented by several official data. The analysis of these difficulties is a main issue in educational research at undergraduate level. It is of particular interest the case of the students who choose mathematics as a major. In fact, for the most part, they are students considered excellent in mathematics during secondary school, they seem to have the cognitive resources to succeed, but, in many cases, they encounter several difficulties during their university experience. Therefore, it appears particularly interesting to study also the affective sources and consequences of these difficulties. With this aim, we developed a qualitative and narrative study focused on students' reflections about their mathematical difficulties in the university experience.

INTRODUCTION

A lot of research documents the fact that students encounter huge difficulties in the transition from secondary to tertiary mathematics education. As Niss (2003) observes, students move from one type of institution with its characteristic culture to another type of institution with a completely different (mathematics) culture. This produces marked discontinuities in the transition process and it is the source of several problems. De Guzmán et al. (1998, p. 756) describe tertiary transition as “*a major stumbling block in the teaching of mathematics*”. Clark and Lovric (2008), drawing on anthropological theories, theorised a *rite of passage* from secondary to tertiary education. This rite of passage is necessary to handle advanced mathematics and it is characterised by an inevitable crisis: the crisis exists and all the actors of the transition (students, teachers, institutions...) must deal with it. Tall (1991, p. 25) underlines that tertiary transition “involves a struggle (...) and a direct confrontation with inevitable conflicts, which require resolution and reconstruction”.

Despite the clear emotional charge of a ‘crisis’, the research about tertiary transition has been characterised by an almost purely cognitive approach (Artigue, 2016). In particular, the focus of the earliest researches in university mathematics concerned the analysis of the cognitive discontinuity between secondary and tertiary mathematics. Within this frame, Tall (1991) has identified, discussed and analysed significant differences between the approach to mathematics at the secondary level and those at the tertiary level: in the use of symbolism and generalisations, in the role given to the definitions of mathematical object, in the relevance given to the formal reasoning and

proof, in the level of abstraction (Hefendehl-Hebeker et al., 2010, coined the evoking expression ‘abstraction shock’).

At a cognitive level, this discontinuity requires students to develop new thinking modes and – according to Tall (1991, p. 25) – this is “an immense personal reconstruction”. This can be particularly hard for high achievers in mathematics at secondary school because they face with an inexplicable fact: reasoning strategies that worked in their previous mathematical experiences suddenly stop working at university level. In the most of the cases, these students fail in mathematics for the first time in their life: they face with as we call ‘the first-time phenomenon’. This is why we believe that the cognitive reconstruction mentioned by Tall strongly involves affective aspects, and they cannot be neglected in the analysis of students’ difficulties in tertiary transition. The relevance of the affective component in the transition is confirmed by a recent empirical study developed by Rach and Heinze (2016). They identify five mixed (cognitive and affective) individual variables that affect successful mathematical learning process at the university level: interest, self-concept, specific prior knowledge, prior achievement, and learning strategies.

Therefore, the cognitive reconstruction involves a deep affective reconstruction: it can result particularly hard for high achievers in mathematics at secondary school. In this case, in their university experience, the freshmen can have to reconsider their view of mathematics, their self-perception in mathematics and finally their emotional disposition towards mathematics. In other words, according to the TMA model for attitude (Fig. 1) developed by Di Martino and Zan (2010), they may have to reconsider their attitude towards mathematics.

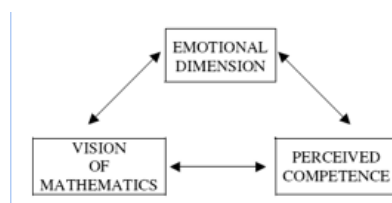


Fig. 1: The TMA-Model for attitude towards mathematics (Di Martino & Zan, 2010)

Within a larger study about students’ difficulties in tertiary transition, we focus our attention on the first-time phenomenon. We are interested in analysing its spread and effects. In particular, how the students’ view of mathematics and self-perception develop and what role emotions play in the arise and management of this phenomenon.

CONTEXT AND METHODOLOGY

Context

The context of our research is the Bachelor of Mathematics in Pisa. It perfectly fit for our purpose of analysing the tertiary crisis of students considered excellent in mathematics at the secondary school level.

On the one hand, the students of the Bachelor in Mathematics in Pisa were high-rated in the final exam of secondary students from AY 2009/10 to 2012/13 (we define high-rated a mark between 90/100 and 100/100). As shown in Table 1, the percentage of high-rated students enrolled in Pisa is much higher than the national one.

	2009/10	2010/11	2011/12	2012/13
Italy	45.0%	42.4%	44.5%	40.1%
Pisa	73.6%	58.6%	65.7%	60.0%

Table 1: Percentage of high-rated students in the Bachelor of mathematics, in Pisa and national wide.

On the other hand, despite this confluence of high-level students, the dropout rates of the Bachelor in Pisa are in the national average (Table 2).

	2009/10	2010/11	2011/12	2012/13
Italy	24.8%	28.4%	19.4%	20.8%
Pisa	17.8%	34.1%	21.4%	17.8%

Table 2: Dropout rates among first-year students in the Bachelor of Mathematics.

Procedure and rationale

We developed our study with the aim to involve two categories of students: *successful students* – students who passed the exams of the first year of the Bachelor program – and *dropout students* – students that left the Bachelor of Mathematics without getting their degree. The study was conducted in two different stages.

In the first stage, we developed an online questionnaire, structured in sections, different for the two categories of student (successful and dropout). The questionnaire was structured in the following 5 sections: S1 – context information; S2 – perceived differences between school mathematics and university mathematics (for dropout students only); S3 – the personal tertiary experience with mathematics; S4 – ways to overcome difficulties (for successful students) and reasons to quit (for dropout students); S5 – Free conclusive comments. The questionnaires had only a few of multiple-choice questions, for the most part concentrated in S1. We choose to use mainly open questions in order to stimulate narratives (for example the following question in S4: *What memory do you have about your experience with mathematics at university?*). The questionnaire was anonymous and the participation voluntary. 75 successful students and 52 dropout students participated in this first stage.

In S5 of the questionnaire, respondents were invited to share their e-mail to participate in the second, non-anonymous stage of the research. 27 successful students and 10 dropout students replied to the call. In the second stage, the second author conducted a semi-structured interview. The schema of the interview has been developed with the aim to explore in depth the main issues raised by the data collected with the question-

naire and therefore it covered the same five issues of the questionnaire. The interviews were audio-recorded, transcribed and analysed.

The two stages organisation of the study is due to the complementarity and relevance of the two research instruments – questionnaire and interview – (Cohen, Manion & Morrison, 2007). Firstly, we wanted to guarantee anonymity, in order to minimise the social desirability pressure in answering: the anonymous questionnaire allows respondents to freely narrate their university experience. The open items give them the opportunity to talk about facts and emotions that they recognise significant, using the words they consider more appropriate for their memories. Besides, the oral interview maintains this important possibility for respondents, but it also is a non-one-way instrument, permitting the interviewer to ask for more details or clarifications.

The method of analysis

In their study on the analysis of autobiographical interviews, Demazière and Dubar (1997) describe two critical possible approaches to the qualitative data: the *illustrative* approach (collected data are used to illustrate the researcher's theoretical standpoints) and the *restitutory* approach (material is returned to its original form, with no comments). Demazière and Dubar underline the researcher's need to catch the *sense* of a story emerging from an interview, tackling the issue of: 'How to analyse in order to understand?'. About this, Lieblich, Tuval-Mashiach, and Zilber (1998) identify two independent dimensions in the process of reading and analysing stories of life: holistic versus categorical and content versus form. Combining these dimensions, we obtain four different modes of approaching qualitative data, each of them provides different kinds of information. In our study, we carried on (in a different time) two of these modes. Firstly, we developed a holistic and content-oriented approach to the data collected in order to identify recurrent themes and to create categories of answers. In this first phase, we identified the first-time phenomenon as particularly recurrent and significant in the participants' memories. Then, on the basis of the categories created, we developed a categorical-content analysis to describe details of this phenomenon.

In the following, we will focus on the results of the first-time phenomenon. All quotes from the narrative data will be translations. We will quote the data using an alphanumeric code composed of three symbols: Q and I (questionnaire/interview); D or S (dropout/successful student); a serial number.

RESULTS AND DISCUSSION

The most part of successful and dropout students described the cognitive and emotional impact of their first failure in mathematics. We define 'first-time phenomenon' exactly the cognitive and emotional reaction to this first experience of failure in mathematics. This phenomenon is well introduced by the following excerpt:

I had never encountered difficulties in the study of mathematics during secondary school, so I did not know how to deal with this unexpected failure both concerning the study methods and the emotional reaction. [QS.69]

The categorical-content analysis of the narratives referring to a first-time phenomenon permitted us to recognise the main features of this phenomenon.

Unexpectedness

As it emerges from the previous excerpt, the first-time phenomenon is characterised by an unexpected failure. The unexpectedness is a fundamental component of the phenomenon. On the one hand, it explains the distinction between the *objective* fact (the mathematical failure) and the *subjective* perception of the phenomenon (the interpretation of the fact, in particular, concerning its unexpectedness). On the other hand, it appears to strongly affect the cognitive and emotional reactions to the failure.

There is a tremendous distance between the experienced failure and what happened immediately before the transition period began:

During secondary school, I was the best one. Here I was considered less than zero. [IS.28]

I thought it'd be easier. I was the number one in my class, and I found myself failing exams. [IS.16]

This distance provokes strong emotional reactions:

If you are used to certain things, as to 'be good at school', end up in a completely different world (...) can shock. [QD.35]

The first time I failed, I felt bad and so I was afraid of experience that feeling again. [IS.35]

During his interview, the student IS.73 generalised and explained this fact:

Most of the students (...) in Pisa were really good in school and they are not used to failure. Accept failure is not trite. There is a university counselling and I found out that the students who access it are mostly from scientific bachelors than from humanities ones. This fact got me unstuck. Looking back, I would call for help before. [IS.73]

In this situation, the students feel the need to interpret the unexpected failure. According to the attribution theory developed by the psychologist Bernard Weiner (1986), individuals tend to explain their own success and failure in terms of three dimensions: locus (*internal* / *external*); stability (*stable* / *unstable*); controllability (*controllable* / *uncontrollable*). By the analysis of our data, it emerges that the first attribution of failure is related to the perceived change in the didactic approach:

University professor presume we know a lot of mathematical facts and they consider 'easy' many arguments that are not so easy to understand for freshmen. [QD.41]

This attribution process causes a clear change in students' view of mathematics and in their perceived competence.

During secondary school, I had the highest grades in school, here I am mediocre. [IS.65]

The difficulty of math is definitely increased [in the comparison between the secondary school and university experience] and I understood that I was not so brilliant. [QD.4]

Students have to handle this changing in self-perception in mathematics, and this is not an easy passage.

I started thinking that maybe I was not as good as I thought, that it had been just an illusion. [QD.44]

Sense of impotence

As we can see from the previous extracts, the distance from the secondary school experience doesn't provide means to the students to face difficulties in mathematics. Students are often stuck in this first and unexpected failure experience and they can't find a way to get out of it.

I always did well and I thought that I would make it if I studied. When I arrived, the first impact was terrible. I was on the same course as people who get bored. (...) For the first time, I wanted to do something and I didn't manage to do it. [ID.34]

Strategies which worked during secondary school stop working at the university:

The first impact was hard, I was used to studying the necessary and obtain good marks. Here, I studied a lot and I didn't manage to get enough. [IS.47]

In addition, this changing appears not to be taken in charge by the Institution, students have the perception having to do everything themselves and this is another clear difference with respect to their secondary school experience:

At the first year, it is a shock for everyone, people don't know what's coming, they are not led to what mathematics requires: formalism, precision... You learn it trying and failing but no one shows you how to do. [IS.38]

Students experience a significant sense of impotence, which doesn't let them react appropriately, as if the failure situation were unavoidable and immutable. It is the germ of what Martin Seligman (1975) defines *learned helplessness*: that is, a perceived lack of control over an event, which results from prior exposure to similar negative events perceived as uncontrollable.

Shame

The unexpectedness of the failure elicits strong negative emotional reactions. In particular, the first negative results are often associated with shame:

I was ashamed because sharing my difficulties meant to admit a personal defeat. [ID.14]

Shame is strongly affected by two factors: the fear of having disappointed parents, and, more generally, close people (here the first-time phenomenon plays a role: the failure is unexpected also for other people), and the comparison with peers. In the most of the narratives, students say that they are not the best in mathematics for the first time in their life. Moreover, they often get the impression to be the only ones in difficulty, with important consequences on self-confidence.

I stopped talking about exams with my parents. [QS.73]

I guess I was the worst and the slowest of the class. [QD.21]

I was not happy when I attended math lessons: I had a sense of inferiority and I was ashamed. [QD.48]

The emotion of shame seems to play a key role in the first-time phenomenon, because it often leads students not to share their difficulties and experience with peers. Overcoming this barrier is crucial since it appears to be the principal differences between dropout and successful students. On the one hand, dropout students avoid sharing their difficulties. On the other hand, successful students – overcoming shame – find the courage to share their difficulties with peers.

So, I made a big improvement when I started asking for help to better students than me (in this respect I was halted by pride and shame). Even now, I'm working on this respect. [QS.6]

To share difficulties, concerns, passions with other students really helped me. [QS.31]

Realising not to be the only one in difficulty has a double effect: on the emotional level, it alleviates the sense of shame; on the attribution level, it permits them to reconsider the locus of control.

CONCLUSIONS

As it emerges in the students' narratives, the most part of the students who enrol the Bachelor of Mathematics experience failure in mathematics for the first time in their life. For the first time, they have to handle not finding a workable way to study, taking bad marks, not being the best in mathematics. The first-time phenomenon causes a recurrent pattern: the first failure provokes strong cognitive and emotional reactions. Moreover, the failure's unexpectedness undermines the view of mathematics and the self-perception: two of the three dimensions of attitude towards mathematics. But also, the emotional dimension suffers a significant repercussion: the main emotion associated with mathematics after the failure is shame. This change in the attitude towards mathematics has devastating effects: a sense of impotence associated with learned helplessness emerges and shame blocks students in seeking help. In particular, shame is an enormous block in sharing difficulties with peers.

By our data, this appears to be a breakthrough moment between success and dropout students. Overcoming the shame of the unexpected failure in mathematics opens to the crucial exchanges of views with peers: a key element for the redemption in transition. On the hand, it allows students to feel like they are not alone and isolated, but like they are part of a community (the community of mathematics students). On the other hand, it leads to re-evaluate their own difficulties and the attributions they made. This is recognised to be an essential point in overcoming difficulties:

I try to ask as many questions as I can and, gladly, I see that they are not stupid questions (before instead, I didn't share my doubts with anyone, I was worried that they were stupid doubts). [QS.6]

In the case of dropout students, shame almost always has the upper hand and learned helplessness is reinforced, quickly leading up to dropouts:

I felt I was wasting my life copying blackboards with symbols (...) I did not understand. So, I preferred to give up and move towards something more accessible. [QD.24]

Our results suggest some implications for practice and some directions for further study (to deepen the knowledge of the phenomenon, its causes and consequences). In particular, it appears necessary to cut off the “first-time chain” giving to the secondary school students the opportunity to face some mathematical difficulties and also failure in a protect environment in order to work on emotional and cognitive aspects, also at a meta-level.

References

- Artigue, M. (2016). Mathematics Education Research at University Level: Achievements and Challenges. In E. Nardi, C. Winslow & T. Hausberger (Eds.), *Proc. 1st Conf. of INDRUM* (pp. 11-27). Montpellier: INDRUM.
- Clark, M., & Lovric, M. (2008). Suggestion for a Theoretical Model for secondary–tertiary transition in mathematics. *Mathematics Education Research Journal*, 20(2), 25–37.
- Cohen, L., Manion, L. & Morrison, K. (2007). *Research methods in education*. London: Routledge.
- Demazière, D. & Dubar, C. (1997). *Analyser les entretiens biographiques* [Analyze biographical interviews]. Paris: Éditions Nathan.
- De Guzmán, M., Hodgson, B., Robert, A., & Villani, V. (1998). Difficulties in the passage from secondary to tertiary education. In A. Louis, U. Rehmann & P. Schneider (Eds.), *Proc. of the ICM*, vol. 3. Berlin, Germany (pp. 747–762).
- Di Martino, P. & Zan, R. (2010). ‘Me and maths’: towards a definition of attitude grounded on students’ narratives. *Journal of Mathematics Teacher Education*, 13 (1), 27-48.
- Hefendehl-Hebeker, L., Ableitinger, C., & Herrmann, A. (2010). Mathematik Besser Verstehen [Mathematics better understanding]. In A. Lindmeier & S. Ufer (Eds.), *Beiträge zum Mathematikunterricht* (pp.93-94). Münster: WTM-Verlag Stein.
- Lieblich, A., Tuval-Mashiach, R., & Zilber, T. (1998). *Narrative research: Reading, analysis, and interpretation* (Vol. 47). Sage.
- Niss, M. (2003). Mathematical competencies and the learning of mathematics: The Danish KOM project. In Gagatsis, A., & Papastavridis, S. (eds.), *Proc. of the 3rd Mediterranean Conf. on Math. Ed.* (pp. 115-124). Athens: Hellenic MS.
- Rach, S. & Heinze, A. (2016). The Transition from School to University in Mathematics: Which Influence Do School-Related Variables Have? *International Journal of Science and Mathematics Education*, 1-21.
- Seligman, M. (1975). *Helplessness: On Depression, Development, and Death*. San Francisco: W.H. Freeman.
- Tall, D. (1991). *Advanced Mathematical Thinking*. Dordrecht: Kluwer.
- Weiner, B. (1986). *An attributional theory of motivation and emotion*. New York: Springer-Verlag.

ROLE OF USING AN ALTERNATIVE CONCEPT DEFINITION IN CONDUCTING MATHEMATICAL TASKS OF TEACHING: THE CASE OF EXPLAINING WHY AN ALGORITHM WORKS

Merve Dilberoğlu and Çiğdem Haser

Middle East Technical University, Ankara, Turkey

This study illustrates the importance of attending to central concept definitions while conducting a particular mathematical task of teaching: Explaining why an algorithm works. Individual task-based interviews were conducted with 14 preservice teachers, in which participants were asked twice to explain why the standard algorithm for obtaining the least common multiple (lcm) of two positive integers works: First by depending on their personal definitions of the concept lcm, and then by considering an alternative definition of the concept presented to them. Upon working on the alternative definition together with the first author, half of the participants improved their explanations. Findings suggest that study of alternative definitions might enhance preservice teachers' understanding of why mathematical algorithms work.

INTRODUCTION

The accurate understanding and use of mathematical definitions constitute an important part of teachers' Mathematical Knowledge for Teaching (Ball, Thames, & Phelps, 2008). Definitions are the building blocks through which concepts are created, distinguished from each other, and communicated (Çakıroğlu, 2013). Once identified in the curriculum, the formal definition of a concept structures the teaching of that concept, ordering of related concepts that are to be introduced and the set of interrelationships among them to be established (Zazkis & Leikin, 2008). Apart from definitions' centrality to the process of concept formation, understanding of definitions are integral to many other tasks of teaching such as "responding to students' why questions," "giving or evaluating mathematical explanations," and "evaluating the plausibility of students' claims" (Ball et al., 2008, p. 400). Therefore, study of definitions in relation to mathematical tasks of teaching could be considered as one of the central tasks of preservice teachers' preparation in teacher education programs.

Another important component of teachers' Mathematical Knowledge for Teaching is reasoning about why mathematical algorithms work. It is one of the tasks that require teachers to have a robust understanding of the concepts involved. Given that studying multiple definitions of the same concept enhances individual's conceptual understanding (Chesler, 2012), this study investigates how preservice middle school mathematics teachers explain why the standard algorithm for calculating *the least common multiple (lcm)* of two positive integers works, depending on a mathematical definition of the concept *lcm* presented to them. Data for this study come from a broader study in which preservice middle school mathematics teachers' use of their mathemati-

cal knowledge from a specific content course in conducting the given mathematical tasks of teaching was investigated. The study presented here focuses on the following research questions:

1. How do preservice middle school mathematics teachers explain why the standard algorithm for obtaining the *lcm* of two positive integers works, depending on their personal definition of the concept *lcm*?
2. How do preservice middle school mathematics teachers explain why the standard algorithm for obtaining the *lcm* of two positive integers works, depending on an alternative definition of the concept *lcm* presented to them?
3. How do preservice middle school mathematics teachers make sense of the given mathematical definition of the term *lcm*?

THEORETICAL BACKGROUND OF THE STUDY

Mathematical Knowledge for Teaching (MKT) is a practice-based theory of mathematical knowledge teachers use “to carry out the work of teaching mathematics” (Hill, Rowan & Ball, 2005, p.373). Among the six domains of the framework, Specialized Content Knowledge (SCK) is the most salient one, identifying the everyday tasks of teaching that are unique to the profession of teaching mathematics. Ball et al. (2008) listed sixteen such tasks under the name of “*mathematical tasks of teaching*”, some of which are “presenting mathematical ideas,” “responding to students’ “why” questions,” and “asking productive mathematical questions” (p.400). Conducting these tasks involves a special kind of unpacking of mathematical knowledge that only teachers may need; for example, in explaining why the invert and multiply algorithm works for dividing fractions. Given that algorithms should be learned with understanding, teachers need to know effective ways of representing the meaning of mathematical algorithms to their students (Ball et al., 2008).

Preservice teachers’ understanding of algorithms

Previous research asked elementary preservice teachers to respond to student work involving algorithmic procedures (Maher & Muir, 2013), to evaluate student generated-algorithms (Salinas, 2009), and to connect student-generated algorithms to their traditional equivalents (Son, 2016). Findings of these studies collectively pointed to preservice teachers’ deficiencies in mathematical understandings, their emphasis on procedural and/or result-oriented interpretations, and weak conceptual explanations. Considering the findings, researchers commonly argued that deficits in preservice teachers’ content knowledge limited their pedagogical strength in responding to the given tasks. For this reason, the current study questions the role of attending to a mathematical definition of the concept *lcm* in assisting preservice teachers’ understanding of the reasons why the standard algorithm works for obtaining it. While most of the studies in the literature addressed addition/subtraction or multiplication/division algorithms with whole numbers or fractions, no studies in the accessible literature investigated the meaning of the *lcm* algorithm whose fluent use in multi-digit com-

putations and in solving of word problems are given ample importance in the middle school mathematics curricula (CCSSI, 2010; MoNE, 2013).

Preservice teachers' knowledge and use of mathematical definitions

Preservice teachers' knowledge and use of definitions has been investigated through a variety of tasks (for example, Chesler, 2012; Kubar & Çakıroğlu, 2017; Zazkis & Leikin, 2008) mostly framed through the theoretical distinction between *concept definition* and *concept image*. Tall and Vinner (1981) described the *concept image* as the collection of all mental representations, properties and processes associated with a concept in one's cognitive structure; and referred to the *concept definition* as the statement that designate it as a concept. Tall and Vinner (1981) indicated that portions of the concept image may not always be coherent with the complementing parts of it or with the actual concept definition. The individual may maintain a contradicting structure until conflicting parts evoke simultaneously and give rise to a *cognitive conflict*. Otherwise, an *evoked concept image* is activated at different times, allowing the individual to do the encountered tasks. Also, individuals' *personal concept definition* may be different from the actual definition of the concept accepted by the mathematics community (Tall & Vinner, 1981). These theoretical distinctions were employed to interpret the findings of this study.

METHOD

Qualitative research methods were employed in order to gain an in-depth understanding of how preservice middle school mathematics teachers explain why a given standard mathematical algorithm works (Merriam, 2009). Task-based interviews were conducted with participants to collect data for the study.

Context and Participants

Participants of the study were 14 preservice middle school mathematics teachers (5 third-years and 9 fourth-years) who were enrolled in a mathematics teacher education program that prepared teachers for 5th to 8th grade levels. The four-year program required preservice teachers to complete nine mathematics content courses until the end of the third year, each offered by the Mathematics Department. Two mathematics teaching methods courses were placed in the third year and one school experience and one practice teaching course were required at the last year. As the data for this study were collected at the end of the spring semester, participants of the study had completed all the mathematics and methods courses. Apart from the 4th-year participants' incidental experiences with teachers and students in middle schools, their knowledge about definitions and algorithms were not expected to differ from 3rd-year participants based on the courses they completed in the teacher education program.

In line with the purpose of the actual study (investigating how preservice middle school mathematics teachers use their mathematical knowledge from a specific content course in conducting mathematical tasks of teaching), participants were selected from preservice teachers who were good at both in the specific mathematics content course

studied -Basic Algebraic Structures course- and in mathematics teaching related courses, especially the two method courses. The former criterion was applied through the course grades that participants had taken from Basic Algebraic Structures course, while the latter was applied through the informed judgement of the second author who by the time of the study taught the method courses in the program and observed the students. Following this way, participants who, compared to their peers, carried the greatest potential for connecting the high-level definition with the middle school algorithm were selected.

Data Collection and Analysis

Data were collected through individual task-based interviews conducted by the first author. Participants were presented the following introductory description about the standard algorithm for obtaining the lcm of two positive integers (see Figure 1). They were asked to explain the reason(s) why the given algorithm works for obtaining the least common multiple of any two numbers.

The least common multiple of the numbers 24 and 36 are found by creating a factors list as demonstrated on the right. In this method, the two numbers are divided continuously, by beginning with the least prime number. Multiplication of the numbers contained in this factor list gives the least common multiple of the numbers 24 and 36.	24	36	2
	12	18	2
	6	9	2
	3	9	3
	1	3	3
		1	
	$2 \times 2 \times 2 \times 3 \times 3 = 72$		

Figure 1: An algorithm for calculating the least common multiple of two numbers. (Adapted from MoNE, 2010, p. 107).

Participants were asked to state their own definitions of the concept lcm before they were introduced the task. Although their statements were not rigorously stated definitions, all of the participants demonstrated a correct understanding of the concept. After they completed their initial explanations regarding the algorithm, they were presented an alternative definition of the concept lcm from their first-year algebra course book (see Figure 2).

- A *least common multiple* of two non-zero integers a and b is an integer m that satisfy the conditions
1. m is a positive integer
 2. $a|m$ and $b|m$
 3. $a|c$ and $b|c$ imply $m|c$.

Figure 2: Definition of the term least common multiple from Basic Algebraic Structures course book (Gilbert & Gilbert, 2000, p.77).

Having studied this definition together with the first author, they were re-asked to explain why the algorithm works, based on this new definition. The whole process was

audio- and video-recorded based on participants' consent. Their written work and supporting explanations, as transcribed from the recordings, were analysed based on the correctness and depth of their mathematical ideas. Frequencies of correct responses are reported and different response types are illustrated in the findings section. Participants are identified by a capital P and a number from 1 to 14.

FINDINGS

Collective analysis of the first and second explanations of the participants yielded a three-item taxonomy for the explanation of why the standard algorithm for obtaining the *lcm* of two integers works: A complete explanation for the mathematical *validity* of the algorithm included explicit attention to the reasons (i) why the obtained number was a multiple of the two given numbers and (ii) why the obtained number was the least of all common multiples. Participants' responses were categorized either as "partial" or "complete" explanations depending on whether they considered both of the reasons or only the former one, because mentioning the latter reason but not the former did not emerge in the current study. In addition to addressing these two essential items, if the explanation also clarified (iii) the reason why the algorithm was built on the division of the two numbers by primes, it was considered an "advanced" explanation. This additional explanation was related to the *efficiency* of the algorithm, which was commonly stated as a criterion for evaluating mathematical algorithms in the educational literature (Salinas, 2009; Son, 2016). The following scripts illustrate a partial, a complete and an advanced level of explanation respectively.

In that case (of standard algorithm) I do like this; I close over one of them (covers the whole column including the number 36 with her finger). I do not see these. I find the same thing. When I do it for this (number 24) some of these (the numbers in the factors list) won't come. For instance, this (3) won't come. Here comes, 23 and 3. When I do the same for this (number 36), these factors come (points to the numbers 2, 2, 3, and 3 in the factors list one by one). Well, I mean, all of these (factors) are into this product (72). (P1, 3rd-year)

By writing side by side, I am eliminating factors of those numbers (24 and 36), I mean, if commonly exist in both of them. If not (common), I take (it) once. Well, after all, the number that I will obtain by taking the product of these numbers (pointing to the factors list), is [...] a number divisible by both of these numbers we obtain, but that the least number we obtain. (P5, 4th-year)

Why do we divide them by primes? Because... hmm... well we do not have to write prime numbers in fact. What if we write here 6 for instance (in the factors list), it is the same as 2×3 (points to the two factors in the factors list) But, if we write prime numbers, you know... What was a prime number? A number that is not divisible by anything, except 1 and itself. Ah-hah. I think, that is why. For instance, let us think about writing 6 there. 6 is divisible by both 2 and 3. Maybe one of these numbers (points to dividends found in the whole algorithm), this 9 for example [...] is not divisible by 2, but is divisible by 3. For this reason, it would be problematic. That is why taking primes is easier. (P6, 3rd-year)

Comparing the first and second explanations of the participants revealed the following results, answering the first and second research questions of the study: Only three of

the participants could give sufficient explanations for why the algorithm works in their first attempts. Five others' explanations were considered as partial because they focused only on the aspect that obtained number was necessarily a common multiple of the input numbers, but overlooked the reason why it was the least of all common multiples. After working on the definition, three of the participants (P4, P8 and P13) who were not able to provide any explanation in their first attempts, provided partial explanations. Two others improved their explanations from partial to complete ones (P10, and P11) while one another (P14) reached the complete explanation from no explanation at all. P6, who previously provided a partial explanation, further questioned herself and reached an advanced level of understanding of the algorithm. Table 1 summarizes the findings.

Participants'	Explanation included the reason why...			Explanation type
	the obtained number is a common multiple of the initial numbers	the obtained number is the least of all common multiples	the algorithm is built on division by primes	
first attempts				
P1, P6, P7, P10, P11	•			Partial
P5, P9, P12	•	•		Complete
second attempts				
P1, P4, P8, P13	•			Partial
P5, P9, P10, P11, P12, P14	•	•		Complete
P6	•	•	•	Advanced

Table 1: Comparison of participants' explanations in the two attempts.

In order to gain a better understanding of the improvements in participants' expressions and to respond to the third research question, participants' interaction with the provided mathematical definition (given in Figure 2) were analysed. Most of the participants had difficulty with understanding the third condition that the least common multiple m had to satisfy. This statement was perceived to be an obvious conclusion about the least common multiple of two integers, rather than indicating an indispensable condition for m 's being the lcm . This common perception of the participants was articulated by P7 (4th-year) as in the following script:

It says: this number, suppose that I have a number (c). It says like, if it is divided by both a and b , this means that it is also divided, in any case, by their lcm . What is there for me to explain about this? [...] Why do we state this here (in the definition)?

In such cases where participants could not make sense of the third condition, the first author guided their understanding through asking questions such as “Well, what would be missing in this definition, if we were to exclude the third condition?” and “Can you find an example for c for the case of 24 and 36?... Then, what does it mean for m to divide c ?” At the end of this guided process five of the participants grasped the meaning of the definition correctly, while six of them did not need any guidance from the author and achieved this comprehension through their own interpretations. Three of the participants accepted the definition without demonstrating any deep conceptual understanding.

DISCUSSION AND CONCLUSION

Although participants’ initial *personal definitions* of the concept *lcm* were correct and they had appropriate *concept images* regarding the *lcm*, their understanding of the concept was not reflected in their explanations about the algorithm. Their examination of the presented definition might have *evoked* one or both of the essential ideas required for establishing the validity of the algorithm, especially the condition that the obtained number must be the least of all common multiples of the two given integers. Particular definition of the concept *lcm* might have facilitated the comprehension of the algorithm, since it separates the two conditions, and hence attracts explicit attention to each of them. Indeed, some of the participants appreciated the way the definition was stated and took the opportunity to transfer their ideas from studying the definition to the work of explaining why the algorithm works. Below script illustrates the case:

When we come to the third one (condition), it says, this (the number c) will be a number different from 72. It may also be 72 but, again it will be a multiple of both 24 and 36; and 72 will be divided this (c). I mean... well, there may be some numbers greater than 72 or different from 72, but 72 must be the least one among them for itself to divide the others (other multiples) Yes, I am also enlightened! (P14, 3rd-year)

As in the case of previous studies (Maher & Muir, 2013; Salina, 2009; Son, 2016), findings of the current study highlighted the importance of preservice teachers’ content knowledge, i.e. knowledge of the *concept definition* in our case, in making sense of mathematical algorithms. Hence, the findings of the study suggest that attending to alternative definitions of central concepts can enhance preservice teachers’ understanding of why algorithms work. Given that half of the participants still could not reach complete explanations at the end of the task-based interview, it can be concluded that the underlying principles of algorithms require special attention in preservice teachers’ preparation. Critical examination of the *least common multiple* algorithm, as well as other mathematical algorithms, should be the part of preservice teachers’ experiences in teacher education programs, especially in mathematics teaching methods courses. Considering that the alternative definition utilized in this study was obtained from a mathematics content course and observing that those who benefited from the definition improved their explanations regarding the algorithm, we point to the importance and power of connecting preservice teachers’ experiences in the content

courses and in the mathematics teaching method courses for attempting to develop their MKT.

References

- Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389–407.
- Chesler, J. (2012). Pre-service secondary mathematics teachers' making sense of definitions of functions. *Mathematics Teacher Education and Development*, 14(1), 27–40.
- Common Core State Standards Initiative (CCSSI). (2010). *Common core state standards for mathematics*. Retrieved August 27, 2015, from http://www.corestandards.org/wp-content/uploads/Math_Standards.pdf.
- Çakıroğlu, E. (2013). Matematik kavramlarının tanımlanması. In İ.Ö. Zembat, M. Özmantar, & E. Bingölbalı (Eds.), *Tanımları ve Tarihsel Gelişimleriyle Matematiksel Kavramlar* (pp. 1–13). Türkiye: Pagem Akademi.
- Gilbert, J., & Gilbert, L. (2000). *Elements of modern algebra*. (5th ed.). Brooks/Cole.
- Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, 42(2), 371–406.
- Kubar, A. & Çakıroğlu, E. (2017). Prospective teachers' knowledge on middle school students' possible descriptions of integers. *International Journal of Education in Mathematics, Science and Technology (IJEMST)*, 5(4), 279–294.
- Maher, N., & Muir, T. (2013). "I know you have to put down a zero, but I'm not sure why": Exploring the link between pre-service teachers' content and pedagogical content knowledge. *Mathematics Teacher Education and Development*, 15(1), 72–87.
- Merriam, S. B. (2009). *Qualitative research: A guide to design and implementation*. San Francisco, CA: Jossey-Bass.
- Ministry of National Education (MoNE) (2010). *İlköğretim matematik 6: Öğretmen kılavuz kitabı*. İstanbul: Devlet Kitapları, Kelebek Matbaacılık.
- Ministry of National Education (MoNE) (2013). *Ortaokul matematik dersi 5, 6, 7 ve 8.sınıflar öğretim programı*, Talim ve Terbiye Kurulu Başkanlığı, Ankara.
- Salinas, T. M. (2009). Beyond the right answer: Exploring how preservice elementary teachers evaluate student-generated algorithms. *Mathematics Educator*, 19(1), 27–34.
- Son, Ji-Won (2016). Moving beyond a traditional algorithm in whole number subtraction: Preservice teachers' responses to a student's invented strategy. *Educational Studies in Mathematics*, 93, 105–129.
- Tall, D., Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies of Mathematics*, 12, 151–169.
- Zazkis, R., & Leikin, R. (2008). Exemplifying definitions: a case of a square. *Educational Studies in Mathematics*, 69, 131–148.

CONCEPTUALIZING AN EXPERT TEACHER'S EXPERTISE IN A LESSON DESIGN STUDY IN SHANGHAI

Liping Ding* and Keith Jones**

*NTNU, Norway; **University of Southampton, UK

This paper proposes a conceptualization of an expert teacher's expertise by coordinating a subject-based behaviour/cognitive analysis and a social-culturally situated analysis. Data from our Lesson Design Study in Shanghai, China, included lesson plans, transcripts of the video-recorded lessons, and transcripts of commentary on the lessons by the expert teacher was analysed. This showed the attunements of the expert teacher to the affordances and constraints of the activity system. This conceptualization of the 'dual nature' of the expert teacher's expertise contributes to a deep analysis of the unique and significant functions of the expert teacher in China.

INTRODUCTION

Li and Kaiser (2011, pp. 6-8) have highlighted three key issues regarding teacher expertise in mathematics education: (1) "identifying teachers with expertise"; (2) "specifying and analyzing aspects of teachers' expertise in mathematics instruction"; and (3) "understanding expertise in mathematics instruction that is valued in different cultures". Subsequently and more recently, Kaiser and Li (2017, p. 81), in a Research Forum at PME41, argued for the need to explore "possible relationships between (subject-based) cognitive and (social-culturally) situated perspectives" in examining and evaluating teachers' competencies and expertise.

In this paper, we make a contribution to conceptualizing the nature of a Chinese expert teacher's expertise in our Lesson Design Study (LDS) in Shanghai (SH) (Ding *et al.*, 2014, 2015) by utilising Greeno's (1998) situative theoretical perspective. Our research question is: in what way does Greeno (1998) model help to conceptualize the nature of the Chinese expert teacher's expertise in the LDS?

RESEARCH BACKGROUND

As pointed out by Pepin *et al.* (2017), the notion of 'expert teacher' is underpinned by the cultural values and different perceptions of the nature of teaching expertise. Whilst an individualistic, and primarily cognitive, perspective on teacher expertise privileges what might be deemed 'rational' factors of proficiency, a situative perspective (using Greeno's, 1998, situative theoretical perspective) might offer a more comprehensive view of the nature of the expertise of an expert teacher.

In research with teachers in China, Gu and Gu (2016) used the term 'teaching research specialist' (TRS) (*jiao yan yuan* in Chinese) to highlight the significant role of the TRS in improving in-service teacher professional development in China. Each TRS, em-

ployed in a specific school district in China, is a didactician in mathematics who works primarily with practicing teachers. While there are a number of studies of mathematics teachers' professional development in China in general, and the mentoring role of TRS in particular (see, for example, Gu & Gu, 2016; Pepin *et al.*, 2016), little is known of the nature of the TRS's work in mentoring practicing teachers.

Two recent findings by Gu and Gu (2016) informed our aim to contribute to conceptualizing the complex nature of a TRS's expertise and the practice of the TRS in mentoring teachers in our LDS. First, Gu and Gu (2016) revealed that Chinese TRSs usually pay a great deal of attention to issues such as setting students' learning goals, designing instructional tasks, formative assessment of students' learning, and improving teachers' instructional behaviors. The TRSs generally pay less attention to mathematics (perhaps because the teachers generally have good mathematics knowledge) and less attention to general pedagogical issues. Second, the TRSs tend to address anticipated problems with a lesson, and with the subsequent lesson, based on their own previous experience. In doing so they may pay less attention to addressing issues raised by the teachers or engaging in dynamic dialogue with them.

The need to understand the interactions of cognitive, situational and social characteristics of the TRS expertise (as demonstrated by the expert teacher in our study) situated in the phenomenon that is the practice of TRS mentoring in China leads us to choose Greeno's (1998) situative perspective as our theoretical framework.

THEORETICAL FRAMEWORK

Greeno (1998) proposed that the main distinguishing characteristic of the situative perspective is its theoretical focus on interactive systems that are larger than the behavior and cognitive processes of an individual agent. One approach to this is to begin within the framework of individual cognition and work outward from the analyses of individual cognition. The alternative is to begin with the situative framework of interactional studies and work inward. In the study we are reporting in this paper, we apply the second approach. Figure 1 illustrates the situative model.

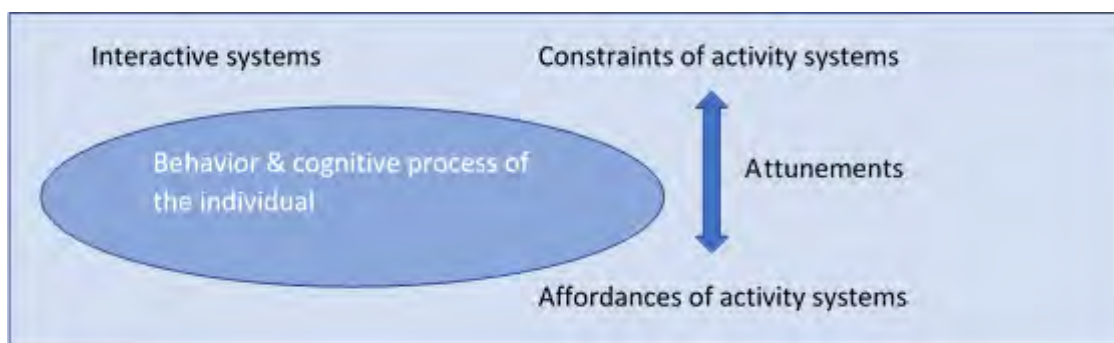


Figure 1: The situative model (adapted from Greeno, 1998).

In Greeno's (1998) framework, he uses the notion of *attunements* to *constraints* and *affordances*. Here *constraints* are the "if-then regularities of social practices and of interactions with material and informational systems that enable a person to anticipate

outcomes and to participate in trajectories of interaction”, *affordances* are “qualities of systems that can support interactions and therefore present possible interactions for an individual to participate in”; and a person’s *attunements* to constraints and to affordances are “regular patterns of an individual’s participation” (Greeno’s, 1998, p. 9). We use the situative model of the attunements to affordances and constraints of activity systems (as illustrated in Figure 1) in our data categories and analysis.

METHODOLOGY

Our school-based LDS was conducted in an international school located in the western suburb of Shanghai (for details see Ding *et al.*, 2014, 2015). The process of our LDS model had three cycles. The first cycle (L1) was the teacher’s initial lesson design, lesson implementation and reflection. The second cycle (L2) entailed implementation of the re-designed L1. The third cycle (L3) was the re-re-designed and re-re-implemented L1.

Each cycle included a set of the school-based teaching research group (TRG) activities, such as the teacher’s classroom teaching, our study members’ observation, and the mathematics TRG meetings. In our LDS there were seven elementary mathematics teachers and three national/regional educators and expert teachers (for more on our LDS project, see Ding *et al.*, 2017). The expert teacher on which we report in this paper was, at the time of the research, a TRS who had worked in the city centre school district of Shanghai for over thirty years (for more on the nature of being an expert teacher in China and why we considered the term applies to two expert teachers in our LDS, see Ding *et al.*, 2017). We refer to the selected expert teacher as Mr Zhang, a pseudonym.

Our data sources include: the teacher’s lesson plans, teaching notes and reflection diary; the transcripts of the video-recorded lessons; the transcripts of the video-recorded comments of Zhang in TRG meetings over the teaching cycles of the LDS model. The lesson topic was investigating the relationship between perimeter and area in the Shanghai Grade 3 textbook, and the lesson title was ‘Which area is bigger?’. The central theme of the TRG meetings was on a participation-oriented lesson design; in Greeno’s words (1998, p. 19), “not only what their students have come to know and understand, but also to how their students are currently able to participate in inquiry, discourse, and reasoning, and how they can help them advance to more successful participation”.

In focusing on the expert teachers’ expertise and mentoring activities in order to develop a framework to conceptualize an expert teacher’s expertise in the LDS, here we focus on two main categories of the expert teacher’s interactions in the TRG before and after the second cycle of the LDS: (1) redesigning the original lesson; and (2) implementation of the redesigned lesson. We apply Greeno’s model (see Figure 1) to develop categories of the expert teacher’s interactions in the LDS. We summarise selected examples in Table 1. Then, in Table 2, we further develop the sub-category of ‘Interactive systems’ according to Greeno’s idea of *attunements* to *constraints* and *affordances*.

Sub-categories	Explanation	Examples of what was said by the expert teacher
Cognitive	e.g., Math knowledge in the textbook; pupil's conceptual understanding.	This textbook topic is not of learning a new concept, but a mathematical proposition.
Behaviour	e.g., Instructional procedure; Pupil's basic knowledge and skills.	<i>What's the intention of this sticks activity?</i> Putting sticks is related to a pupil's skill; <i>how would you embed the instructional intention into this activity?</i>
Interactive systems	e.g., The relations of individual learning with small group and a whole class discussions; pupils' interactions with materials, activities and teacher.	In terms of classroom discussion, we need to be aware of the fact that students' discussion is based on each individual's experiences and sense making of the activities.

Table 1: The categories of the expert teacher's interactions in LDS.

Sub-categories	Explanation	Examples of what was said by the expert teacher
Constraints	e.g., If-then regularities of a student's early experience of a classroom activity and of interactions with the tasks and other pupils to enable the student to anticipate outcomes and to participate in trajectories of interaction.	You need to know clearly about the relation of the following three lines: ① <i>what's students' previous experience with the putting sticks activity</i> ; ② <i>what's the intention of this sticks activity?</i> ... ③ <i>the changes you made from a rectangle problem to a square problem</i> ; I wonder <i>whether these two learning situations were the best situation for students' learning?</i>
Affordances	e.g., Teacher's lesson design, together with activities and tasks design and knowledge connection.	I think that the lesson can be designed in a way to enable students to experience the whole process of plausible reasoning.
Attunements	e.g., The pattern of teacher's teaching instruments, questions and language to enable student to participate in learning.	<i>Why we chose this textbook topic to study?</i> ... <i>Whether the lesson design suits our fundamental theory and main educational value nowadays?</i>

Table 2: The sub-categories of 'Interactive systems'.

CONCEPTUALIZING THE EXPERT TEACHER'S EXPERTISE

Coordinating multiple levels of analysis of Mr Zhang's expertise

We constructed the coding in Tables 3, 4 and 5 according to the categories in Tables 1 and 2. The coding in Table 3 focuses on the components of behaviour/cognitive (B/C), the coding in Table 4 on those of the interactive systems (IS) of Mr Zhang's expertise, and the coding in both Table 4 and Table 5 on the attunements to constraints and affordances.

First, the coding in Tables 3 and 4 show two levels of analysis of the dynamic interactional process of the expert teacher with the teachers in the LDS. As explained above, we started from the dynamic interactions in the TRG meetings and then worked inward to the parts in each teaching cycle (e.g., lesson design, lesson practice, teacher's reflection) in the LDS model in order to identify the codes in the general category of B/C at this stage of our data analysis. The term 'level' does not mean hierarchy in the analysis, but inward or outward layers of the analysis.

The coding in Table 3 captures the behaviour and cognitive components of both individual pupils and the teacher addressed in Zhang's explanations. For instance, the teacher and her pupils' ways to teach and learn the mathematics topic in the textbook.

Behaviour/ cognitive	Explanation	Examples of what was said by Zhang
Understanding textbook	e.g., Teacher's knowledge and skills of crafting textbook; pupils' conceptual understand of the textbook.	First, to understand the textbook. <i>Why we chose this topic to study?</i>
The type of mathematical proposition and its learning	e.g., Teacher's mathematics and pedagogy	It's not to learn a new concept, but to learn a new proposition. It's to discover a rule or a relationship in the process of learning the proposition.

Table 3: Codes of the behaviour and cognitive components in Zhang's explanations.

The coding in Table 4 recognizes Zhang's target to draw the junior teacher's awareness to the important interactions between mathematical proposition in elementary textbook and the assessment of teaching and learning. Data examples were chosen from Zhang's explanations to the junior teacher of redesigning L1 of the LDS (see Tables 3 as well).

Concurrently, the coding in Tables 3 and 4 can also be considered together in the analysis of the constraints of activity systems that were made visible by Zhang to the junior teacher in supporting the teacher to be aware of the factors that may play a role in the participation-oriented lesson design in the LDS. For instance, to enable pupils to participate actively in the mathematical practices and classroom discourse in the lesson, Zhang emphasised the interactions between a deep understanding of the teaching

and learning goal of the mathematic topic in the textbook and the assessment of teaching and learning, as illustrated by the example in Table 4.

Interactive systems	Explanation	Example
Assessment of teaching and learning	e.g., lesson structure and procedure; pupil's cognitive nature and their learning methods and processes; social and cultural values in education; etc.	Secondly, to distinct the deep learning from the surface teaching from the perspective of teaching and learning assessment. The deep learning addresses the cognitive process of students. The surface teaching refers to the teacher's instructional structure/procedure of the lesson that was advocated ten years ago. ... But it is very important for us to assess the lesson from students' learning perspective. That is, <i>whether the idea of the lesson design suits our fundamental theory and main educational value nowadays?</i>

Table 4: Codes of the interactive systems in Zhang's explanations.

The coding in Table 5 was identified for the level of analysis of the affordances of activity systems that were observed in Zhang's explanations in supporting the junior teacher's professional learning of how to implement the participation-oriented lesson design in the LDS. Data examples in Table 5 were chosen from Zhang's explanations to the junior teacher of re-implementing L2 of the LDS.

We consider that the types of Zhang's questions, such as '*what?*', '*how?*', '*why?*', '*whether?*' (we used *italic* to highlight them in the examples in Tables 1-5) play a significant role as a scaffolding of *attunements* for the junior teacher to be aware of, and then be able to participate in the trajectory of teaching and learning in the re-designed lesson that was explained by Zhang and then be able to reflect on her own instructional intentions and practice from this specific perspective.

The dual nature of Zhang's expertise in the LDS

From our analysis, we propose that there is a *dual nature* of Zhang's expertise in our LDS. On the one hand, Zhang's explanations with teachers in the TRG make the hidden constraints and affordances in the interactive systems of the LDS visible to teachers in order to engage pupils into actively participate and to be able to attune by themselves to the mathematical practices and classroom discourse in the designed lesson; on the other hand, Zhang's questions to teachers (see examples highlighted in *Italic* in Tables 3, 4 & 5) plays a kind of scaffolding role to enable the teacher to learn and understand how and why to attune to the constraints and affordances in the interactive systems of the LDS.

Affordances	Explanation	Examples
Lesson design	e.g., Mathematical inquiry lesson; lesson structure; instructional coherence.	There is an obvious gap in the first two activities in the lesson: The first activity is to draw rectangles with constant perimeter 10 cm. The second activity is to draw rectangles with constant perimeter 20 cm. You (the junior teacher) need to build up the connection of the two activities to support pupils to develop an understanding of the two activities. That is, <i>why the second activity is necessary after the first one?</i>
Teacher's teaching language	e.g., Teacher's explanation of teaching and learning goals of a classroom activity; teacher's questions, etc.	The rough thought and judgement of the first activity is that, given the same perimeter, the areas of rectangles can be different. The first activity is a stepping stone for the second activity. <i>Why?</i> The more precise judgement and finding of the regularity of the operation is from the second activity. That is, the closer the length and width of a rectangle, the larger the area. The teacher must play a leading role in explaining to pupils to enable them to participate in the coherence of the two activities.

Table 5: Codes of the affordances in Zhang's explanations.

DISCUSSION AND CONCLUSION

In this paper, we propose a conceptualization of the expert teacher's expertise in the LDS by coordinating the level of teacher and pupils' subject-based behaviour/-cognitive analysis (the two ellipses in Figure 2) and the level of the social-culturally situated analysis of the LDS from Greeno's model (1998). We propose that there is a dual nature of Zhang's expertise in our LDS, as indicated by the two overlapping rectangles in Figure 2.

The first nature of Zhang's expertise is scaffolding the teachers to learn concurrently the act of the multiple theoretical ideas (e.g., behaviour, cognitive and situative theories) through the participation-oriented mathematics lesson design study. The second nature of Zhang's expertise is scaffolding the teachers to learn to reflect on their own beliefs about the subject, pedagogical thinking and action, and to develop their identity as mathematics teachers in their long-term professional life.

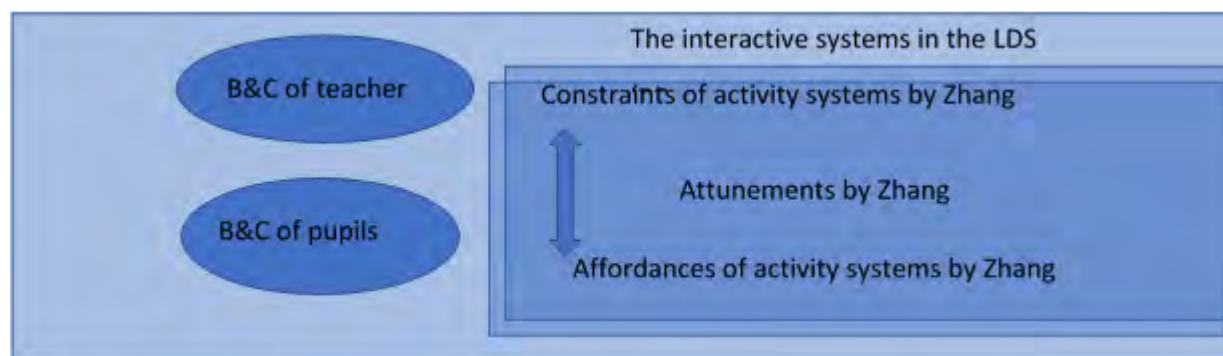


Figure 2: The model of the dual-nature of Zhang's expertise in the LDS.

In the next step, we plan to apply the categories and codes presented in this paper into a close analysis of the expert teacher's expertise. We also aim to make the scaffolding functions of Zhang's *attunements* in the LDS visible in our analysis. In so doing, we aim to contribute a deep analysis of the unique and significant functions of Chinese expert teacher's explanations and questions in the teacher professional development in China; something that outsiders may see as "monologues rather than dialogic in nature" (Gu & Gu, 2016, p. 451) that as such, remain a challenge to be explored by insiders to the education system in China.

References

- Ding, L., Jones, K., Pepin, B., & Sikko, S. A. (2014). An expert teacher's local instruction theory underlying a lesson design study. In Nicol, C., Liljedahl, P., Oesterle, S., & Allan, D. (Eds.) *Proceedings of the Joint Meeting 2 - 401 of PME 38 and PME-NA 36*, Vol. 2, pp. 401-408. Vancouver, Canada: PME.
- Ding, L., Jones, K., Mei, L., & Sikko, S.A. (2015). "Not to lose the chain in learning mathematics": Expert teaching with variation in Shanghai. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 2, pp. 209-216. Hobart, Australia: PME.
- Ding, L., Jones, K., & Sikko, S. A. (2017). An expert teacher's use of teaching with variation to support a junior mathematics teacher's professional learning. In R. Huang & Y. Li (Eds.) *Teaching and learning mathematics through variation* (pp. 241-266). Rotterdam: Sense Publishers.
- Greeno, J. G. (1998). The situativity of knowing, learning and research. *American Psychologist*, 53(1), 5-26.
- Gu, F. & Gu, L. (2016). Characterizing mathematics teaching research specialists' mentoring in the context of Chinese lesson study. *ZDM Mathematics Education*, 48, 441-454.
- Kaiser, G. and Li, Y. (2017). Perspectives on (future) teachers' professional competencies. *Proceedings of PME41*, Vol. 1, pp. 81-108.
- Li, Y., & Kaiser, G. (2011). Expertise in mathematics instruction. In Y. Li & G. Kaiser (Eds.), *Expertise in Mathematics Instruction* (pp. 3-15). New York: Springer.
- Pepin, B., Xu, B., Trouche, L., & Wang, C. (2016). Developing a deeper understanding of mathematics teaching expertise. *Educational studies in Mathematics*, 94(3), 257-274.

MATHEMATICIANS' CRITERIA FOR ACCEPTING THEOREMS AND PROOFS – AN INTERNATIONAL STUDY

Anika Dreher¹ and Aiso Heinze²

¹ Freiburg University of Education, Germany

² IPN – Leibniz-Institute for Science and Mathematics Education, Kiel, Germany

Argumentation and proof are crucial for the mathematics discipline and should thus permeate mathematics education. In particular proof validation lately became a focus of attention in mathematics education research. Since expert practice is seen as an important frame of reference for instructional goals regarding proof validation, it was emphasized that empirical research on mathematicians' criteria for accepting proofs is needed. However, such empirical research based on reliable quantitative data is still scarce. Consequently, this study analyzes criteria for accepting proofs in their daily work held by $N = 243$ highly respected mathematicians from all over the world. The results indicate three types of mathematicians who rely on certain criteria to various degrees as well as differences between status groups.

INTRODUCTION

As mathematics is a proving science, proof is widely seen as essential especially in secondary and tertiary mathematics education (e.g., Hanna, 1997; Marriotti, 2006). Consequently, there is a broad base of educational research on how students engage with mathematical proofs. Although this research focuses mainly on the construction of proofs (Sommerhoff, Ufer, & Kollar, 2015), there is a growing interest in practices of proof validation (e.g., Selden & Selden, 2003; Weber, 2008). Several scholars in mathematics education emphasized that corresponding goals for instruction should be informed by expert practice of mathematicians (e.g., Inglis & Alcock, 2012; Weber, Inglis, & Ramos, 2014). Since this requires a thorough understanding of such expert practice, research focusing on mathematicians' professional practices regarding proofs was called for (Weber et al., 2014). Against this background there were in particular some studies conducted exploring how mathematicians validate proofs (e.g., Weber, 2008) as well as their criteria for accepting mathematical theorems as being valid (Heinze, 2010; Mejía-Ramos, & Weber, 2014). Based on corresponding findings Weber and colleagues (2014) argued that current instructional recommendations in mathematics education are “oversimplified and not based on an accurate understanding of mathematical practice” (p. 54). However, as these studies were so far mostly explorative and based on relatively small sample sizes, further research with better-developed instruments and better-quality data is necessary to get more insight into expert's criteria for accepting mathematical theorems and proofs as being valid (Heinze, 2010). Consequently, this online survey study explores mathematicians' ac-

ceptance criteria in their daily work based on a high-class international sample by means of quantitative methods.

THEORETICAL BACKGROUND

The naïve idea that the deductive nature of mathematics allows determining the correctness of mathematical proofs with absolute certainty turned out to be wrong (e.g., Hanna, 1983). Consequently, the question as to how new mathematics results get accepted by mathematicians gives rise to a complex field of research which is interesting not only from a mathematics education point of view, but also from the perspective of philosophy of mathematics (e.g., Geist, Loewe, & van Kerkhove, 2010). Reflecting upon this question, among others, Heinze (2010) looked at the situation in science, where a criterion for accepting a new result based on experiments is that this result must be replicated independently under the same conditions by different researchers. Considering proofs as thought experiments, he asked whether mathematicians rely on experiments and replications by others or whether they have to replicate it themselves in order to accept a corresponding result. The question raised, namely to what degree knowledge by testimony (Geist et al., 2010) or authoritarian evidence (Weber et al., 2014) can lead mathematicians to accept a new theorem in the sense of relying on journals or other mathematicians is central to the discussion of mathematicians' acceptance criteria. There are mathematicians for whom it is crucial that they check a proof of every mathematical result which they apply in their work (e.g., Geist et al., 2010). However, often this is hardly possible, since the proofs of some theorems are extremely long and the diversity of the mathematics discipline involves that many mathematicians are unable to follow the proofs of theorems that come from another area of research (Auslander, 2008). Hence, there is a consensus that social processes play a role for the acceptance of new mathematical results (Hanna, 1983). Moreover, findings from first empirical studies in this area indicate that some mathematicians rely on authoritarian evidence to accept theorems and proofs as black boxes in their own research (Heinze, 2010; Weber et al., 2014). In their online survey with 118 American mathematicians, Mejía-Ramos and Weber (2014) found for instance that 72% of the participants agreed with the statement "It is not uncommon for me to believe that a proof is correct because it is published in an academic journal" (p. 166). In view of such results Weber and colleagues (2014) argued that authors in mathematics education "who believe that students should not accept claims as true because an authority told them that this was the case and that one should not consider who wrote the argument while evaluating its validity" (p. 45) should rethink the grounds for their instructional suggestions. The findings of Heinze's (2010) exploratory online survey with 40 German mathematicians also indicate that there is a substantial amount of reliance on the mathematics community and peer-reviewed journals with respect to the acceptance of theorems and proofs. He found, for instance, that on average the participants stated to frequently accept a theorem in their daily work as being valid, if a proof was published long ago and there was no contradiction so far. However, the results also suggested that full professors relied less frequently on the mathematics

community and peer-reviewed journals than PhD students and postdocs. This could be a result of the professors' experience, in particular with reviewing papers for journals, since the mathematical refereeing process is sometimes not as trustworthy as one may think (Geist et al., 2014; Nathanson, 2008). Thus, the participants' high reliance on authoritarian evidence in the study by Mejía-Ramos and Weber (2014) might be partly due to the small share of full professors in their sample (28% faculty members) and the fact that more than half of the participants had never refereed a paper for a journal.

If mathematicians lack time and other resources to check a proof step by step of every theorem they use in their daily work, but still do not rely on authoritarian evidence, another solution could be to gain conviction by “partly” checking a proof. Indeed, acceptance criteria for some mathematicians can be that they checked the key arguments of a proof, are convinced that the main ideas of a given proof are correct (Heinze, 2010), or checked the theorem for carefully chosen examples (Weber, 2008). In particular in view of the latter criterion, Weber and colleagues (2014) “challenge[d] instruction that aims for students to never seek conviction in this way”.

It can be summarized that mathematicians' criteria for accepting theorems and proofs in their own daily work may be individual checking – where a distinction can be made between “step by step” and “partly” checking – but also authoritarian evidence which may refer to respected journals or the assumption that “enough” mathematicians in the community have verified a proof. However, there is still little evidence for answering the question to what extent mathematicians rely on these different kinds of acceptance criteria in their daily work. Empirical findings indicate that there is substantial heterogeneity among mathematicians regarding their acceptance criteria (e.g., Heinze, 2010; Mejía-Ramos & Weber, 2014). This heterogeneity could be a result of external factors (e.g., different status groups, areas of mathematics, culture) or of more individual characteristics in the sense of different types of mathematicians. This is not clear, yet. Thus, there is a need for research into questions whether there are differences between the acceptance criteria of certain groups of mathematicians, what types of mathematicians exist and how dominant these types are in the mathematics community. The quantitative studies on mathematicians' criteria for accepting theorems and proofs that exist so far focus mainly on the level of PhD students and postdocs instead of full professors who are experienced and esteemed members of the community and on national samples. In view of the assumed heterogeneity, it might however be crucial to focus on an adequate sample in order to get the full picture. Moreover, since so far this research was based on analysis regarding single items, a study using more developed instruments is needed.

RESEARCH QUESTIONS

According to the need for research pointed out in the previous sections the study presented here aims to provide evidence for the following research questions:

1. *To what extent do mathematicians rely on different criteria for accepting a theorem and proof as valid in their daily work?*

2. *Are differences with respect to external factors (status groups, refereeing experience) associated with differences regarding acceptance criteria?*
3. *How can different types of mathematicians regarding their acceptance criteria be characterized?*

SAMPLE AND METHODS

For answering these research questions an online survey was designed using the software “Unipark”. The questionnaire was completed by a sample of 243 research mathematicians (177 male, 30 female, 36 without data) who have been participants of workshops at the highly reputable Mathematical Research Institute of Oberwolfach during the last years and are thus esteemed members of the international mathematics research community. The sample is international and not even restricted to European countries (151 from Europe, 39 from North America, 11 from Asia, 3 from Australia, 3 from South America, 36 without data). Most of the participants are full professors (114 full professors, 30 associate professors, 28 assistant professors, 27 postdocs, 1 PhD student, 2 professors emeritus, 3 senior lecturers (UK), 38 without data). The large majority has been referee for a journal paper multiple times (at least three times: 191, once or twice: 11, not yet: 7, without data: 34).

Corresponding to the research questions for this study, the participants were asked under which conditions they accept a mathematical theorem as valid in their daily mathematical work. The mathematicians could express their approval or disagreement regarding statements of the form “In my mathematical work I assume that a mathematical theorem is valid, if...” on a six-point Likert scale with endpoints “entirely disagree” and “entirely agree”. The statements were completed by criteria as identified in the previous section (for details and sample items see Table 1).

RESULTS

We started the data analysis by conducting a factor analysis with oblique rotation. After excluding one item that could not be assigned to any factor, the Kaiser criterion yielded 4 factors, where each item loads with > 0.4 on one factor (51% explained variance). The clustering of items suggests that the four factors represent the four kinds of acceptance criteria identified in the theoretical background. Hence, four scales could be formed (see Table 1). For three of the scales the reliability is good and in view of only two items forming the remaining scale, its reliability is acceptable.

The means and standard errors for these scales displayed in Figure 1 show that on average the mathematicians reported high agreement with the acceptance criterion individual checking “step by step” and medium agreement with the other three kinds of criteria. Regarding these other three kinds of criteria on average authoritarian evidence in the sense of “enough” other mathematicians have checked the proof received most and authoritarian evidence in the sense of peer-reviewed journals received least agreement.

Scale	Sample items	# items	Cronbach's α
Individual checking "step by step"	... I verified a given proof step by step.	2	.63
Individual checking "partly"	... I am convinced that the main ideas of a given proof are correct. ... the theorem is valid for all examples that I know.	7	.82
authoritarian evidence "journals"	... the theorem was published with a proof in a refereed journal.	5	.88
authoritarian evidence "enough" mathematicians	... I know that a proof has been available for a long time and has been checked by many mathematicians.	6	.86

Table 1: Scales regarding different acceptance criteria

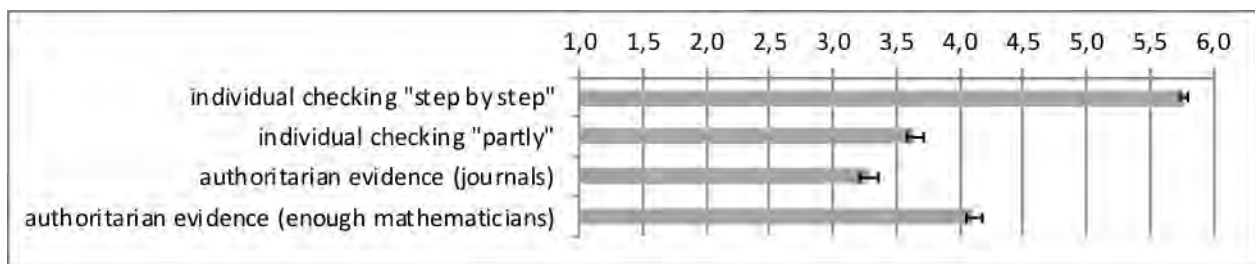


Figure 1: Agreement with acceptance criteria (means and their standard errors), six-point Likert scale from 1 = entirely disagree to 6 = entirely agree

In view of the second research question, we explored next whether mathematicians who differ regarding certain external factors also differ regarding their acceptance criteria. Due to space limitations, only selected results can be presented. For getting insight into whether the status group is a factor that accounts for heterogeneity among mathematicians regarding acceptance criteria, we consider the lowest and the highest status group in our sample with more than one representative: postdocs and full professor. Comparing these two status groups yields no significant differences regarding the two criteria referring to individual checking, but reveals that full professors agreed significantly less with authoritarian evidence "journals" ($t(138) = 2.96, p < .01, d = 0.73$) and with authoritarian evidence "enough mathematicians" ($t(53) = 2.72, p < .01, d = 0.52$) than postdocs. Both differences represent medium effect sizes. To investigate whether experience in refereeing for journals is associated with less reliance on peer-reviewed journals, those mathematicians with no refereeing experience were compared to those who had been refereeing for at least three times. Indeed, the former

agreed significantly and substantially more with the criterion authoritarian evidence “journals” than the latter ($t(195) = 2.68, p < .01, d = 1.03$).

For answering the third research question and exploring different answering patterns a hierarchical cluster analysis using Ward’s method was conducted. The cluster analysis was based on the four scales representing different kinds of acceptance criteria. This analysis yielded three clusters showing distinct answering patterns which are presented in Figure 2.

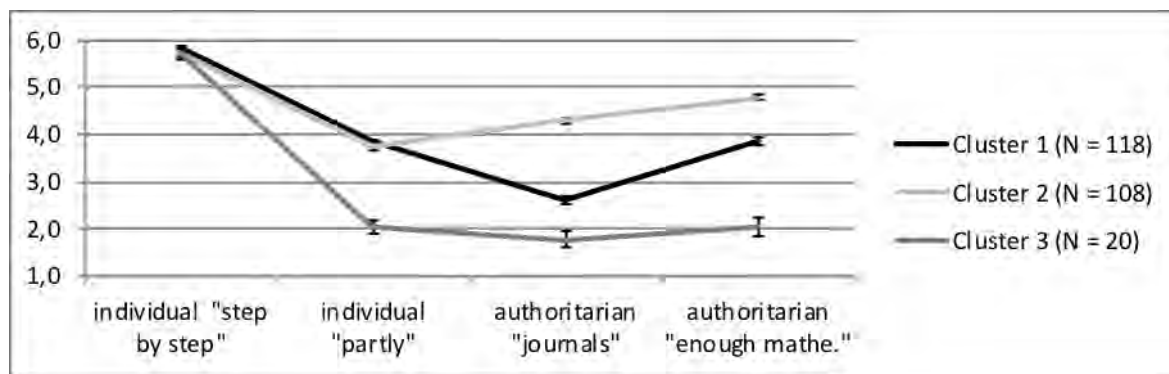


Figure 2: Means and their standard errors of the three clusters, six-point Likert scale from 1 = entirely disagree to 6 = entirely agree

The three clusters do not differ with respect to the criterion of individual checking “step by step”, but there are substantial differences regarding the other three kinds of criteria. Cluster 3 consists of a minority of mathematicians who disagreed with any other acceptance criterion. Cluster 1 and 2 are almost of the same size and are characterized by the same level of medium agreement with the criterion individual checking “partly”. However, the mathematicians of cluster 1 stated to rely less on journals than on their own partly checking, whereas this is vice versa for cluster 2. Moreover, the mathematicians of cluster 1 showed less conviction by authoritarian evidence “enough mathematicians” than their colleagues of cluster 2. Consistent with the results regarding the second research question, cluster 1 and 3 consist of relatively more full professors and cluster 2 consists of relatively more postdocs compared to the proportions in the full sample.

DISCUSSION AND CONCLUSIONS

Based on reliable scales and an international sample of highly esteemed members of the mathematics research community, this study provides further evidence that most mathematicians do not exclusively rely on individual step by step verification for accepting theorems and proofs as valid in their daily work. They may also use “partly” checking and authoritarian evidence referring to respected journals or the assumption that “enough” mathematicians have verified a proof as acceptance criteria. Concerning the heterogeneity within the mathematics research community with respect to the extent to which mathematicians rely on these different kinds of criteria, the findings of this study can give new insights: On the one hand, according to the second research

question, it was explored whether external factors account for such heterogeneity. Indeed, full professors showed significantly less acceptance of both kinds of authoritarian evidence compared to postdocs. This could be a result of the full professors' experience in the mathematics research community. The assumption that in particular experience with the mathematical refereeing process could lead to less reliance on peer-reviewed journals is supported by the result that experience in refereeing was associated with significantly and substantially less agreement with the acceptance criterion referring to peer-reviewed journals. These results challenge Mejía-Ramos and Weber (2014), who concluded from their findings that there were no differences between the acceptance criteria of less experienced and more experienced mathematicians. However, their sample included only a small share of full professors and more than half of the participants had never refereed a paper for a journal. Consequently, the mathematicians' high reliance on authoritarian evidence reported from this study should be interpreted with care.

On the other hand, according to the third research question, different types of mathematicians regarding their acceptance criteria were explored by means of a cluster analysis. In line with Geist and colleagues (2010) the results indicate that there is a type of mathematician that clearly disagrees with every acceptance criterion except for individual step by step checking of a proof. However, this type appears to account for a minority in the mathematics research community. According to our findings the community is dominated by two types that do not strictly reject other acceptance criteria: one of them is characterized by relying more on individual "partly" checking than on peer-reviewed journals and the other one is characterized by relying more on authoritarian evidence than on "partly" checking. The fact that the three types represented the status groups by different proportions suggests that further analyses and research is necessary to investigate to what extent these types are based on external factors or more individual characteristics of mathematicians.

We would like to recall that the findings of this study should be interpreted with care, since the data is based on self-reports and thus social desirability may play a role. Hence, these results should be corroborated by means of other methods. Comments by some participants indicate that they often use a combination of acceptance criteria (see also Hanna, 1983). Thus, further research taking into account such combinations is necessary. Follow-up research should moreover focus on further external factors and also on the context of refereeing a paper complementing the context of mathematicians' daily work as considered in this study (e.g., Mejía-Ramos & Weber, 2014).

Acknowledgements

We would like to thank Benedikt Loewe for his support in initiating this study and in the item development. Moreover, we are thankful to Juan Pablo Mejía-Ramos for his helpful comments regarding the survey.

References

- Auslander, J. (2008). On the roles of proof in mathematics. In B. Gold & R. A. Simons (Eds.), *Proofs and other dilemmas: Mathematics and philosophy* (pp. 61–77). Washington, DC: Mathematical Association of America.
- Inglis, M., & Alcock, L. (2012). Expert and novice approaches to reading mathematical proofs. *Journal for Research in Mathematics Education*, 43, 358–390.
- Inglis, M., & Mejia-Ramos, J. P. (2009). The effect of authority on the persuasiveness of mathematical arguments. *Cognition and Instruction*, 27, 25–50.
- Geist, C. Löwe, B., & Van Kerkhove, B. (2010). Peer review and testimony in mathematics. In B. Löwe & T. Müller (Eds.), *Philosophy of Mathematics: Sociological Aspects and Mathematical Practice* (pp. 155–178). London: College Publications.
- Hanna, G. (1983). *Rigorous proof in mathematics education*, OISE Press, Toronto.
- Hanna, G. (1997). The ongoing value of proof in mathematics education. *Journal für Mathematik Didaktik*, 97(2/3), 171–185.
- Heinze, A. (2010). Mathematicians' individual criteria for accepting theorems and proofs: An empirical approach. In G. Hanna, H. N. Jahnke, & H. Pulte (Eds.), *Explanation and proof in mathematics: Philosophical and educational perspectives* (pp. 101–111). New York: Springer.
- Marriotti, M. (2006). Proof and proving in mathematics education. In A. Gutierrez & P. Boero (Eds.) *Handbook of research in mathematics education: Past, present, and future* [PME 1976-2006]. (pp. 173–204). Rotterdam: Sense Publishers.
- Mejía-Ramos, J. P., & Weber, K. (2014). How and why mathematicians read proofs: further evidence from a survey study. *Educational Studies in Mathematics*, 85, 161–173.
- Nathanson, M. B. (2008). Desperately seeking mathematical truth. *Notices of the American Mathematical Society*, 55(7):773.
- Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education*, 34(1), 4–36.
- Sommerhoff, D., Ufer, S., & Kollar, I. (2015). Research on mathematical argumentation: A descriptive review of PME proceedings. In K. Beswick, T. Muir, & J. Wells (Eds.), *Proceedings of the 39th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 193–200). Hobart, Australia: PME.
- Stylianides, A.J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education*, 38, 289–321.
- Weber, K. (2008). How mathematicians determine if an argument is a valid proof. *Journal for Research in Mathematics Education*, 39, 431–459.
- Weber, K., Inglis, M., & Mejia-Ramos, J.P. (2014). How mathematicians obtain conviction: implications for mathematics instruction and research on epistemic cognition. *Educational Psychologist*, 49(1), 36–58.

FROM EVERYDAY PROBLEM TO A MATHEMATICAL SOLUTION — UNDERSTANDING STUDENT REASONING BY IDENTIFYING THEIR CHAIN OF REFERENCE

Jonas Dreyøe¹, Dorte Moeskær Larsen², and Morten Misfeldt³

Aalborg University^{1,3} and University of Southern Denmark²

This paper investigates a group of students' reasoning in an inquiry-oriented and open mathematical investigation developed as a part of a large-scale intervention. We focus on the role of manipulatives, articulations, and representations in collaborative mathematical reasoning among grade 5 students. In this analysis, we apply the idea of the chain of reference from the studies of Bruno Latour (1999) to the exploration, generation, and formalization of scientific knowledge. This framework allows us to combine knowledge from mathematics education about language and representations, manipulatives, and reasoning in a way that allows us to follow the material traces of students' mathematical reasoning and hence discuss the possibilities, limitations, and pedagogical consequences of the application of Latour's (1999) framework.

THE MATERIAL TRACES OF MATHEMATICAL REASONING

This paper concerns how to understand students' work process and thinking in the course of progressing from an everyday problem to a formal mathematical solution in the mathematics classroom by focusing on students' material artefacts, articulations, and representations and the transformations between them.

We aim to emphasize that the division of the world into a mathematical and an empirical world is problematic, too formalistic, and out of step with how students learn and make sense in the mathematics classroom. We propose an approach that involves developing mathematical reasoning in continuity with students' everyday understanding and empirical inquiry, rather than as a categorical distinction between everyday knowledge and mathematical knowledge. To do so, we utilize Latour's (1999) description of the process that generates discursive scientific facts from investigations of material reality: *circulating reference*.

Latour (1999) takes outset in the radical gap between “words and the world” that he argues is hardwired into western philosophy and must be reduced. He suggests there is neither correspondence nor a significant gap, but instead he presents a new model: chain of reference (Latour, 1999). Using this model, he argues that the distinction between words and the world is not a fundamental gap, but rather a chain of small fundamental distinctions between references that are both representative and material.

Materials, manipulatives, and utterances are all crucial elements for mathematical reasoning (e.g., Pimm, 1987; Duval, 2006; Defreitas & Sinclair, 2014). According to Duval (2006), mathematical objects cannot themselves be perceived or observed di-

rectly with instruments, but only by using signs and semiotic representations. His research therefore often focuses on the semiotic representations mobilized in mathematical processes. Though his research has given attention to troubles in the conversion of representations, he does not elaborate on the process of developing these mathematical representations.

In the literature, we also find a distinction between reasoning in mathematics and reasoning in everyday life. An example of this is Harel and Sowder's (1998, 2007) distinction between empirical proof schemes and formal proof schemes. The majority of students in elementary school have empirical proof schemes, and research has shown that changing students' empirical schemes to formal proof schemes is highly non-trivial (Education Committee of the EMS, 2011). Nevertheless, the change is an important process in mathematics education. Unfortunately, this literature does not elaborate on how this change takes place.

Using Latour's (1999) thinking, we aim to combine mathematical reasoning and proof with mathematical work processes and the use of representations in a single model, thus overcoming the distinction between deductive and empirical reasoning and mathematical objects and semiotic representations. This leads us to the following research question:

What are the potentials and limitations of using the concept of circulating reference to study the manifestation of mathematical reasoning in student articulations, manipulatives, and representations when solving an everyday problem mathematically?

METHODOLOGY

We employ a microethnographic design, as such an approach is well-suited for describing, analyzing, and interpreting a specific aspect of a group's shared behavior, beliefs, and language in a specific setting (Creswell, 2014; Garcez, 1997).

The study is embedded in a three-year design-based research, mixed methods, and randomized controlled trial program ($n = 177$ schools) that includes the development of a didactical design for inquiry-based learning over a four-month period. To achieve the aim, we employ a case study wherein we study one of the collected cases of the teaching activity called "What do the boxes weigh?" In the analysis, we conduct a within-case analysis and develop a case description (Yin, 2002).

Data collection and coding

The data used for constructing this case include field notes from non-participant observations, interviews, and audiovisual materials. To broaden our understanding, we visited four schools three times to observe specific classrooms for 90 minutes, subsequently interviewing students. The corpus of this paper is one of the observations and interviews conducted. In collecting the data, we focused specifically on one student per visit. The students were chosen by the teacher as being industrious and emotionally robust. These selection criteria were chosen to ensure the students would work with the assignment and not be overwhelmed by the researcher following them and their work.

The 15-to-35-minute unstructured interviews were conducted immediately after the observation as one-on-one interviews (Creswell, 2014) to allow the students to elaborate and reflect on their choices of materiality with their representations and reasoning fresh in their memory. They were asked questions, such as “Why did you express this in that way?” They were also shown alternative representations and asked in what ways they differ from their own.

We also collected audiovisual materials by video recording the teaching activities in full, by following the students and taking pictures of their work, and by recording the interview. All of the recordings have been transcribed in full.

Regarding data coding, we took an exploratory approach by reading the data end-to-end several times and discussing general trends. After, we developed codes based on the theoretical framework utilized in this paper (Latour, 1999). The first and second authors of the paper then coded the observation and interview individually and thereafter compared and discussed the coding with the last author. Individual “double” coding was conducted to avoid subjective bias in the analysis and to increase the inter-coding reliability (Russell, 2018).

CHAIN OF REFERENCE

In this section, we present the theoretical basis for the paper, which serves as a means to describe holistically the development of representations in a material form. Latour (1999) distinguishes between words and the world, but he does not perceive them as “...disjointed spheres separated by a unique and radical gap that must be reduced through the search for correspondence, for reference, between words and the world” (Latour, 1999, p. 69).

On the other hand, in Latour’s (1999) framework, matter is at one end and form at the other. This is disjunct from the stage that follows by a gap, which he claims no resemblance can fill. This means that knowledge is not located in the mind when the subject is confronted with an object and that a reference in a language does not *designate* an object by means of words verifying the object.

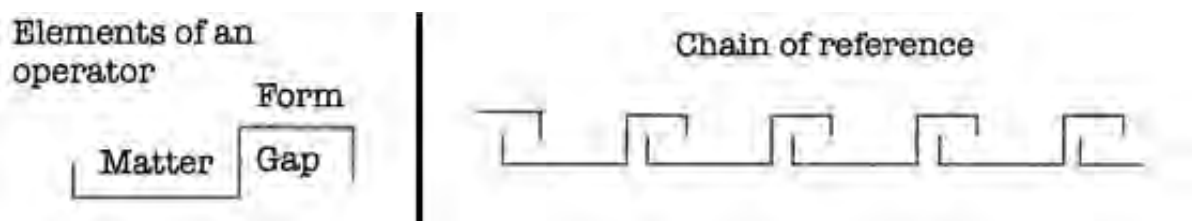


Figure 1: Elements of an operator and chain of reference (Latour, 1999, p. 70).

These elements, formed by *matter*, *form*, and a *gap*, are called *operators*. These operators are linked in a reversible *chain*. The essential property of a chain is its traceability, allowing you to travel in both directions. Latour (1999, p. 69) describes it as follows: “If the chain is interrupted at any point, it ceases to transport truth — ceases, that is, to produce, to construct, to trace, and to conduct it.”

This conception of reference has no limit at either end, as opposed to the traditional understanding of the two disjunct spheres. This chain of reference can be extended at both ends by adding operators. However, it is not possible to cut the chain or skip any of the operators. To comprehend the chain of reference, Latour (1999) introduces two new terms: *reduction* and *amplification*. These two processes occur while moving from one operator to another. There is a reduction in locality, particularity, materiality, multiplicity, and continuity when moving from one operator to another. While reducing some parts of the operator's complexity, other parts are amplified. The amplification regards compatibility, standardization, text, calculation, circulation, and relative universality. This means in the end, we are able to understand and explain the dynamics of a system, but we are also able to trace back to the complexity we meet at first. Latour (1999) depicts this reduction and amplification in a model in which two triangles overlap.

In this tradition, it is never possible to understand a phenomenon fully, as it grows from the middle toward the ends, and there are no ends to the amplification and reduction we can perform (Latour, 1999).

THE CASE OF AMY AND DAN

In the following, we present the case of Amy and Dan working with the activity “What do the boxes weigh?” This task concerns six boxes that have been weighed in pairs. All the paired combinations of boxes weigh 6, 8, 10, 12, 14, and 16 kg, respectively. The students are to determine the individual weights of the boxes by inquiry.

To understand the chain of reference and to grasp the dialectical processes of the amplification and reduction of the complexity of the representations that characterize each stage in the reasoning process, we will investigate and describe the chain of operators (Latour, 1999).

First operator: Boxes and weights

In this case, the teacher begins his narrative by telling a contextual story in which he is in need of help because he has four boxes that fell out of his car earlier that morning. The problem to solve is the need to determine what each weighs separately. He brings four different-sized boxes into the classroom, which he showed the class. The boxes were labelled A, B, C, and D. He tells the class explicitly that these boxes are empty. Additionally, he writes on the board that the paired weights of the boxes are 6, 8, 10, 12, 14, and 16 kg, respectively.

This denominates the first operator: the empty boxes in their physical form and the paired weights written on the board.

Second operator: Boxes and drawings

The two students, Amy and Dan, begin the task by drawing the boxes side by side onto Amy's paper and writing the names of the actual boxes — A, B, C, and D — inside the drawn ones. At the bottom, Amy writes the paired weights. While drawing the boxes,

Amy says, “Ehhh...It is just to be able to remember that two of the boxes’ weights combined is this; if not, we would be completely confused.” While trying to figure out how to solve the task, the students still use the actual boxes. Matilde keeps pointing at them, and Dan goes to the boxes and tries to weigh them by hand.

By writing the names (A, B, C, and D) of the boxes and drawing them on her paper, the second operator begins, which seems an adaptation of the teacher’s presented narrative using the same names for the boxes and the same boxes. However, now they have added a new representation of the boxes on Amy’s paper.

Third operator: Boxes, drawings, and equations

Meanwhile, Amy sits at the table talking to Dan across the classroom. Amy says, “So, if we say that...if we take A and B — they do not look very big (...) [long pause] — but I do not think it’s those that weigh six; I would imagine four or something.” Subsequently, Amy notes the agreed-upon resulting weight, i.e., writing $A + B = 4$.

This representation thus marks a shift in the operators, where the students write the agreed-upon paired weights as an equation onto their paper. Thus, the third operator is the boxes in their physical form, the weights on the board, the drawings of the boxes, and the addition of the boxes’ names equaling the paired weights assigned.

Fourth operator: Drawings and equations

The teacher approaches Dan, as he is still by the boxes and weighing them by hand.

Teacher: (...) you can’t just weigh them by hand, I already told you. See if you can explain to me how. (...) But would it make sense to try and choose some number and then see if you can combine your way out of it?

Dan: No [shakes his head]. (...)

Teacher: Okay, maybe it could be so that one of them has a weight of one and another one with a weight of five. [Dan nods and looks at Amy over his shoulder]. Good, but then try to think more about that. (...) Now we need to know what the last two boxes weigh, but we should be able to make the combination to get all of these up here [the weight results on the blackboard]. Try to talk it over with Amy.

They now stop using the boxes in their physical form; instead, they use the drawings of the boxes on Amy’s paper and write the paired weights on the paper as well.

Amy: C and A are roughly the same size.

Dan: Maybe A weighs 3 kg.

Amy: And then C could weigh — well they are pretty much the same size.

The situation when Dan speaks to the teacher leads the students to shift operators, as Amy and Dan discuss after what the teacher spoke about to Dan. They then *exclude* the boxes in the physical form from their reasoning. The fourth operator thus consists of the equations of the boxes, the drawings of the boxes, and the notation of the paired weights. However, even though the drawings of the boxes are the same size, it is still

important to Amy that the large boxes weigh more than the small boxes. Thus, the sizes of the boxes are remembered, even though they are not depicted in the representations used in the operator.

Fifth operator: An attempted solution

While working with the drawings and equations, they stop mentioning the sizes of the boxes and start to think about them as numbers instead.

Amy: Perhaps we have to think about which numbers we can get before we write them down. If I try, I will just make another one with A, B, and C instead of erasing. Okay?

Dan: So, we need to swap [some of the numbers].

Amy: I think we need the number four... because we have already used it three times.

Amy and Dan now try a solution with four different weights by writing them below the drawn boxes. However, the solution is incorrect.

While working with this operator, the sizes of the boxes in their physical forms are still an obstacle for them in solving the task. The reduction of complexity offered by this operator, however, helps them to focus not on the sizes of the physical shapes, but instead on the amplified attributes: that they are each an entity needing to be assigned a weight that can make up the paired weights, as identified by the task. The fifth operator is the concrete example that did not work.

Sixth operator: A set of attempted solutions

Amy and Dan try two additional sets of attempted solutions that differ from the first, and the fourth solution is one of the two correct solutions. The sixth operator evolves from the fifth operator by creating a set of attempted solutions as a structure for systematizing, enabling them to eliminate wrongful combinations. This systematization of more than one different set is thus the sixth and final operator. This systematization structure helps them to identify the result quickly, as they are not repeating wrong earlier sets.

Amy and Dan's chain of reference

The focal point and first operator of the chain of reference is the contextual narrative presented by the teacher. In the second operator, the students adapt the narrative and supplement it with their own material structure. The third element in the chain involves an addition where the equations are written, thus enabling them to remember what they have done. The fourth operator requires the students to stop using the boxes in their physical form, but to still remember and use the sizes of the boxes. In the fifth operator, the sizes of the boxes are entirely uninvolved, thus reducing the complexity of the physical shape and amplifying the fact that they are entities assigned a weight. The fifth operator is thus the one attempted solution that did not work out. The sixth oper-

ator is a set of attempted solutions that enables them to eliminate already tried and unsuccessful solutions and that leads ultimately to the correct combinations of weights.

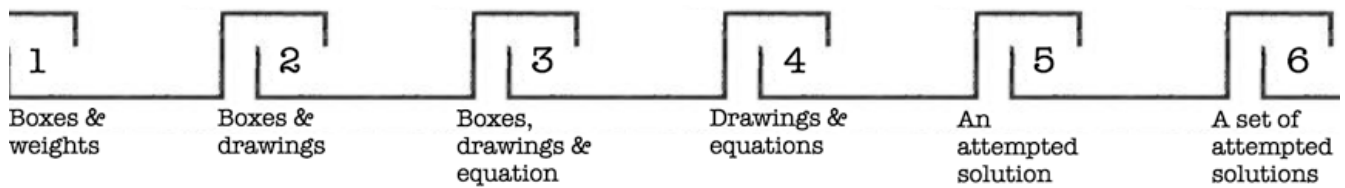


Figure 2: Amy and Dan's chain of reference for the task: "What do the boxes weigh?".

Based on the case, it seems apparent that for Amy and Dan, it was imperative that they, disregard the importance of the boxes' sizes, namely overcoming a gap in the chain of reference. This structured and transparent transgression to the more abstract seems necessary to be able to generate cognitively an overview of the problem. This insight would be hard to explain without the framework of circulating reference.

Discussion and conclusion

Using Latour's (1999) thinking, we aimed in this paper to coincide mathematical reasoning and proof with mathematical work in a single model, thus overcoming the distinction between deductive and empirical reasoning and mathematical objects and semiotic representations. Research focusing on student reasoning (Harel & Sowder, 1998; Education Committee of the EMS, 2011) describes the importance of understanding deductive proof schemes. However, Latour's (1999) model does not make any distinction between different proof schemes, instead focusing on describing the process. Latour's (1999) model allows us to acknowledge the quality of the specific arguments and systematizations, without matching them to normative categories, such as deductive and inductive.

Latour's (1999) model provides us with a chain of small fundamental distinctions between the object outside of the mathematical domain and the representations inside the domain. By utilizing Latour's (1999) model, it is thus possible to gain a broader insight into how the transformation from everyday object to mathematical solution takes place.

In the described case of Amy and Dan, we found that by using manipulatives, articulations, and representations and by making transformations between them, the two students created a chain of reference to go from an everyday problem to constructing a mathematical solution. In our data, we have observations from three other classrooms working with the same problem, and in our analysis, we found that these students all followed a similar process; however, the students used other operators in the process, such as, e.g., colors as representations or descriptions of boxes as names.

From the introduction of the everyday problem to the students' mathematical solutions, we can observe that stage by stage, the students' representations lose locality, particularity, materiality, and multiplicity, such that in the end, there is almost nothing left but numbers on their paper; thus, a reduction in complexity occurs. However, this reduction has not only removed complexity, but also yielded greater standardization,

calculations, relative universality, and formalized knowledge, which is an amplification of the desired properties

In general, Latour's (1999) model has the potential to focus on the work process and thinking of students in the transformation from everyday problem to mathematical solution, as it makes the process noticeable. Teachers need to be aware that the construction of the chain is an important process, but it often needs some guidance, and the guidance should be aimed toward the point in the chain causing trouble, that is, students should be allowed to overcome gaps, such as the sizes of the boxes shown in the case.

It is thus important to let the gaps in the chain remain, to reduce complexity, and to amplify particularity and generalizability, as well as to maintain the students' ability to walk along the chain of reference to go from an everyday problem in mathematics to a more formal mathematical solution.

References

- Creswell, J. W. (2014). *Educational research: Planning, conducting and evaluating quantitative and qualitative research*. Harlow, Essex: Pearson.
- De Freitas, E., & Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the classroom*. Cambridge University Press.
- Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61, 103–131.
- Education Committee of the EMS. (2011). Do theorems admit exceptions? Solid findings in mathematics education on empirical proof schemes. *Newsletter of the European Mathematical Society*, 81, 50–53.
- Garcez, P. M. (1997). Microethnography. In N. H. Hornberger & D. Corson (Eds.), *Encyclopedia of Language and Education* (Vol. 8), (pp. 187–196). Dordrecht: Springer.
- Harel, G., & Sowder, L. (1998). Students' proof schemes: Results from exploratory studies. *Research in collegiate mathematics education III*, 234–283.
- Harel, G., & Sowder, L. (2007). Toward comprehensive perspectives on the learning and teaching of proof. In F. K. Lester Jr. (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (pp. 805–842). Charlotte, NC: Information Age Pub.
- Latour, B. (1999). Circulating reference — Sampling the soil in the Amazon forest. In *Pandora's hope essays on the reality of science studies* (pp. 24–79). Harvard University Press.
- Pimm, D. (1987). *Speaking mathematically—Communication in mathematics classrooms*. London: Routledge.
- Russell, B. H. (2018). *Research methods in anthropology: Qualitative and quantitative approaches* (6th ed.). Lanham, Maryland: Rowman & Littlefield.
- Yin, R. (2002). *Case Study Research - Design and Methods* (3rd ed.). United States of America: SAGE.

CONCEPTUALISING TRANSLATIONS BETWEEN REPRESENTATIONS

Anneli Dyrvold

Department of Education, Uppsala University, Sweden

Representations and translations between them are central in mathematics education. For example, in the NCTM standards it is emphasized students need to be able to “select, apply, and translate among mathematical representations to solve problems” (NCTM 2000, p.67). A variety of research studies have contributed to the knowledge about translations the last decades. This variety is both an asset and an obstacle when this research is used to implement new strategies in the school practice or as a base to plan new research studies. To enable an accumulation of the emerging knowledge there is a need to categorize studies that focus on similar questions and that conceptualizes translation similarly. The current paper suggests some classifications that such a categorization can be based on in an emerging framework.

INTRODUCTION

In school mathematics students can be engaged in translations between representations, between modes, or between semiotic resources. The translation can be part of a solution process or the task given in a test. In short, *translation* can refer to different processes in different contexts and this comprehensive use of the term calls a need to clarify. The purpose of the current paper is to present an emerging framework for the study of a concept used with slightly different meanings: translations. Such a framework is useful both as a structure to categorise previous research and when planning studies about translations. It is suggested future studies are more explicit about how, where, and why the translation of interest is conducted. The paper as a whole contributes to the research field on translations by highlighting some important concepts, by incorporating those concepts in structure useful as a backbone to a framework on translations, and by relating the structure to Pierce’s theory on signs.

TERMINOLOGY

Representations

The term *representation* is used in a variety of ways and the conceptualisation of the term contributes to what is meant by translations between representations. Von Glasersfeld (1987) argues the term representation is fuzzy. He gives four distinct different meanings of the term. Representation can refer to i) depicting something like a flower, ii) to mentally represent something, iii) to denote an unknown quantity, and iv) for a person to replace someone. The first three are representations that can be translated and the distinction made is therefore important for how a particular translation is conceptualized. A difference between i) and ii) is that in the first case there is something

existing that are represented but in the second case there is no prior object that is re-presented. In iii) the representation is pure mental. It applies to the use of a representation such as x as a substitute for something, a common usage in mathematics texts. It is obvious the translations are of different types depending on if a representation is a depiction of a visual object (i) or an abstract substitution for something (iii). Besides this difference the representation can be depicted in different ways something that also affect how a particular translation can be understood. The next section addresses this issue of kinds of representations.

Semiotic resources

In mathematics education research one way to distinguish between types of semiotic resources used in printed text is the divide between natural language, images, and mathematical notation (see e.g., O'Halloran, 2008; Dyrvold, 2016). The different semiotic resources are often interwoven in printed text, for example in images with words and mathematical notation. Therefore referring to these different parts of a printed text as representations may obscure the intricacy in how they are composed, and thus, the distinction between different semiotic resources is useful. Natural language consists of letters that constitute the words and sentences. For images the function of the 'marks' are of a much bigger variety. A line in an image can represent a similar line that exists in reality, such as the horizon, but the line can also represent a continuous pattern of dots representing $y=x$. For mathematical notation there is also a large variety in how the 'marks' shall be read. Pimm (1987) distinguishes four types of symbols used as mathematical notation: logograms that represent whole concepts (e.g., π and \div), pictograms (e.g., \parallel and \angle), letters (e.g., μ and β), and punctuation marks (e.g., $!$ and $]$). Letters, punctuation marks and logograms, are all truly symbolic in that they are arbitrarily chosen and there is a necessary agreement that they shall stand for something. Pictograms on the other hand are iconic since they refer to another item by means of sensory motor similarity (see von Glasersfeld, 1987) something that is not the case with non-iconic signs.

The differences between representations in the group *images* as well as *mathematical notation* is worth to consider in research about translations in mathematics since the variety in for example how an image is represented means the translations from/to images are of different types. A translation between a mathematical expression with an iconic sign such as \angle and a pictorial image of the angle of a soffit is a translation between representations that partly uses the same means to represent. Therefore the translation can be perceived as less pervading than if the representations differ more.

Multimodality and multisemiotics

Two concepts of importance for a particular use of the term translations are multimodality and multisemiotics. Selander and Kress (2010) explain the communication is *multimodal* when different modes are used in communication. The variety of resources that make the communication multimodal can for example be objects, gestures, words

and symbols. That is, different types of media *and* signs. O'Halloran (2008) addresses the problem of terminology in studies of multimodality. She emphasizes that the use of the terms *mode* and *semiotic* causes confusion, especially in the terms multisemiotic and multimodal. O'Halloran suggests the term *mode* is used for the channel used to represent, for example auditory, visual or tactile. Medium on the other hand may be used to refer to the material resources for the channel, for example a radio or a newspaper. *Semiotic* is used to refer to the semiotic resources that have unique grammatical systems through which they are organized, for example visual images, natural language, and mathematical notation (O'Halloran, 2008). With these distinctions *multisemiotic* is used for texts with more than one semiotic resource and *multimodal* for discourses with more than one mode of semiosis. A website can be both multimodal (visual and auditory) and multisemiotic (natural language and visual images) but the medium is one, the screen.

In research about translations the conceptualisation of a particular communicative act or representation as multisemiotic or multimodal can have big impact on what translation means in that context. Since the terms multimodal and multisemiotic is used in such a variety of ways, knowing that for example a classroom study focus on multimodal translation does not tell exactly what is studied. It is important when a study is reported and for readers to be aware of the need to communicate and to carefully read what the translation actually concerns.

Summing up about terminology

The request to translate can be communicated through different media in different modes and a particular representation can consist of several different semiotic resources. Therefore the term *translation* can refer to a plethora of different communicative acts. If essential terms are clearly defined more is revealed about the translation but as part of an emerging framework on translation is valuable also to focus on *where* something is translated, *how* the translation emerges and *why* something is translated. These issues are addressed in next section.

TRANSLATIONS

In mathematics a concept, a mathematical object, or a process can often be represented in different ways. Janvier, Girardon, and Morand (1993) suggest it is in such cases translations between representations are possible. The avoidance to define translation in an exhaustive way in the current paper is deliberate since some of the issues raised here correspond to different definitions of translation. A broad definition of translation is Kress' (2010) description of translation as a shift in meaning that can occur for example between modes, genres or cultures. In many occasions however, it is of necessity to distinguish exactly what kind of translation that is of interest and where the potential boundaries between different types of translations are.

Where

One issue that distinguishes translations is *where* the translation takes place. This

question depends on how translations are defined. Both Kress (2010) and Duval (2006) do from distinct different theoretical perspectives separate between different types of translations. Kress suggests the term *transformation* for translations without change in mode. One mode can for example be words and another images. A translation between different images is with Kress terminology a transformation, but if the translation occur between an image and words Kress would categorize it as transduction. A similar division is made by Duval (2006). This divide is relevant since images, mathematical notation, and natural language differ in grammar and syntax and representations such as gestures have other means that in many ways differ from those that occur in print. In addition there is representations that is visualised on computer screens, which have the means of movement and 3D.

The grammar and syntax of natural language and mathematical notation are similar in some issues; for example the decoding from left to right, and the use of parentheses to signal order. A 3D image on screen and an image in print also share parts of the grammar for how it shall be interpreted. Taking that into account, it seems reasonable to assume that a translation between natural language in print and a 3D object on screen entails a larger change than for example a translation between natural language and mathematical notation, both in print. Thus, referring to both as just translations obscure how different the two are.

How

Another issue in relation to translations is the question whether any active reformulation take place, the question of *how* the translation is present. The question posed is whether the translation/s addressed is something requested by the representations in the act of interpretation or whether an active agent perform the translation when a new representation is created. This variety is also, implicitly, evident from Kress' description of translation, in the use of both 'shift in meaning' and 'meaning is moved'. The former indicate existence, and the latter some action (Kress, 2010). 'Meaning is moved' indicate that meaning is actively moved between representations whereas 'shift in meaning' may refer to what the text offer. This distinction between active/passive translations is not posed by Kress, presumably since in his view the text is partly created by the reader in the reading process.

The point here is what this issue may mean for the phenomenon that is studied. It is possible to distinguish two different foci when the issue of *how* is considered: translations can be i) performed by the reader when offered by the representations, or ii) performed by the reader when a representation that translates meaning posed in another representation is *created*. The difference is noteworthy because there are different things that are studied with these different conceptualisations of translations. The first type of translations is focused in many studies on the relation between a function and its graph (e.g., Zazkis, Liljedahl, & Gadovsky, 2003). In such studies the phenomena studied is the translation process as part of the reading. In the other type of studies students actively construct translated representations. A difference between these two

types is also the object of study. In studies on translations as part of the reading process the translation is done mentally, compared to when a representation is constructed and therefore possible to visually evaluate.

Why

At last, there is also a question of *why* a translation is conducted. This issue is related to the question of how; that is, which kind of activity does the student engage in. The question of why regards an agent who does the translation but this issue regard *why* the translation is needed. This issue is significant in relation to what is actually studied since being asked to translate between representations evokes a reading behaviour that is potentially different than if the translation is needed in the reading of a text but that is not explicitly said. Several studies on translations analyses success rate for particular groups of participants who are *asked to* translate between representations (e.g., Capraro & Joffrin, 2006). If the tasks used are of that type the issue studied is whether students are able to correctly translate when asked.

If not evoked by a request to translate, translations may be conducted as part of the solution process. Translations can for example be needed in the interpretation of the mathematics. The answer to *why* a translation is conducted may also be that the solution of a problem demands a translation. Consider the task *Leo and Ben started reading the same book on Monday. Leo read four pages a day and Ben read nine pages a day. What page was Ben on when Leo was on page one hundred?*. In the solution of such a task, the presentation of the solution would reasonably utilize mathematical notation and therefore the solver translates the problem. In addition, the more complex a task is the more likely it is that several different translations are needed. Since mathematical problems often are presented using several representations and several semiotic resources, the interpretation as well as the solution of a task may result in several translations done by the solver. Part of the competence in problem solving is actually the ability to choose the most apt representation in the solution.

The backbone to a framework on translations: a diagram

The concepts and issues in relation to translations in mathematics mentioned previously are related in a complex pattern. A reminder of the main points and how they are related is given in a diagram (Figure 1). The arrows represent a path of choices. The first choice regard representations and after that each issue concern types of translations (TR). One important point is that it is impossible to study only internal representations, the possible first choice is whether to include *also* mental representations as part of the object of study, something that will be discussed further in relation to theory on signs. Every new step in the diagram represent two options or more, which means that if every step is taken into account the number of different types of translations are huge. In some instances the choice can also be to include several of the options. In addition, there are other perspectives that are of importance for studies on translations; for example the choice of theoretical perspective.

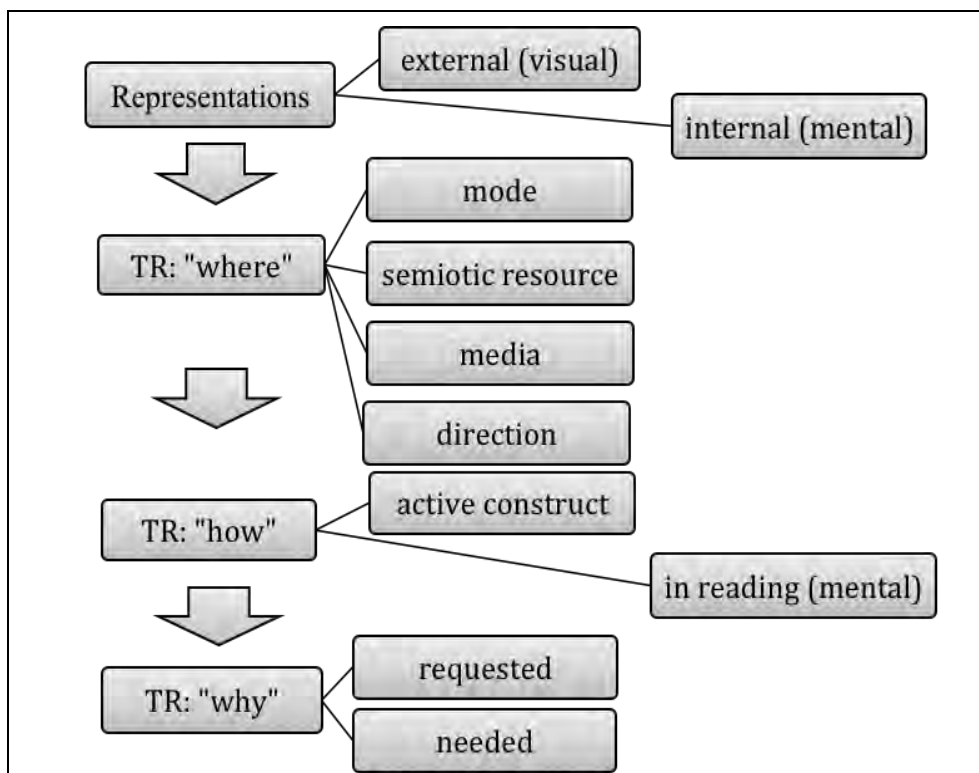


Figure 1: From representations to issues about types of translations.

In the remaining part of the paper Pierce's theory of signs will be discussed to illustrate the interrelation between theoretical issues and methodical choices. In two instances in Figure 1 a focus on mental representations and processes is emphasized. These options are moved to the rightmost part of the figure to illustrate the relation to a choice of theoretical emphasis; the rightmost vertex of the angle in Figure 2.

Influence from theory on the object of study in research on translations

One essential ingredient in translations is the sign, in its most broad definition, and therefore this theoretical discussion takes its starting point in Pierce's definition of sign. In this paper the italicized '*sign*' is used to refer to Pierce's comprehensive notion of sign, which includes three sign components. The first component, the representamen, is what we usually mean by the word "sign" for a symbol as π or an image. An important issue within this theory of signs is the relation between the three components. The representamen exist in relation with the second component, the semiotic object (or semiotically real object). This object is called the *semiotic* object since our perception of it is never absolute; it is what the sign stands for. The third component of the sign is the *interpretant*. The interpretant can be understood as the meaning of the sign and it is related to the other two components. Merrell (2001) argues the interpretant "relates to and mediates between the representamen and the semiotic object in such a way as to bring about an interrelation between them at the same time and in the same way that it brings itself into interrelation with them" (p.28). Every component acts as an intermediary between the other two, and thus there is no '*sign*' if any of the three components is missing since they all are dependent of the other two. This interdependence is visualized in Figure 2. For example, an object such as a *protractor* ex-

ists, but the semiotic object *protractor* is dependent of the interpretant (the agreed meaning) and the representamen (any representation) of a protractor. The complex action or process where the relations between the components are established is called semiosis, also visualised in Figure 2. If we consider that in translations several processes of semiosis are, at least partially, simultaneously present the complexity of translations is very apparent.

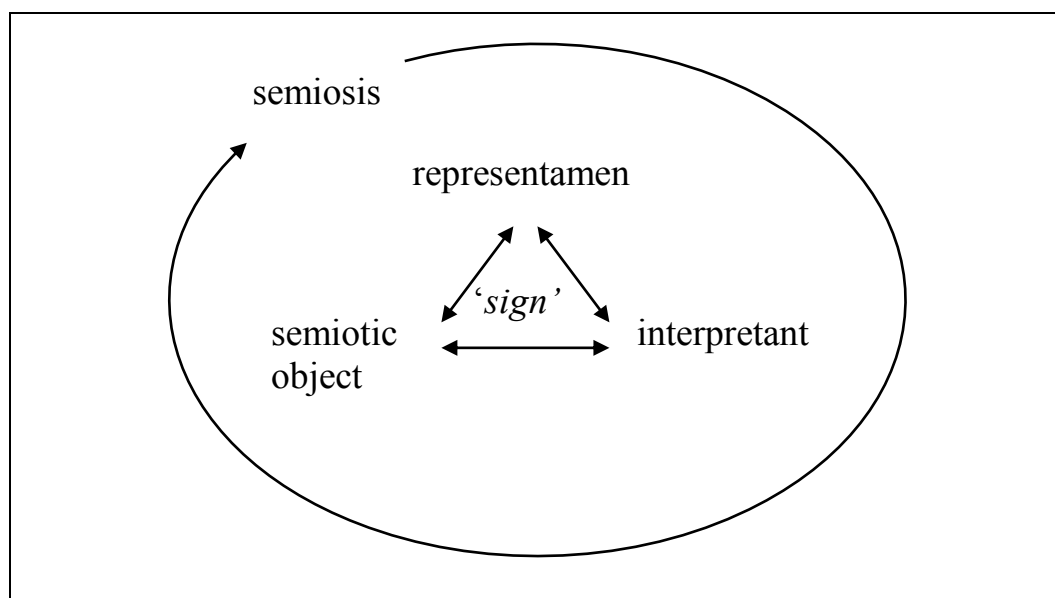


Figure 2: Pierce's model of the 'sign' and the process of semiosis.

In the diagram (Figure 1) choices leading to an emphasis on mental issues are deliberately made visible. We cannot study thoughts, but in research (e.g. about translations) we may concentrate on visual data or on processes of comprehension; for example by analyses of utterances made by the solver. It is useful to reflect over the model of the 'sign' (Figure 2) in relation to this difference. Essential is of course that all three sign components are needed because of the interdependence between them, but there is a difference in how pronounced the different sign components are in studies on translations. For example in a study on student's constructions of solutions requested in a particular semiotic resource (translated), an emphasis on the accuracy of the representation corresponds to a pronounced interest in the representamen. If the emphasis in the study instead is on the student's construction of understanding in relation to their translated representation this indicates a more pronounced interest in the interpretant. All sign components are in play in both studies but it is valuable, both when reading research and when studies are designed, to evaluate the research interest in relation to the sign components.

The perspective of context is also important in relation to studies on translations, in particular in relation to the sign component *interpretant*. Representations in mathematics may mistakenly be seen as definite and exact but as in all semiotic processes 'signs' are construed and re-constituted. This aspect is taken into account by Steinbring (2006) whose epistemological triangle can contribute with the perspective of context to a framework on translations.

In summary, representations and relations between them are important to focus on in teaching. Engaging in relations between representations in mathematics means engagement in translations that can differ substantially in what they demand from the student. Therefore it is necessary to implement knowledge gained from research on translations in schools, and also to further study different aspects of translations. A framework on translations would contribute to such a development since it provides a structure that facilitates an accumulative development of the collected results. Hopefully the emerging framework and the issues stressed in the current paper can contribute to a nuanced discussion about the concept translations, and to enlighten some aspects of importance in relation to the developing field.

References

- Capraro, M., & Joffrion, H. (2006). Algebraic equations: Can middle-school students meaningfully translate from words to mathematical symbols? *Reading Psychology*, 27 (2-3), 147–164.
- Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61 (1-2), 103–131.
- Dyrvold, A. (2016). The role of semiotic resources when reading and solving mathematics tasks. *Nordisk matematikdidaktikk*, 21(3), 51–72.
- Janvier, C., Girardon, C., & Morand, J. C. (1993). Mathematical symbols and representations. In P. S. Wilson (Ed.). *Research ideas for the classroom: High school mathematics* (pp. 79–102). New York: Macmillan.
- Kress, G. (2010). *Multimodality: A social semiotic approach to contemporary communication*. Milton Park, Abingdon, Oxon: Routledge.
- Merrell, F. (2001) Charles Sanders Pierce's concept of the sign. In P. Cobley (Ed). *The Routledge Companion to Semiotics and Linguistics*. New York: Routledge.
- NCTM. (2000). *Principles and standards for school mathematics*. Reston, VA, USA: National Council of Teachers of Mathematics.
- O'Halloran, K. (2008). *Mathematical Discourse: Language, symbolism and visual images*. London: Continuum.
- Pimm, D. (1987). *Speaking Mathematically: Communication in mathematics classrooms*. London: Routledge Kegan & Paul.
- Selander, S., & Kress, G. (2010). *Design för lärande: Ett multimodalt perspektiv*. Lund: Studentlitteratur.
- Steinbring, H. (2006). What makes a sign a mathematical sign? – An epistemological perspective on mathematical interaction. *Educational Studies in Mathematics*, 61, 133–162.
- Von Glasersfeld, E. (1987). Preliminaries to any theory of representation. In C. Janvier (Ed.). *Problems of representation in the teaching and learning of mathematics*. Hillsdale, New Jersey: Lawrence Erlbaum Associates.
- Zazkis, R., Liljedahl, P., & Gadowsky, K. (2003). Conceptions of function translation: Obstacles, intuitions, and rerouting. *Journal of Mathematical Behavior*, 22(4), 435–448.

STUDENTS' CONCEPTIONS FOR CURVE LENGTH: A HYPOTHETICAL LEARNING TRAJECTORY APPROACH

Cheryl L. Eames

Southern Illinois University Edwardsville

The purpose of this paper is to report on a larger program of research, aimed at extending and evaluating a hypothetical learning trajectory for length measurement. Prior research focused on young children's conceptions related to unit for straight or rectilinear path tasks. The present study increased the scope of this research by including middle and secondary level students conceptions related to unit for curved paths. Two individual task-based interviews were conducted with 16 participants. Codes related to students' strategies related to unit were generated through a constant comparative method and the frequency of each code was tracked to explore developmental patterns and inform extensions to the hypothetical learning trajectory.

INTRODUCTION

The hypothetical learning trajectory (LT) approach has emerged as promising program of research for investigating and supporting children's understanding length measurement (Clements et al., 2017). Within this approach, mathematics educators have primarily focused on the big ideas of length measurement in elementary mathematics: establishing a correspondence between a unit and an object to be measured, equal partitioning, a relationship between the size and number of units, the need for identical units, iteration of same-size units, accumulation of distance, and an understanding of the zero point on the ruler (Lehrer, 2003). The consensus of this body of research is that children develop these concepts over time (see Clements et al., 2017; Curry, Mitchellmore, & Outhred, 2006). With a primary focus on elementary children, only a scant body of literature exists to inform hypotheses about how children's knowledge of length measurement grow beyond elementary school as students encounter new big ideas in geometric measurement in middle and secondary mathematics.

Osborne (1976) outlined four problems of length and distance: (a) comparing lengths of segments on two different lines, (b) measuring lengths of bent paths, (c) finding the shortest distance between two points, and (d) determining the length of a curve. Osborne claimed that determining the length of a curve "is a step beyond school mathematics" because "the solution depends on limit processes, the additivity property, extended to allow for adding an infinite number of segments" (p. 24). However, using informal limit arguments is an approach to measurement in secondary mathematics that has been recommended in the United States as part of the Common Core State Standards for Mathematics (Common Core State Standards Initiative, 2010), and determining the lengths of a curve is a curricular goal in many countries at the middle

level. This suggests that research on students' conceptions for determining the length of a curve is overdue (Grugnetti, Rizza, & Marchini, 2007).

Given the rich research basis of the LT approach to investigating children's conceptions for length measurement, it makes sense to ground the exploration of children's conceptions for determining curve length in an existing LT framework for length measurement (Clements et al., 2017). The present study seeks to address the following question: How do children's strategies for comparing curvilinear paths by length change across levels of an LT for length measurement?

THEORETICAL FRAMEWORK

The LT for length measurement in this study is viewed from the lens of hierarchic interactionism (Clements et al., 2017). This LT originates from an assumption of HI, which postulates that children progress through domain-specific levels of understandings in ways that can be characterized by concepts and processes that build hierarchically on previous levels. The LT for length measurement describes a "sequence of knowledge about quantity, based on ratio between a unit and the measured object, and other measured lengths as ratios" (Barrett et al., 2012, p. 51). Table 1 summarizes the levels relevant to this study.

Level	Summary of Operating Characteristics
Consistent Length Measurer (CLM)	possess integrated counting schemes that allow for the concurrent iteration of a unit and subdivision of the unit
Conceptual Ruler Measurer (CRM)	Children have an "internal" measurement tool (e.g., employ explicit strategies to estimate lengths, such as mentally iterating internal units or partitioning a length into equal-length parts
Integrated Conceptual Path Measurer (ICPM)	when reflecting on the measure of a bent path or the perimeter of a polygon, regard a group of units as a flexible object, a "string" of units wrapped along the entire path; begin to coordinate other measures with linear measures, such as curve, and construct smaller units to increase precision
Abstract Length Measurer (ALM)	Children have developed a continuous sense of length and engage in dynamic imagery to coordinate and operate internally on collections of units of units

Table 1: Relevant levels of the LT for length measurement (Clements et al., 2017).

METHODOLOGY

Data were collected as part of a dissertation study on children's conceptions of length measurement (Eames, 2014). The design of this study was informed by methods used in previous research that was focused on extending LTs for length, area, and volume measurement (see Beck, Eames, Cullen, Barrett, Clements, & Sarama, 2014; Kara, 2013). Key elements of this method include a) tasks that reveal student thinking for an

aspect not addressed in the LT, b) sample of students that include some students at the same LT levels and some at adjacent LT levels, c) describing and differentiating students' responses to each task, and d) comparing the strategies of students within the same LT level and across adjacent LT levels to inform extensions to the LT.

Procedures

A written length LT-based assessment was administered to a sample of 71 children in Grades, 4, 6, 8, and 10 to identify a sample that includes some students at the same LT levels and some at adjacent LT levels. Based on the results of this assessment, a sample of 16 participants – four each in Grades 4, 6, 8, and 10 – was recruited for two semi-structured task-based interviews. Four LT level groups were formed, each consisting of four students: CLM group (all Grade in 4), CRM group (3 in Grade 6 and 1 in Grade 8), ICPM group (1 in Grade 6 and 3 in Grade 8), and the ALM group (all in Grade 10). Interviews were video recorded to capture students' gestures as well as verbal and written responses to five tasks.

Tasks

Three tasks involved comparing a single curve to a straight object (Figure 1). Students were provided with an image of a curve printed on a standard piece of paper, a 4-in. stick, and a pen. Students were asked to compare the length of the curved path and the stick. Students were not told that the length of the stick was 4-in.

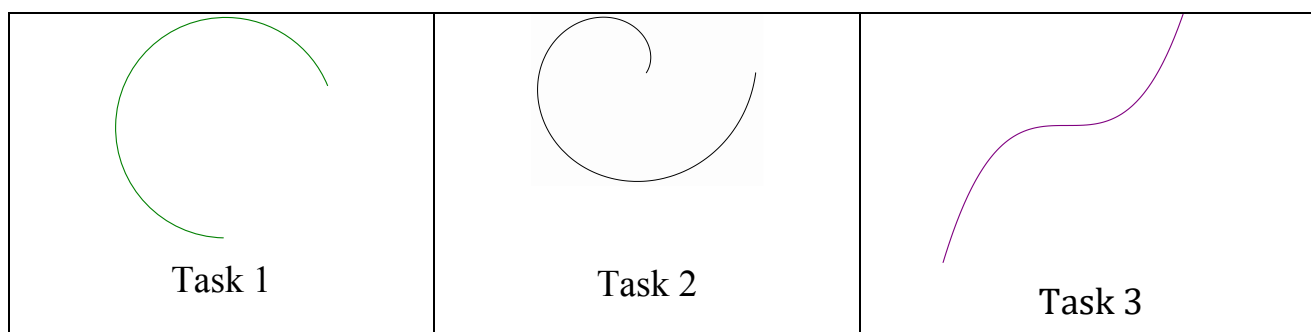


Figure 1: Curves for tasks involving comparing a curved path to a straight object.

Two involved indirectly comparing curved paths using a straight object (Figure 2). Each curve was presented on a standard piece of paper. Students were first asked to compare the curves by length without tools. Next, students were given the same 4-in. stick and a pen, and were then asked to use the stick to help them check.

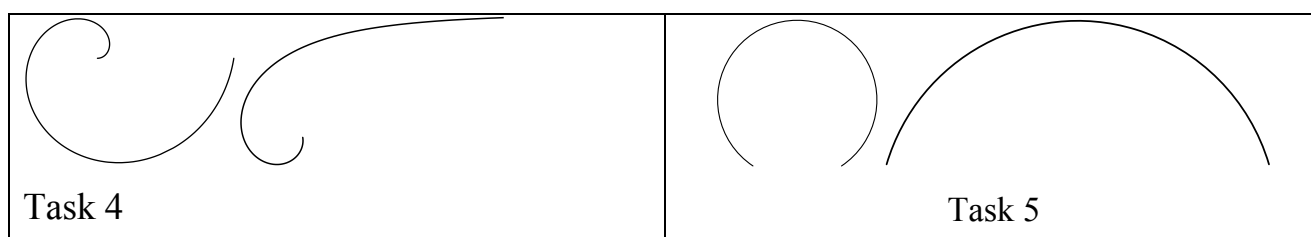


Figure 2. Curves for tasks involving indirect comparisons using a straight object.

For all tasks, the interview protocol included a series of pre-planned follow-up questions designed to elicit students' conceptions for their comparisons.

Analysis

Analysis proceeded according to two phases. Phase 1 involved developing and applying codes for qualitatively different strategies related to unit that reoccurred with regularity. Phase 2 involved tracking the instances of each code for the groups of students who represented the various length LT levels. Developmental patterns across these groups of participants were then examined.

RESULTS AND DISCUSSION: PHASE 1

Strategies for fracturing units. Four strategies related to the ways in which students fractured units (see Figure 3).

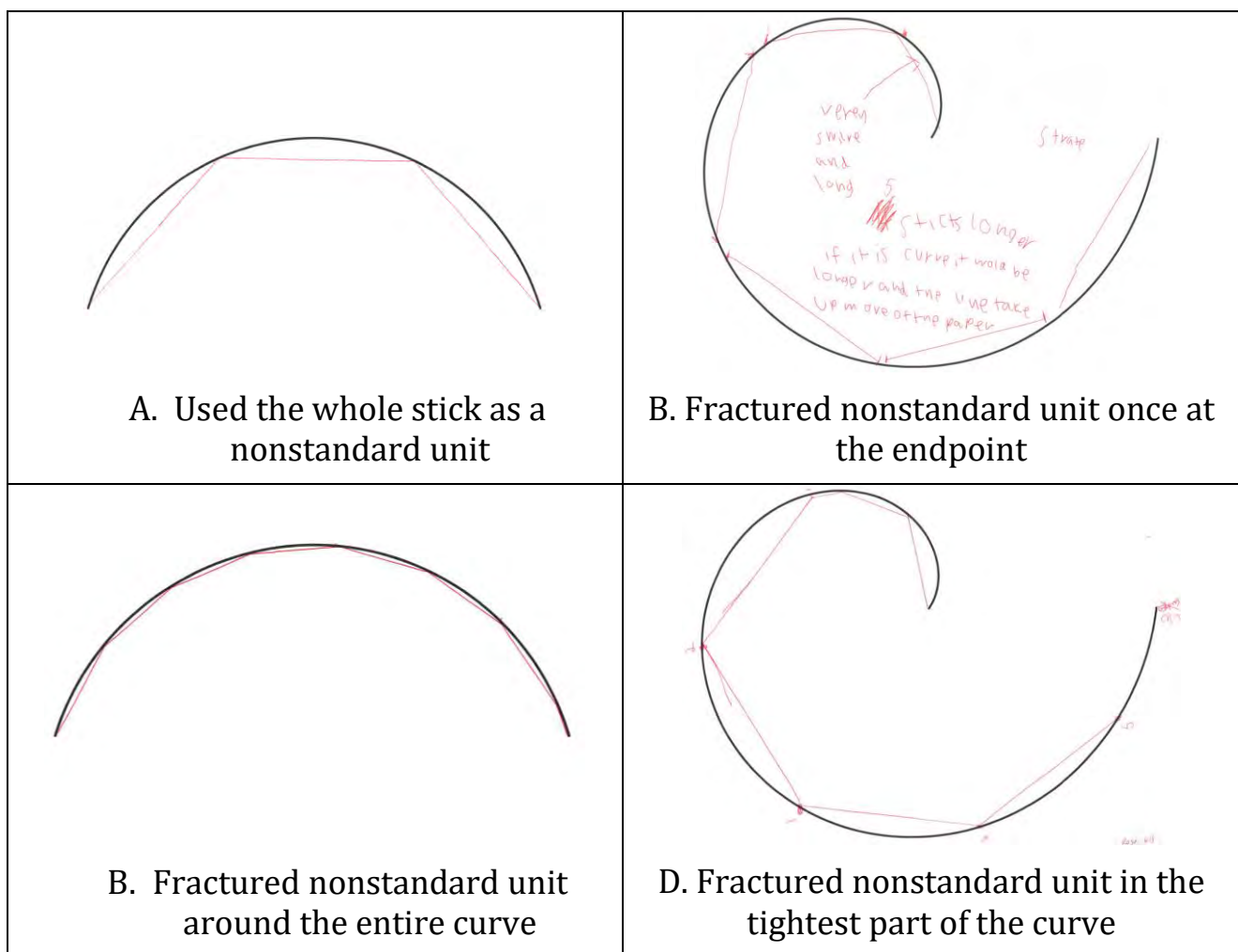


Figure 3. Strategies related to fracturing units.

Counted a partial unit as a whole. Students' counting strategies did not necessarily match their comparison strategies. For example, Kevin (see Figure 3b above) fractured the nonstandard unit once at the endpoint; however, he the final partial segment in the tightest part of the curve as a whole.

Compensated for curvature. The strategy of compensating for curvature was observed as students explained their thinking after directly comparing nonstandard unit to a curve or indirectly comparing two curves using a nonstandard unit. For example, after fracturing the non-standard unit in the tightest part of the curve Marie (Grade 10, ALM) said “OK. So, I when I measured it I got like six and three quarters it looks like. But, again, since it would be pulled up I guess it would be around seven or eight to cover. It would be a little bit more since the curves like here would pull a little bit more in some spots than others.” Marie rounded the number of stick units needed to span the length of the curve from six and three quarters. That is, she compensated for curvature.

Applied mental units. Trent (Grade 6, CRM) used a mental unit strategy by applying a benchmark in Task 3. He first placed the stick at the 8.5-in. side of the paper. Next, he drew a tick mark to represent the end of the stick and iterated the stick, aligning the end of the stick with this tick mark and drawing another tick mark at the end of the stick. When asked what he was thinking he said, “an average sheet of computer paper's about eight and a half inches long, so this took about two...two times it would be about four and a half inches.” Although Trent’s calculation of half of eight and a half as four and a half was incorrect, he remembered the length of a standard piece of paper in inches and used this to determine the length of the stick in inches. For him, the length of the short side of a standard sheet of computer paper as eight and a half inches was a benchmark.

Mental transformation of the unit or curve. Ned (Grade 6, CRM) exhibited a curved unit iteration strategy in Task 4. He first placed the stick on the outside of the curve as a tangent aligned with one end of the curve. Next, Ned drew a tick mark at the end of this first stick unit interval and realigned the stick with this tick mark, placing it again as a tangent. He then drew a second tick mark to indicate the end of this second stick unit interval. For the third and fourth iterations of the stick along the tightest part of the curve, he placed the stick as a tangent, aligned with the tick mark representing the end of the previous stick unit interval and allowed the stick to extend beyond the curve. He then drew a tick mark further along the curve than the point at which the stick departed from the curve. A small part of the curve was still extending beyond the fourth iteration. Ned placed part of the stick along this small part of the curve and wrote four and one third sticks. He then explained how he thought about using the stick to help him check saying, “I laid the stick by trying to line it up as...at about as straight as it can go (laid the stick as a tangent on the outside of the spiral-shaped curve) against the line and then...I figured out since it was curving, I would try to straighten it out and then figure out about where it would be if it was straight.” Ned’s explanation of “figure out about where it would be if it was straight” indicates that he was imagining mentally straightening parts of the curve, at least for the third and fourth tangent stick unit iterations.

Other students exhibited the curved unit iteration strategy by describing a way of mentally straightening segments of the curve to match the straight unit.

RESULTS AND DISCUSSION: PHASE 2

I tracked patterns of the strategies related to unit described above both within and across LT groups representing the CLM, CRM, ICPM, and ALM levels of the length LT. The relative frequency of each strategy is shown in Figure 6 below.

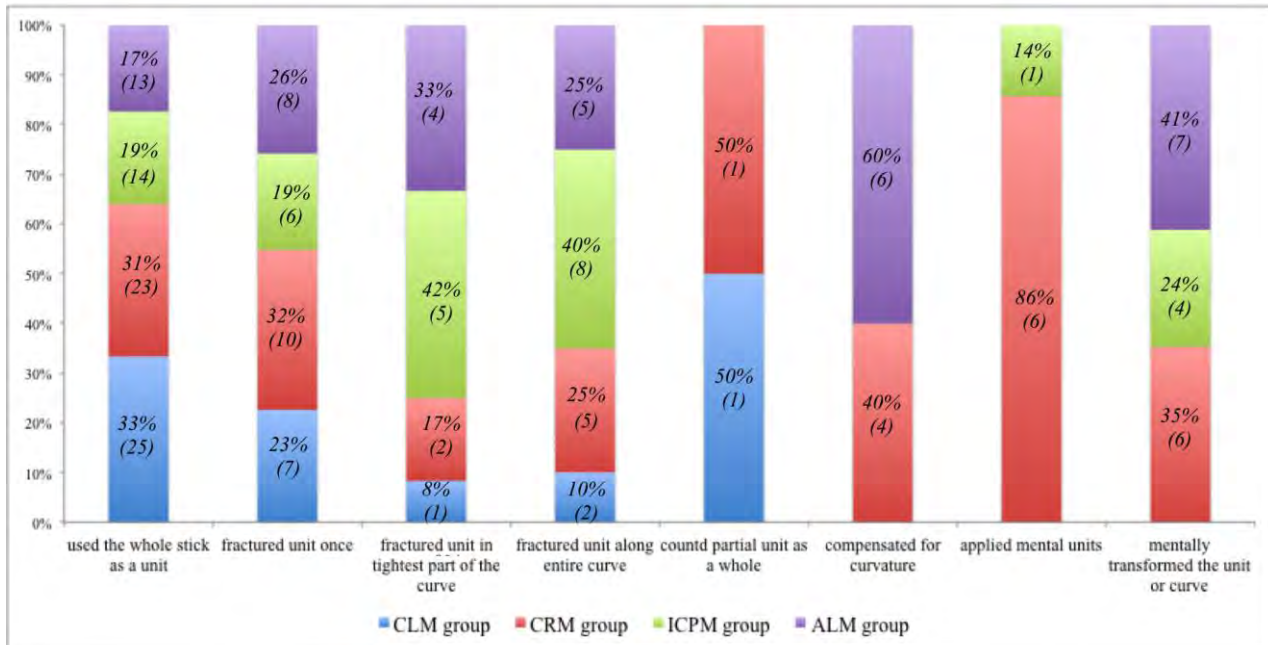


Figure 6. Relative frequency of strategies related to units

Figure 6 indicates that the instances of the strategy of using the whole stick as the unit occurred most often within the lowest level group included in the study, the CLM group, with 25 occurrences. Figure 6 also illustrates a trend of decreasing instances of using the whole stick as the unit as the level groups increased in sophistication. Overall, the fewest instances of fracturing units occurred within the CLM group, and there exists an overall trend of increasing instances of fracturing units as the level groups increase in sophistication. Within the CLM and CRM groups, participants mainly fractured units when a whole unit could not fit at the end of the curve. By the ICPM and ALM levels, participants exhibited more instances of fracturing units in the tightest part of the curve and along the entire curve. The strategy of fracturing units in the tightest part of the curve or along the entire curve constitutes evidence of coordinating linear extent with curve. Therefore, this suggests that, by the ICPM level, students coordinated linear extent with curve. Counting partial units as whole units occurred only in the CLM and CRM groups, whereas the application of mental units and mentally transforming the curve or the unit occurred only within the CRM, ICPM, and ALM level groups. The strategy of comparing a curve and a straight object by applying mental units (either a benchmark or a conceptual standard unit) occurred most often at the CRM level.

CONCLUSIONS

Results reported here confirm some of the conjectured concepts and processes outlined at different levels of the LT (Clements et al., 2017). For example, it was conjectured that CLM level students possessed integrated counting and iterating schemes that allow for the concurrent iteration of a unit and subdivision of the unit. This was confirmed as CLM level students typically exhibited instances of operating with a combination of units and parts of units when measuring a curve with a nonstandard unit. At the CRM level of the LT for length measurement, it was hypothesized that students mentally partition lengths by projecting a mental unit, a ruler, or a sequence of units onto an unpartitioned object. This was supported by the results of the present study; the highest frequency of the appearance of the application of mental units occurred within the CRM level group. At the ICPM level of the length LT, it was conjectured that students would coordinate other measures with linear measures, such as angle, curvature, or time. In the present study, the ICPM level group exhibited increased instances of fracturing the unit in the tightest part of the curve and fracturing the unit along the entire curve, showing evidence of coordinating linear measures with curvature. Finally, at the ALM level of the length LT, it was hypothesized that students had developed a continuous sense of length. This was confirmed by the results of the present study as the participants within the ALM level group increasingly relied on mentally transforming rectilinear or curvilinear paths into the same shape for the purpose of comparing by length.

IMPLICATIONS

Osborne (1976) noted that determining the length of a curve “is a step beyond most school mathematics” (p. 24) because the solution involves limit processes, or the additivity principle extended to allow for the addition of infinitely many segments. However, the results reported here indicate that measurement tasks involving determining the length of a curve have potential instructional value for eliciting and discussing measurement from a mathematical perspective (Osborne, 1976) using informal limit arguments, an approach that has been recommended for secondary students in the Common Core State Standards for Mathematics (Common Core State Standards Initiative, 2010). In the present study, when measuring a curve with a nonstandard unit, participants exhibited 20 instances of fracturing the nonstandard unit around the entire curve (Figure 6). These instances occurred most often in Grades 6, 8, and 10. This suggests that by middle school, in an instructional setting, students may be ready to use and make sense of informal limit arguments by discussing processes in which a curve is represented by increasingly large numbers of segments of decreasing lengths to decrease the error in measuring and approach a true length of the curve. Future research is needed to explore the instructional affordances of the tasks presented here.

References

- Barrett, J. E., Sarama, J., Clements, D. H., Cullen, C., McCool, J., Witkowski-Rumsey, C., & Klanderma, D. (2012). Evaluating and improving a learning trajectory for linear measurement in elementary grades. *Mathematical Thinking and Learning*, 14, 28–54.
- Beck, P., Eames, C. L., Cullen, C. J., Barrett, J. E., Clements, D. H., & Sarama, J. (2014). Linking children's knowledge of length measurement to their use of double number lines. In C. Nicol, P. Liljedahl, S. Oesterle, & D. Allan (Eds.), *Proceedings of the 38th Conference of the International Group for the Psychology of Mathematics Education and the 36th Conference of the North American Chapter of the Psychology of Mathematics Education*, Vol. 2. (pp. 105–112). Vancouver, CA: PME.
- Clements, D. H., Barrett, J. E., Sarama, J., Cullen, C. J., Van Dine, D. W., Eames, C. L., Kara, M., Klanderma, D., & Vukovich, M. (2017). Length – A summary report. In J. E. Barrett, D. H. Clements, & J. Sarama (Eds.), *A longitudinal account of children's knowledge of measurement* (JRME Monograph No. XVI). Reston, VA: National Council of Teachers of Mathematics.
- Common Core State Standards Initiative. (2010). *Common core state standards for mathematics*. Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. Retrieved from <http://www.corestandards.org>.
- Curry, M., Mitchelmore, M., & Outhred, L. (2006). Development of children's understanding of length, area and volume measurement principles. *Mathematics in the centre: proceedings of the 30th conference of the International Group for the Psychology of Mathematics Education Charles University, Faculty of Education, Prague, Czech Republic 16-21 July 2006*, 2, 377–384.
- Eames, C. L. (2014). *Investigating children's intuitive and analytical thinking about path length as a developmental phenomenon* (Order No. 3670556). Available from ProQuest Dissertations & Theses Global. (1650641457). Retrieved from <http://libproxy.lib.ilstu.edu/login?url=http://search.proquest.com/docview/1650641457?accountid=11578>.
- Grugnetti L., & Rizza, A., Marchini, C. (2007). A lengthy process for the establishment of the concept of limit starting from pupils' pre-conceptions. *Far East Journal of Mathematics Education*, 1(1), 1–32.
- Kara, M. (2013). *Students' reasoning about invariance of volume as a quantity*. (Order No. 3592420, Illinois State University). ProQuest Dissertations and Theses, 332. Retrieved from <http://search.proquest.com/docview/1436986351?accountid=11578>. (1436986351).
- Lehrer, R. (2003). *Developing understanding of measurement*. In J. Kilpatrick, W. G. Martin, and D. Schifter (Eds.), *A Research Companion to Principles and Standards for School Mathematics* (pp. 179–192). Reston, VA: National Council of Teachers of Mathematics.
- Osborne, A. R. (1976). Mathematical distinctions in the teaching of measure. In D. Nelson (Ed.), *Measurement in school mathematics* (pp. 11–34). Reston, VA: National Council of Teachers of Mathematics.

PREDICTORS OF DEMAND FOR MATHEMATICS SUPPORT

Antony Edwards¹ and David Carroll²

¹Swinburne University of Technology, ²The University of Western Australia

This report analyses usage data for a well-established undergraduate mathematics support centre, exploring the relationship between time of visit, duration of visit and various demographic groupings such as age, gender, residency status, first language and a constructed variable for “at risk of failing”. We found, and measured, surges in demand for mathematics support as summative assessments approached. Visits further in advance of assessments tended to be of a shorter duration, and were more likely to be made by female and older students. Our results are important for those involved in mathematics support, especially in its administration and evaluation, but they are also of interest for anyone wishing to explore or predict times where their own students may require additional support.

INTRODUCTION

It is believed by many policymakers, practitioners and researchers in education that there is a “mathematics problem” of incoming undergraduates being mathematically underprepared for their courses (Savage & Hawkes, 2000; Marr & Grove, 2010). As more subjects become quantitative, the “mathematics problem” has broadened and affects more and more students (Johnson & Blanksby, 2014). Thus, there has been an increase in need for mathematics support provision in European, Australasian and North American universities (Cronin, Cole, Clancy, Breen, & O’Sé, 2016; Grove, Croft, Lawson, & Petrie, 2017).

Institutions promote mathematics support provision in marketing materials, and report on its existence and effects to funding agencies and in response to external and internal teaching evaluations. Evaluation demands effective resourcing of the support offered, and to meet this demand we need to know more about the students who seek support and the types of support requested. Such research is useful not just for practitioners of support; analyses such as the one presented here can offer insights into spontaneous study behaviours that are of use to mathematics lecturers and tutors, as well as the wider research community interested in student engagement and learning behaviour.

Swinburne University of Technology (Melbourne, Australia) mathematics support sees over 2000 student visits each 12-week semester. The support operates on a drop-in basis, with tutors employed Mon-Fri throughout the day to assist students. This paper considers student visitors who were enrolled in first-year undergraduate mathematics courses in Semester 2, 2017. Our analysis initially considers the relationship between attending a mathematics support centre and students’ final exam results. It then goes on to explore predictors of demand for undergraduate mathematics support.

LITERATURE REVIEW

There has been considerable research on the effects of students accessing additional mathematics support during their degree programmes. Quantitative research studies have found that students who visit mathematics support centres are twice as likely to complete their engineering course (Cuthbert & MacGillivray, 2007), obtain results 8% higher than their peers after controlling for the results of a diagnostic test administered on entry (Lee, Harrison, Pell, & Robinson, 2008), and that increases in mathematics test results are more pronounced for students studying an Arts majors compared with those studying a Science major (Mac an Bhaird, Morgan, & O'Shea, 2009).

Less conclusive are studies exploring the behavioural patterns of students who visit. One study found that students who were the most “at risk” of failing tended to attend support services more frequently than their peers (Mac an Bhaird & O'Shea, 2009); but other researchers have found that the majority of users tend to be higher achievers working to improve their grades (Pell & Croft, 2008). Compared with males, female students have been found to be more than twice as likely to seek additional support (Fhloinn, Fitzmaurice, Mac an Bhaird, & O'Sullivan, 2016).

Much less is known about usage patterns. We might expect increases in demand during the semester as course content becomes more difficult. Credit-bearing assessments are related to extrinsic motivation and help-seeking behaviour (Ryan & Pintrich, 1997), so we would expect spikes in demand near assessments. But as far as we are aware no quantitative research has explored this in the context of mathematics support. We do not know if these (or other) trends exist or if they differ amongst different groups of students. Are usage patterns of at-risk students different from those who already expect a good grade? Do male students access mathematics support in a different way to female students? And what trends exist for other demographic groups?

This paper contributes to existing literature by first examining whether the above trends are present for Swinburne University's mathematics support service. It then contributes original research by exploring predictors of demand for support, and considers whether these are consistent across different demographic groups.

METHODS

Participants

The data for this study were collected Semester 2 of the 2017 academic year (July – October, 2017) from cohorts enrolled in four first-year undergraduate mathematics courses ($n = 59, 44, 122$ and 432). Because a whole-cohort approach maximised statistical power and involved negligible risk for participants, we were granted a consent waiver from the University's ethics board to collect and analyse whole-cohort data comprising 657 student-course enrolments. We note that 39 students were enrolled into two concurrent mathematics courses; all statistical tests reported here were also run with these 39 students removed from the analysis with no significant changes in the magnitude or statistical significance of results.

Variables

For each student, we collected the following demographic indicators from the student record database: gender, date of birth, main language spoken at home, and residency status. The final two variables were recoded as 0/1 dummy variables: English/English as a second language (ESL) and Domestic/International, respectively.

Course results were linked to each student-course identifier, including final exam result and assessments during semester. Only summative assessments were considered as data for this paper (e.g. midterms), rather than those where course credit was given for demonstrating mastery of techniques (e.g. a “try as many times as you like” online assessments where the majority of students scored full marks).

For mathematics support usage, data consisted of 498 visit records from 129 first-year mathematics students. These records were generated by students scanning their university card on a reader indicating the course for which they were seeking help. Attached to 385 records was also a scan-out time (optional for students), and so the variable “duration of visit” could be calculated for these records.

Research Questions

The results are split into two parts. The first aims to verify associations identified in the literature (RQ1 and RQ2). The second, to explore and measure time-trend data (RQ3).

- RQ1 Did students who used mathematics support perform better than their peers?
- RQ2 Were at-risk students who sought mathematics support more likely to complete their courses?
- RQ3 What demographic and time-series trends exist in the data that predict demand?

RESULTS

Did students that used mathematics support perform better than their peers?

Treated as a single cohort, students who attended mathematics support performed better than those that did not by 8.5 percentage points, which was a significant result ($t(229.450) = 3.417, p = 0.001$). When we controlled for potential differences across courses by including three course dummies in a linear regression (see Table 1), the effect increases to 9.1 percentage points and remains significant ($p = 0.001$).

	<i>B</i>	<i>SE B</i>	β	<i>p</i>
Constant	0.555	0.038		.000
DV Course 1->2	-0.031	0.059	-.026	.596
DV Course 1->3	-0.020	0.047	-.026	.667
DV Course 1->4	0.131	0.040	.213	.001
Mathematics Support accessed	0.091	0.028	.129	.001

Table 1: Linear regression of final exam mark for all students. Model $R^2 = .072$, $\Delta R^2 = 0.016$ when including the “Mathematics Support accessed” binary variable.

Considering not just attendance, but the number of times a student attended, a separate linear regression with course dummies and the variable “number of instances of support accessed” found that the “value add” of each visit to mathematics support is 1.2 percentage points ($p = 0.006$). Space does not permit the full output of this model to be included here, but it can be obtained from the authors upon request.

Are at-risk students who seek mathematics support more likely to complete their courses?

To identify students at risk (of failing a course), we considered those who had attempted in-semester assessments and scored less than 50% (the cut-off for a fail at this institution). We then coded for the presence of mathematics support visits after the student’s last in-semester assessment in their respective course. Of 256 at-risk students, 7% sought mathematics support, compared with 10% of the 361 students deemed not at-risk. This was not a significant difference ($\chi^2(1) = 1.62, p = .203$).

	<i>B</i>	<i>SE B</i>	β	<i>p</i>
Constant	0.317	0.082		.000
DV Course 1->2	-0.011	0.103	-.011	.916
DV Course 1->3	0.062	0.088	.107	.482
DV Course 1->4	0.182	0.084	.354	.031
Mathematics Support accessed after final assessment	0.114	0.057	.127	.047

Table 2: Linear regression of final exam mark for at-risk students only. Model $R^2 = .086$, $\Delta R^2 = 0.016$ when including the “Mathematics Support” binary variable.

For the at-risk students, the effect associated with seeking mathematics support was large: as reported in Table 2, attending mathematics support was associated with an 11 percentage point difference in final exam mark for these students ($p = .047$) on average, which corresponded to an average difference of 8.1 per visit ($p = .005$). Both of these statistics are from linear regressions with course dummies included. A logistic regression indicated that at-risk students who attended mathematics support were more likely to pass their course ($B = .407$), but this result was not statistically significant ($p = .223$), possibly due to low statistical power given the small sample size.

What demographic and time-series trends exist in the data that predict demand?

We found that female students were significantly more likely to seek mathematics support during the semester compared with male students (a logistic regression coefficient of $B = 0.62, p = 0.008$), but they were not significantly more likely to be at-risk of failing than male students. Older students were also significantly more likely to seek support ($B = .064, p = 0.004$), but not more likely to be at-risk than their younger peers. The duration of visits was practically identical for all students, whether male (108 min)

or female (107 min), irrespective of age ($r_s = -.029$, $p = .764$). Other demographic indicators collected were not related to likelihood of accessing support.

When exploring time-series predictors of demand for mathematics support our analysis began with testing for two trends that, anecdotally, the authors believed to be true. The first hypothesised that demand increases during a semester. This could be due to word-of-mouth between students, taught material becoming more difficult or impending final exams. The second was that surges in support demand are seen close to assessment deadlines. As far as our literature search shows, such trends in undergraduate mathematics support have not been identified, measured or published.

When data were analysed at a per-week level there was no significant increase in demand for support during the semester. A regression line fitted to the weekly frequencies showed a slight upward trend in visits across the course of the semester (2.1 visits per week, on average); however, this trend was not significantly different from zero ($F(1,10) = 1.06$, $p = 0.327$). Although no semester-long trend was found in usage, plotting a histogram did indicate surges in visitor numbers before assessments. We therefore created a variable that, for each visit record, stated the number of business days until the next assessment in a students' course.

As shown in Figure 1, the closer the assessment, the more the demand for mathematics support. As (business) days until the next assessment (x) decreased, instances of support (y) increased, as modelled by the exponential function $y = 27.9e^{-0.089x}$ ($R^2 = .732$, $F(1,34) = 92.9$, $p < 0.001$). This model has a half-life of 7.69 days, i.e. it predicts that for every eight days before an assessment, there is an associated halving in student demand for support.

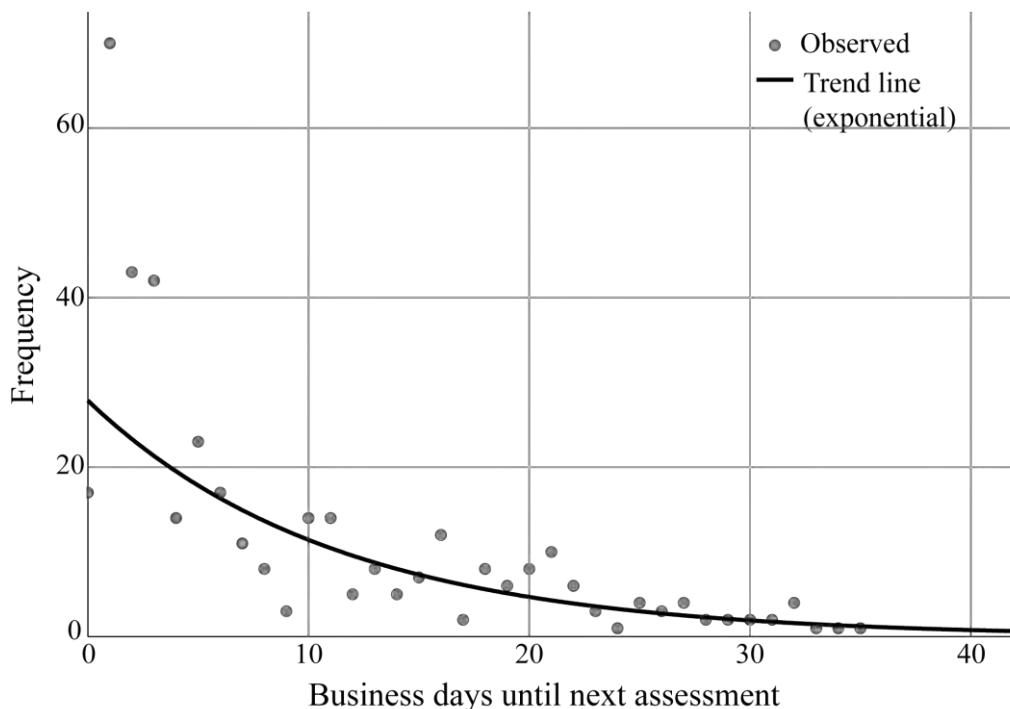


Figure 1: Frequency of visits by proximity to next summative assessment, with the exponential model of best fit drawn.

We found a significant relationship between mean duration of visit (where recorded) and days before an assessment ($r_s = -0.257, p < 0.001$). To explore the effect size of the difference, a chi-square automatic interaction detection (CHAID) was computed to see if the duration variable could be split into categories. This process found a significant difference between the duration of visits that were three business days or closer to an assessment ($n = 221, M = 126$ minutes, $SD = 86.5$) and those four days away or further ($n = 164, M = 84$ minutes, $SD = 55.5$), $F(1,301) = 22.9, p < 0.001$.

To explore differences in behaviour of different demographic groups, we constructed a linear regression with dependent variable “number of business days before an assessment” against available demographic markers, our earlier-constructed “at-risk” indicator, and course dummies. Model coefficients are presented in Table 3.

	<i>B</i>	<i>SE B</i>	β	<i>p</i>
Constant	6.921	2.359		.004
DV Course 1->2	-1.613	2.475	-.048	.515
DV Course 1->3	-0.001	2.296	.000	.998
DV Course 1->4	-4.987	2.078	-.261	.017
Age (years)	0.299	0.051	.293	.000
Gender (0=female, 1=male)	-2.563	0.975	-.128	.009
International (0=domestic, 1=int’)	-2.125	1.526	-.084	.165
Language (0=English, 1=ESL)	0.888	1.161	.046	.445
At-risk (0=not at-risk, 1=at-risk)	0.873	1.003	.046	.384

Table 3: Linear regression exploring how different demographic indicators accessed support in advance of assessments. Positive coefficients correspond to students seeking support further in advance of assessments.

The regression results in Table 3 suggest gender and age are significant predictors of demand relative to assessment date. Earlier, we found that female students were significantly more likely to seek mathematics support during the semester. Here we find that they tend to do so earlier too: female students seek help on average 2.6 business days before male students. Older students were more likely to seek support further in advance of assessments, our model indicating that for every three years older the student, support was sought on average a day earlier.

For the other demographic indicators, differences were seen but were not statistically significant. International students (those on a temporary visa, $n = 68$) on average sought support two days closer to assessments. Those who spoke English as a second language ($n = 141$) tended to visit around a day earlier than those who spoke English at home. We showed earlier that at-risk students who attended support did considerably

better in final exams; they also tended to seek mathematics support on average a day earlier than students who were already passing their in-semester assignments.

DISCUSSION

This paper began by considering if results seen in the literature were present for Swinburne University's mathematics support service. We found that students who sought mathematics support scored on average 9.1 percentage points higher in their final exams and that female and older students were more likely to seek support.

Such results do not take into account any other information about the prior performance of students and so group together at-risk students together with those performing at a higher level and potentially using mathematics support to boost grades. As Pell and Croft (2008) noted, from a funding perspective the emphasis of mathematics support is often on student retention, and so a primary goal is to assisting students at-risk of failing their courses.

Worryingly, our data suggested that at-risk students were unlikely to seek additional mathematics support in the first place. Of 256 at-risk students, only 18 sought support. Such disengagement in mathematics support is sadly commonplace and well-known in the literature (Pell & Cook, 2008; Symonds, Lawson, & Robinson, 2008) and is due to a variety of cognitive and social factors (Grehan, Mac an Bhaird, & O'Shea, 2013), and are part of a wider suite of issues including low attendance and lack of engagement in mathematics more generally (Gill, O'Donoghue, Faulkner, & Hannigan, 2010). Those at-risk students who did seek support scored, on average 11 percentage points higher on their final exam. They were also more likely to pass their courses, but not significantly so.

We also explored visiting behaviour in an attempt to determine predictors of demand. If seeking mathematics support is a proxy for engagement, we have strong evidence that engagement is highest near assessment times, with a halving of engagement approximately each week beforehand. Female students and those who were older were more likely to engage further in advance of assessments. This paper provides quantitative evidence for a behaviour that might be expected; it implies future research opportunities to consider the motivation of these groups of students, potentially with an aim to encourage engagement and visits to mathematics support.

References

- Cuthbert, R. H., & MacGillivray, H. L. (2007). Investigation of completion rates of engineering students. In: *Proceedings of DELTA 07* (pp. 35–41), El Calafate, Argentina.
- Cronin, A., Cole J., Clancy, M., Breen, C., & O'Sé D. (2016) An audit of mathematics learning support provision on the Island of Ireland in 2015. *An Irish Mathematics Learning Support Network Report*. ISBN: 978-1-910963-07-4.

- Fhloinn, E. N., Fitzmaurice, O., Mac an Bhaird, C., & O'Sullivan, C. (2016). Gender differences in the level of engagement with mathematics support in higher education in Ireland. *International Journal of Research in Undergraduate Mathematics Education*, 2(3), 297–317.
- Gill, O., O'Donoghue, J., Faulkner, F., & Hannigan, A. (2010). Trends in performance of science and technology students (1997–2008) in Ireland. *International Journal of Mathematical Education in Science and Technology*, 41(3), 323–339.
- Grehan, M., Mac an Bhaird, C., & O'Shea, A. (2016). Investigating students' levels of engagement with mathematics: critical events, motivations, and influences on behaviour. *International Journal of Mathematical Education in Science and Technology*, 47(1), 1–28.
- Grove, M. J., Croft, T., Lawson, D., & Petrie, M. (2017). Community perspectives of mathematics and statistics support in higher education: building the infrastructure. *Teaching Mathematics and its Applications: An International Journal of the IMA*. Advance online publication. <https://doi.org/10.1093/teamat/hrx014>.
- Jackson, D. C., Johnson, E. D., & Blanksby, T. M. (2014). A Practitioner's Guide to Implementing cross-disciplinary links in a Mathematics Support Program. *International Journal of Innovation in Science and Mathematics Education*, 22(1), 67–83.
- Lee, S., Harrison, M. C., Pell, G., & Robinson, C. L. (2008). Predicting performance of first year engineering students and the importance of assessment tools therein. *Engineering Education*, 3(1), 44–51.
- Mac an Bhaird, C., Morgan, T., & O'Shea, A. (2009). The impact of the mathematics support centre on the grades of first year students at the National University of Ireland Maynooth. *Teaching Mathematics and its Applications: An International Journal of the IMA*, 28(3), 117–122.
- Mac an Bhaird, C., & O'Shea, A. (2009). What type of student avails of mathematics support and why? In *CETL-MSOR Conference 2009* (pp. 48-51), The Maths, Stats & OR Network.
- Marr, C. and Grove, M. (Eds) (2010) *Responding to the Mathematics Problem: The Implementation of Institutional Support Mechanisms*, MSOR Network, Birmingham.
- Pell, G., & Croft, T. (2008). Mathematics support—Support for all? *Teaching Mathematics and its Applications: An International Journal of the IMA*, 27(4), 167–173.
- Ryan, A. M., & Pintrich, P. R. (1997). “Should I ask for help?” The role of motivation and attitudes in adolescents' help seeking in math class. *Journal of Educational Psychology*, 89(2), 329–341.
- Savage, M. D., & Hawkes, T. (2000). *Measuring the Mathematics Problem*, London (UK): Engineering Council.
- Symonds, R., Lawson, D., & Robinson, C. (2008). Promoting student engagement with mathematics support. *Teaching Mathematics and its Applications: An International Journal of the IMA*, 27(3), 140–149.

ONE STUDENT'S DISCURSIVE DEVELOPMENT ON ROTATION IN RELATION TO INSTRUCTION FROM A COMMGNITIVE PERSPECTIVE

Elçin Emre-Akdoğan

Gazi University

Beste Güçler

University of Massachusetts
Dartmouth

Ziya Argün

Gazi University

The aim of this study is to explore one Turkish high school student's discursive development on rotation in relation to his teacher's discourse through a commognitive framework. The data sources for examining the teacher's and student's discourses included classroom observations and task-based interviews. The data was analyzed in terms of participants' word use, visual mediators, routines, and narratives from a commognitive perspective. The results indicated that teacher's discourse was based on an algebraic-formal approach during his instruction. The student imitated the teacher's algebraic approach but he also adjusted the teacher's discourse with his own previously existing discourse, possibly due to a lack of clarity about the reason and logic behind the teacher's discourse.

INTRODUCTION

In recent years, transformation geometry has been emphasized by various curricula (e.g., MoNE, 2010; NCTM, 2000) as a powerful topic that enables students understand congruence and similarity (Seago et. al, 2013). Despite its importance and being considered as more challenging compared to other geometry concepts (e.g. shapes, topology measurement) (Mammarella, Giofrè, Ferrara, & Cornoldi, 2012), researchers argue that there is limited research on transformation geometry (Boulter & Kirby, 1994; Hollebrands, 2003; Xistouri & Pitta-Pantazi, 2011).

Existing literature on geometric transformations focus on issues such as student understanding (e.g., Clements, 2004; Flagan, 2001; Hollebrands 2003, 2004, 2007; Portnoy, et. al, 2006) and pre-service teachers' knowledge and thinking on geometric transformations (e.g., Ada & Kurtulus, 2010; Harper, 2002; Yanik, 2006). Among the concepts of transformation geometry, rotation is considered as challenging by scholars (Ada & Kurtuluş, 2010; Edwards, 1991; Xistouri & Pitta-Pantazi, 2011). Ada & Kurtuluş (2010) found that the students in their study operated with the algebraic meaning of rotations but they did not understand the geometric meaning of them. Researchers noted the position of the center of rotation as a challenging aspect of rotations (Harper, 2002; Yanik, 2006). Hollebrands (2004) found that students have difficulties with rotations if the center of rotation is not on the figure. Students have also been reported to have difficulties while rotating a shape in cases where the pre- and post-images of a rotated shape match (Harper, 2002). Students may not consider that the points on the

pre-image and post-image have equal distances from the center of rotation (Cave, 2008). It is not easy for students to understand the physical connection between the rotated object and the center of rotation (Edwards, 2003). Studies show that students may also consider transformations as procedures instead of mathematical objects (Cave, 2008; Edwards, 1991). Another concept that makes rotation challenging for students is the concept of angle (Clements, 2004; Mitchelmore & White, 2000; Yazgan, Argün, & Emre, 2009). Researchers note that students should think about degree as a measure of angle in order to understand rotation (Harper, 2002; Panorkou & Maloney, 2015). Research findings reveal that students perform better when they use 0, 90, 180, and 270 degrees as angles while rotating (Hollebrands, 2004).

Most of the existing studies on rotation are based on cognitive perspectives and few of them explore student learning in relation to instruction. Further, there is a scarcity of research focusing on rotation at the high school level. In order to examine student learning within its context, utilization of social theories may offer additional insights. Therefore, our aim is to explore one Turkish high school student's discursive development on rotation in relation to his teacher's discourse. In our study we use a commognitive perspective because it highlights the communicational nature of learning and provides us with the analytical tools with which to examine student development and juxtapose it with the teacher's discourse in the classroom. We address the following question: How does one student develop his discourse on rotation in relation to the teacher's discourse in a high school classroom?

THEORATICAL BACKGROUND

Commognitive is a combination of the words of "cognition" and "communication" (Sfard, 2012). As a sociocultural approach, commognitive perspective eliminates the dilemma between thinking and communication by formulating thinking as self-communication (Sfard, 2008). Sfard (2008) defines discourse as a "special type of communication made distinct by its repertoire of admissible actions and the way these actions are paired with re-actions (p. 297)". If the discourse is related to mathematics, it is called mathematical discourse (Sfard, 2008). Sfard (2008) notes that mathematical discourse can be identified through four elements: word use, visual mediators, routines and endorsed narratives. Word use refers to the mathematical words used in the discourse. Visual mediators refer to graphs, diagrams, algebraic notations, figures, and shapes in the discourse. Routines are repetitive patterns that provide us to analyse actions in detail. Endorsed narratives are utterances participants consider as true as substantiated by the other three elements of discourse.

Sfard defines four developmental stages in participants' word use: passive, routine-driven, phrase-driven and object-driven use. Passive use is the stage where participants would not utter the mathematical word in their speech. For instance, students would not utter "rotation" in their word use; they could just say "it", "this" or "that" instead of "rotation". In the stage of routine-driven use, participants use mathematical words only in relation to specific and limited procedures and actions they perform. For

example, when asked about rotation, they may describe all the actions with which they rotate a geometric shape. The next stage is phrase-driven use, where “entire phrases rather than the word as such constitute the basic building blocks” of participants’ utterances (Sfard, 2008, p. 181). At this stage, students may not refer to the actions they perform when rotating a shape but instead use phrases such as “rotation is when we turn a given geometric object” in their discourses. The last stage is object-driven use where participants utter mathematical words as if they refer to objects or end states by using a noun. For example, if students utter “rotation is a geometric transformation”, then they refer to rotation as a specific mathematical object.

Consistent with the commognitive approach, we conceptualize development as change in discourse in this study. Sfard (2012) explains development as a “modification of activity, not as an inner change in the actor (the activity may be public – communication with others – or private – thinking) (p. 2)”. Sfard (2012) suggests that students should be “encouraged to engage in a thoughtful imitation of expert participants’ discursive moves (p. 6)”, where thoughtful imitation refers to understanding the reasons behind the experts’ discourses Sfard (2012).

METHODOLOGY

This is a case study investigating one 10th grade (16-year-old) high school student’s discursive development on rotation in comparison to the teacher’s discourse in a medium-size urban public high school in Turkey. We selected Okan (a pseudonym) as the student for our study, since he was a talkative student who communicated his experiences in an expressive and reflective way and was also willing to participate in the study—a form of purposeful sampling for rich and in-depth data collection (Patton, 2002).

The data sources for examining the teacher’s (Mr. Can, a pseudonym) discourse on rotation included the two video-taped classroom sessions during which he talked about rotation and a video-taped task-based interview that was administered three weeks after the instruction. The purpose of the interview was to gain information about Mr. Can’s discourse on rotation. The interview consisted of four tasks on rotation. In the first task, we asked Mr. Can to give an example for rotation. In the second task, which included geometric and algebraic visual mediators, we asked him to rotate a given rhomboid around the origin. The third task involved geometrically rotating a polygon around a point that was outside the polygon. The last task was based on the identification of the geometric transformation (rotation) that transformed a given triangle to another triangle.

The data sources for examining Okan’s discourse on rotation included three video-taped task-based interviews. Okan was interviewed individually and each interview lasted about 20 minutes. We conducted the first interview before Mr. Can introduced rotation in the classroom. The second interview was conducted right after instruction and the third interview was conducted 20 days after instruction. Each interview consisted of four tasks on rotation, which were mathematically equivalent to those used in

Mr. Can's interview, but with different follow-up questions. To ensure the equivalence and validity of the tasks (Patton, 2002), we sought and incorporated feedback from five experts (one professor of mathematics, one professor of mathematics education, two Ph.D. candidates in mathematics education, and one high school mathematics teacher) regarding the parallelism and content of the tasks.

We conducted the interviews in participants' native language and then translated them from Turkish into English. The transcripts of the interviews and classroom observations included participants' utterances as well as their visual mediators and actions. The data was analyzed in terms of participants' word use, visual mediators, routines, and narratives (Sfard, 2008).

RESULTS

In this section, due to space constraints, we only provide a comparative analysis that outlines the differences and similarities between Mr. Can's and Okan's mathematical discourses according to the four components of their discourses (word uses, visual mediators, routines, narratives).

In the classroom, Mr. Can's approach to the concept of rotation was formal. He defined rotation as follows:

"The point obtained by rotating a point $P(x, y)$ on a plane around the point O at an angle α is $Q = R_\alpha(P) = (x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha)$. R_α here is called a rotation. $R_\alpha: R^2 \rightarrow R^2$ is a rotation as $R_\alpha(P)$ can be performed for each point P on the plane."

During the study, Mr. Can's word use on rotation was consistently object-driven with some phrase-driven use. Okan's word use on rotation was routine-driven before the lesson. In the second interview after instruction, his word use was mainly phrase-driven (with some rare object-driven word use). Okan's word use during the last interview was mainly phrase-driven with some rare phrase-driven word use.

We observed that Mr. Can predominantly used algebraic notation as visual mediators. Although less frequently, he also used some geometric mediators when he drew the pre- and post-images of a geometric shape. He often used algebraic equations and notations while rotating shapes. On the other hand, Okan's visual mediators were mainly geometric during all of the three interviews. However, Okan also used the algebraic approach during the interviews conducted right after the lesson on rotation.

Consistent with his use of algebraic visual mediators, one algebraic routine we identified in Mr. Can's discourse was *rotating each vertex of a geometric shape by using the equation $R_\alpha(P) = (x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha)$* . We also observed a geometric routine in Mr. Can's discourse during the interview: *rotating a geometric shape at degrees of 45, 90 and/or 180 and drawing, on an eyeball estimate, the rotated shape*. We did not identify a repetitive routine in Okan's discourse during the first interview. In the second interview, Okan used two different routines, one geometric and one algebraic routine. Okan's geometric routine was *rotating a geometric shape at degrees of 45, 90 and/or 180 and drawing, on an eyeball estimate, the rotated shape* and his

algebraic routine was *rotating each vertex of a geometric shape by using the equation $R_\alpha(P) = (x\cos\alpha - y\sin\alpha, x\sin\alpha + y\cos\alpha)$* . Although Okan did not use an algebraic routine before the lesson on rotation, he adopted the teacher's algebraic routine after instruction. In the third interview, Okan used the same two routines he used in the second interview.

Some of the narratives Mr. Can endorsed about rotation were his formal definition of rotation, “rotation is a transformation”, “rotation is a congruence transformation”, and “in rotation, sizes of shapes do not change; their locations and directions do.” Okan did not specifically endorse any narratives about rotation in the first interview. Instead he talked about his actions with which he rotated geometric shapes. His narratives on rotation during the second and final interviews showed some similarities with those of the teacher. “Rotation is a transformation”, and “In rotation, sizes of shapes do not change; their locations and directions do”, which were among Mr. Can's endorsed narratives, were also among Okan's endorsed narratives. However, Okan endorsed such narratives less frequently compared to the teacher; instead he tended to endorse narratives that expressed rotation through various geometric approaches rather than the formal-algebraic approach the teacher used in the classroom (the formal definition). It is possible that Mr. Can taught rotation with the assumption that the students were already familiar with it since the Turkish curriculum has a spiral nature and students are exposed to the concept of rotation in earlier grades before 10th grade.

CONCLUSION AND DISCUSSION

In this study, we investigated one Turkish high school student's discursive development on rotation in relation to his teacher's discourse. The results indicated that Mr. Can's discourse was based on an algebraic-formal approach during his instruction. Mr. Can only used a geometric approach during the interview, possibly due to the prompts in the tasks we provided. Mr. Can's word use on rotation was mainly object-driven with some phrase-driven use. Okan's word use was primarily routine-driven in the first interview and phrase-driven in the second and third interviews. During all of the three interviews, Okan used geometric routines with uncertainty whereas the teacher did not use a geometric routine in the classroom. In the second and third interviews, Okan used the algebraic routine that the teacher used in the classroom. These findings indicate that Okan was actively adjusting his discourse by taking into account Mr. Can's discourse while also making it compatible with his own thinking about rotation. The findings also show the close relationship among the four elements of mathematical discourse as they inform and are informed by each other (Emre, Güçler, & Argün, 2013; Güçler, 2013).

During the study, Okan was more successful in rotating 45, 90, 180 and 270 degrees as angles; he struggled with rotating shapes geometrically; and also with the concept of angle, which are results consistent with the existing literature (Ada & Kurtulus, 2010; Harper, 2002; Hollebrands, 2004). Despite Okan's tendency to use a geometric approach in his discourse, he did not demonstrate a clear, certain, and logical discourse

about it in the context of our study. One possible reason for this may be the lack of a geometric approach in the teacher's discourse during instruction. Another reason may be the lack of teacher's transparency and explanation behind his reasons to use a formal-algebraic and objectified discourse rather than a geometric one in the classroom.

Although Okan seemed to imitate his teacher during his discursive development, his uncertainty about some elements of his discourse indicates that such imitation may not be considered as "thoughtful imitation" from a commognitive perspective since, according to Sfard (2012), "thoughtful imitation" requires understanding of the reasons and logic behind the experts' discourses. Studies have shown that if the teachers do not have consistent, clear, and transparent mathematical discourse in the classroom, students may have difficulties understanding the teachers' discourses and may imitate the teacher inaccurately or without thinking about the reasoning behind the teachers' discourses (Güçler, 2013, 2014). It is important for teachers to make their discourses clear and relevant for the students to prevent communicational failures in the classroom. Further studies using sociocultural approaches to explore student learning in relation to classroom discourse are needed to shed additional light on how to improve classroom communication and support student learning of geometric transformations.

Acknowledgements

This study is supported by the Scientific and Technological Research Council of Turkey (TUBITAK) and University of Massachusetts Dartmouth.

References

- Ada, T., & Kurtulus, A. (2010). Students' misconceptions and errors in transformation geometry. *International Journal of Mathematical Education in Science and Technology*, 41(7), 901–909.
- Boulter, D. R., & Kirby, J. R. (1994). Identification of strategies used in solving transformational geometry problems. *Journal of Educational Research*, 87(5), 298–303.
- Cave, M. D. (2008). *Impact of Community Service Learning on Middle School African and Latino Americans' Understanding of Mathematics* (Unpublished Doctoral Dissertation). North Carolina State University.
- Clements, D. H. (2004). Geometric and spatial thinking in early childhood education. In D. H. Clements, J. Sarama, & A.-M. Di Biase (Eds.), *Engaging young children in mathematics: Standards for early childhood mathematics education* (pp. 267–298). Mahwah, NJ: Lawrence Earlbaum Associates Inc.
- Edwards, L. D. (1991). Children's learning in a computer microworld for transformation geometry. *Journal for Research in Mathematics Education*, 22(2), 122–137.
- Edwards, L. D. (2003). The nature of mathematics as viewed from cognitive science. *Proc. 3rd Conference of the European Society for Research in Mathematics Education*, Bellaria, Italy.

- Emre, E., Güçler, B., Argün, Z. (2013). One High School Student's Development of Mathematical Discourse on Translation. In Martinez, M. & Castro Superfine, A (Eds.), *Proc. of the 35th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 207–210). Chicago, USA: PME-NA.
- Flanagan, K. A. (2001). *High school students' understandings of geometric transformations in the context of a technological environment* (Unpublished Doctoral Dissertation). Pennsylvania State University, Pennsylvania.
- Güçler, B. (2014). The role of symbols in mathematical communication: The case of the limit notation. *Research in Mathematics Education*, 16(3), 251–268.
- Güçler, B. (2013). Examining the discourse on the limit concept in a beginning-level calculus classroom. *Educational Studies in Mathematics*, 82(3), 439–453.
- Harper, S. R. (2002). *Enhancing elementary pre-service teachers' knowledge of geometric transformations* (Unpublished Doctoral Dissertation). University of Virginia.
- Hollebrands, K. F. (2003). High school students' understandings of geometric transformations in the context of a technological environment. *Journal of Mathematical Behavior*, 22, 55–72.
- Hollebrands, K. F. (2004). High school students' intuitive understandings of geometric transformations. *Mathematics Teacher*, 97(3), 207–214.
- Hollebrands, K. F. (2007). The role of a dynamic software program for geometry in the strategies high school mathematics students employ. *Journal for Research in Mathematics Education*, 38(2), 164–192.
- Mammarella, I. C., Giofrè, D., Ferrara, R., & Cornoldi, C. (2012). Intuitive geometry and visuospatial working memory in children showing symptoms of nonverbal learning disabilities. *Child Neuropsychology*, 19(3), 235–249.
- Mitchelmore M. C. & White, A. P. (2000). Development of angle concepts by progressive abstraction and generalisation. *Educational Studies in Mathematics*, 41, 209–238.
- Ministry of National Education (MoNE) (2010). *Talim ve terbiye kurulu başkanlığı, ortaöğretim geometri dersi (9-10.sınıflar) öğretim programı [Board of Education, Secondary school geometry curriculum (9-10th grades)]*. Ankara: MEB.
- National Council of Teachers of Mathematics [NCTM] (2000). *Professional Standards for Teaching Mathematics*, NCTM, Reston, VA.
- Panorkou, N., & Maloney, A. (2015). Elementary Students' Construction of Geometric Transformation Reasoning in a Dynamic Animation Environment. *Constructivist Foundations*, 10(3), 338–347.
- Patton, M.Q. (2002). *Qualitative research and evaluation methods*. Thousand Oaks, CA: Sage.
- Portnoy, N., Grundmeier, T. A., & Graham, K. J. (2006). Students' understanding of mathematical objects in the context of transformational geometry: Implications for constructing and understanding proofs. *Journal of Mathematical Behavior*, 25, 196–207.

- Seago, N., Jacobs, J., Driscoll, M., Nikula, J., Matassa, M., & Callahan, P. (2013). Developing teachers' knowledge of a transformations-based approach to geometric similarity. *Mathematics Teacher Educator*, 2(1), 74–85.
- Sfard, A. (2008). *Thinking as Communicating: human development, the growth of discourses, and mathematizing*. Cambridge: Cambridge University.
- Sfard, A. (2012). Introduction: Developing mathematical discourse—Some insights from communicational research. *International Journal of Educational Research*, 51–52, 1–9.
- Xistouri, X., & Pitta-Pantazi, D. (2011). *Elementary students' transformational geometry abilities and cognitive style*. Proc. 7th Conference of the European Society for Research in Mathematics Education. Rzeszów, Poland.
- Yanik, H. B. (2006). *Prospective elementary teachers' growth in knowledge and understanding of rigid geometric transformations* (Unpublished Doctoral Dissertation). Arizona State University.
- Yazgan, G., Argun, Z., & Emre, E. (2009). Teacher Sceneries Related to “Angle Concept”: Turkey case. *Procedia-Social and Behavioural Sciences*, 1(1), 285–290.

FROM PRINCIPLES OF VISION AND DIVISION TO A SYSTEM PREMISED ON AND SUBJECT TO INTERANIMATED DIMENSIONS: SOME REFLECTIONS ON IDENTITY IN MATHEMATICS EDUCATION

Osnat Fellus

University of Ottawa, Canada

The concept of identity in mathematics education has gained much attention in scholarly work in the last two decades. However, in spite of growing momentum in seeing learners' identity as part and parcel of students' level of engagement and success in school mathematics, the concept remains under-specified and under-theorized. This paper presents a critical overview of the concept of identity and draws on Ricoeur's (1992) differentiation between idem identity and ipse identity, on Brubaker and Cooper's (2000) distinction between identity as a category of practice and identity as a category of analysis, and on Ivanič's (1998) four-dimensional model to foreground a shift in mathematics education from seeing identity as principles of vision and division to seeing it as an inter-animated, multi-dimensional system.

INTRODUCTION: IDENTITY IN VOGUE

The concept of identity became a zeitgeist in the 1960s and has since been used as a conceptual and analytical tool in a wide array of scholarly research (Brubaker & Cooper, 2000). Indeed, a simple search of the key word *identity* in Google Scholar, a publicly accessible search engine of academic literature, yields more than four million hits. Refining the search using the custom-range feature, one can see that the number of scholarly publications on identity more than tripled between 1950 to 1980 and more than doubled between 1980 and 2017. This avalanche of research can be explained by the fact that identity has been recognized as a construct that plays a consequential and critical role in one's development (McAdams & Olson, 2010) and wellbeing (McLean, Breen & Fournier, 2010). It is not surprising then that the field of education embraced the concept of identity as a tool to explore learning and development (Wenger, 1998) and that widespread attention to the construct has yielded important work in different content areas including mathematics education. However, in spite of this increased attention to *identity*, there remain under-articulated and under-addressed problems that include the overproduction and devaluation of the meaning of identity; the theoretical and empirical under-specified treatment of the concept; and the incompatibility between how identity is conceptualized and how it is explored. Whereas these are not trivial issues to answer, they are rather compelling especially in the field of mathematics education.

THEORIZING IDENTITY: A SHIFT FROM IDEM TO IPSE IDENTITY

In spite of its being a word everybody, everywhere, uses, identity seems to generate different meanings and interpretations that heavily depend on context, purpose, and user (Brubaker & Cooper, 2000)—a phenomenon dubbed *an amoeba word* (Cayley, 1992; Fellus & Glanfield, 2017). Looking at the plethora of work on identity, Brubaker and Cooper (2000) argue that—more often than not—identity is used in research studies as both a *category of practice* and as a *category of analysis*, even though these are qualitatively different as each points to a different theoretical and empirical trajectory. Identity is employed as a category of practice, Brubaker and Cooper (2000) explain, when it is treated in terms of readily observable terms such as race, class, and gender. These render the concept of identity as terminally static, flat, and one-dimensional. On the other hand, identity is used as a *category of analysis*, when researchers turn attention to its action-oriented, continually negotiated, and multi-dimensional nature. Harré and Gillett (1994) allude to this distinction as “the difference between the old idea of the self as something inside a person and the new idea of the self as a continuous production” (p. 110).

This shift in perception from perceiving identity as a substance to perceiving it as an idea that a person constructs is put forth by Polkinghorne (1988) who calls for a transition from treating identity as “a flattened reality that consists of physical objects in time and space along with their relations” (Polkinghorne, 1988, p. 149), to seeing it as a “construction built on other people’s responses and attitude toward a person” (Polkinghorne, 1988, p. 145). Thinking of identity in terms of “other people’s responses” allows us to explore the construct as a co-constructed, negotiated phenomenon. This distinction also speaks to the etymology of the word *identity* that embeds, simultaneously, two different meanings (Ricoeur, 1992). One references sameness (*idem* in Latin), the other selfhood (*ipse* in Latin). The former implies a sense of static continuity over time, the latter of continual development of sense of self; the former is oriented toward the past, the latter toward the future; the former elides individuals to create a unified façade, the latter brings forth the diversity between individuals. It is identity as selfhood and as a category of analysis that this paper aims to put forth in the context of mathematics education. To understand how this *ipse* identity takes form, and how identity as a category of analysis can be explored, this discussion is framed within the sociocultural framework that contends that people develop in and through language and other semiotic artefacts.

Importing Ivanič’s (1998) four-dimensional identity model

Even though Ivanič’s (1998) research was conducted within the field of academic writing, combining—simultaneously—four dimensions to explore the developing identities of students who were learning how to write for academic purposes, I find it helpful for the purposes of exploring identity work among learners of mathematics. I am fully cognizant that by importing Ivanič’s (1998) work, I am pulling down the disciplinary barriers between the two fields. Nonetheless, such import, I believe, is timely and necessary. In her work, Ivanič (1998) demonstrates how writers continually

shape their respective writer's identity through four distinct—however, inseparable—dimensions. She shows how her students structure and construct their autobiographical identity that they choose to put forth, how they appropriate ideas and take ownership over what they say and write, how they position themselves in relation to others through discourse, and how, and why, they choose to align their writing with others. Ivanič (1998) convincingly shows that the four identity-related dimensions of autobiographical self, authorial self, discorsal self, and the socioculturally available selfhoods are complementary rather than contesting.

Autobiographical identity has been found to contribute “to the constitution of self” (Ricoeur, 1992, p. 114) and to a person's understanding “regarding how he or she came to be and where he or she is going in life” (McAdams & Olson, 2010, p. 527). Ochs and Capps (1996) argue that autobiography is useful in understanding the motivation behind and relationship between people's actions. While these are important, I find that McAdams and Bowman's (2001) work on contaminating and redemption self-narratives is especially instructive as they claim that the framing of identity narratives as a reconstruction of “positive” or “negative” past events impact the present and shape the future.

Authorial identity draws on Bakhtin's notion of authorship that positions speakers as ‘owning’ their words when they populate them with their own accents and intentions. According to Bakhtin, says Holquist (1983), “A speaker is to his utterance what an author is to his text” (Holquist, 1983, p. 315). Indeed, Bakhtin (1981) writes:

The word in language is half someone else's. It becomes ‘one's own’ only when the speaker populates it with his own intention, his own accent when he appropriates the word, adapting it to his own semantic and expressive intention. Prior to this moment of appropriation, the word does not exist in a neutral and impersonal language (it is not, after all, out of a dictionary that the speaker gets his words!), but rather it exists in other people's mouths, in other people's contexts, serving other people's intentions: it is from there that one must take the word, and make it one's own (pp. 293-294).

Holland, Lachiotte, Skinner, and Cain (1998) have extended Bakhtin's idea of authorship. They state that “[i]n the making of meaning, we ‘author’ the world” (p. 170). Understanding authorial identity, then, concern's the individual's voice as they express their “position, opinions, and beliefs” (Ivanič, 1998, p. 26) echoing other people's words.

Discorsal identity is concerned, according to Ivanič (2006), with how one's identity is constructed by address, (i.e., how individuals are talked *to* by others) by attribution, (i.e., how individuals are talked *about* by others) and by affiliation and alignment (i.e., how individuals *sound like* others). The use of discourse as a tool to construct one's identity was identified as a turning point in research and practice as far back as two decades ago by Harré and Gillett (1994) and was since developed theoretically and supported empirically in a wide variety of research. In considering discorsal identity, one's identity does not exist ontologically in separation from semantically loaded expressions. Rather, works such as Heath (1983) show that identity is continually gen-

erated by how and what we speak, how we are talked *to* and *about*, and what we affiliate ourselves with through our speech.

The construct of *socioculturally available identities* is the fourth dimension Ivanič (1998) proposes for the investigation of identity. Here, she claims, that socioculturally available selfhoods can empower or constrain one's imagined selfhood. She describes how such available selfhoods are co-constructed by cultural, historical, and sociological contexts, which may include cultural heritage, family stories, historical events, and literary characters that take forms and shapes in fiction, literature, movies, social and cultural narratives, and meta-narratives. Socioculturally available selfhoods function as repositories for identities that are contextually and socioculturally available. The unique affordances of socioculturally available identities are that they are scripted roles open for selection and appropriation.

These four identity-related dimensions are relevant to the work on identity in mathematics education as I show below. For the purpose of this paper, I have limited my search for scholarly work to identity on K-12 mathematics education. These studies, I will show, mark a shift in mathematics education toward treating identity in as a category of analysis to highlight *ipse* identity.

AUTOBIOGRAPHICAL IDENTITY IN MATHEMATICS EDUCATION

In a recent work, Graven, Hewana, and Stott (2013) collected students' written autobiographical records of their experience of school mathematics since the beginning of elementary school and concluded that the stories told by learners about their experiences of school mathematics formulate students' dispositions toward the subject and their identities as learners and doers of school mathematics. Specifically, these autobiographical identities were painted by the children in either negative or positive colours, which corresponds with McAdams and Bowman's (2001) concept of impact contamination and redemption autobiographies carry on people's beliefs.

In another research Di Martino and Zan (2010) collected essays from 1,496 students in Italy. The essay was titled *Me and mathematics: My relationship with mathematics up to now*. In analysing the essays, the authors formulated the following three main findings: that most students reported on negative experiences with mathematics, that most students thought that mathematics was theoretical mostly, and that students had a clear perception of themselves as learners and doers of mathematics indicating a strong association between students' mathematics-related autobiographical identity and their dispositions toward mathematics.

These works not only provide promising empirical evidence to support the recognition that autobiographical identity is integral to understanding learners' mathematical identity but also reveal a picture that depicts a vast terrain of uncharted paths in research on autobiographical identity within K-12 learners of mathematics (Towers, Hall, Rapke, Martin, & Andrews, 2016).

AUTHORIAL IDENTITY IN MATHEMATICS EDUCATION

Authoring and mathematics may seem to some as an oxymoron. It cannot be further from the truth if we consider the act of authoring as the driving force behind doing real mathematics. Going beyond procedural school mathematics, when students receive opportunities to embark on a quest of mathematical meaning making or to experience “knowledge ownership” (Hogan, 2008, p. 110), they get a chance to author mathematical ideas (Langer-Osuna, 2017) and develop their authorial identity.

Hogan (2008) demonstrates how opportunities to formulate one’s own ideas provide one student the space to take ownership for learning of mathematical ideas and for developing conceptual connections. Taking away these opportunities made the student feel “far less ownership in her learning than she did” before (Hogan, 2008, p. 107). Similarly, Andersson, Valero, and Meaney’s (2015) found that when students took ownership over topics of mathematical investigation, they saw themselves as being more mathematically abled. Jorgensen (2014) alludes to this authorial identity in mathematics when he calls for a paradigm shift that will allow for the adaptation of a “knowledge-making epistemology” (p. 316) so that “theorizing the identities of learners will help advance mathematical learning for all students” (p. 314).

DISCOURSAL IDENTITY IN MATHEMATICS EDUCATION

Framing discorsal identity within Ivanič’s (2006) concept of address, attribution, and affiliation, points to relevant work that reveals how institutional practices of labelling propagate and continually reproduce discorsal identities that realize the very characteristics assigned in the labelling (Boaler, 2005). But it is not only institutionally propagated labelling that constructs learners’ discorsal identity. It is also the way people talk of themselves to themselves and others. Bishop (2012) for example, shows how labelling is also self-assigned through classroom discourse.

Curriculum and classroom practices as well generate discorsal mathematics identities. Hogan (2008) demonstrates how curriculum content translates into classroom practices that animate teachers’ attribution of students’ mathematical ability and, in turn, frame the way students address, and are addressed by, their teachers. Hogan (2008) provides empirical support to demonstrate how students’ academic achievements in mathematics are premised on discorsal identity as it is “less about [their] intrinsic abilities and more about how [they are] constructed as learners by [their] teacher” (p. 95).

SOCIOCULTURALLY AVAILABLE IDENTITIES IN MATHEMATICS

Learners of mathematics emulate others who do mathematics. These others can be depicted through cultural or family stories (Sfard & Prusak, 2005) or through popular media (Epstein, Mendick, & Moreau, 2010; Mendick, 2004). Hogan (2008) explains how sociocultural and historical scripted roles dictate and continually reproduce what one can be, do, or say mathematically. Similarly, Betz and Sekaquaptewa (2012) show how available role models impact girls’ identities as learners of science and mathe-

matics. Such role models can be depicted in movies and other forms of entertainment as well. Indeed, in her analysis of mathematics-related movies, Mendick (2004) found that movies perpetuate the myth that mathematics is gender specific, and that it is typified by isolation, obsession, and social awkwardness. Work on socioculturally available selfhoods is instructive as it shows how these possibilities sometimes offer very limited and limiting sources for identification.

CONCLUSION

Identity has been long identified as consequential in learners' engagement with and success in mathematics. However, in spite of its pivotal role in mathematics education, it has been perceived as, mostly, enigmatic. In this paper, I aimed at highlighting the differences between treating the concept of identity as a category of practice (read *idem* identity) and as a category of analysis (read *ipse* identity). I argue that if consciously, or unwittingly, the term is used as the former, it brings forth one-dimensional snapshots of who learners of mathematics are and pushes the work of identity as a process into the shadows.

In order to reflect the complex, inter-animated nature of identity work, this paper imports Ivanič's (1998) work to suggest a simultaneous investigation of four distinct—but inseparable—dimensions in identity-related studies so that it can gradually gain empirical substance. By importing Ivanič's work and tracing the use of each of the identified identity dimensions, I respond to Lutovac and Kaasila's (2017) call to draw on research on identity outside of mathematics education and hope to continue the conversation about research on identity in mathematics education that moves away from principles of vision and division and gets closer to treating identity as a system that is premised on and subject to simultaneously operating inter-animated dimensions.

References

- Andersson, A., Valero, P., & Meaney, T. (2015). I am [not always] a maths hater": Shifting students' identity narratives in context. *Educational Studies in Mathematics*, 90(2), 143-161.
- Bakhtin, M. (1981). *The dialogic imagination* (C. Emerson & M. Holquist, Trans.). Austin: University of Texas Press.
- Betz, D. E., & Sekaquaptewa, D. (2012). My fair physicist? Feminine math and science role models demotivate young girls. *Social psychological and personality science*, 3(6), 738-746.
- Bishop, J. P. (2012). "She's always been the smart one. I've always been the dumb one": Identities in the mathematics classroom. *Journal for Research in Mathematics Education*, 43(1), 34-74.
- Boaler, J. (2005). The 'Psychological Prison' from which they never escaped: The role of ability grouping in reproducing social class inequalities, *FORUM*, 47(2-3), 135-144.
- Brubaker, R., & Cooper, F. (2000). Beyond "identity." *Theory and society*, 29(1), 1-47.

- Cayley, D. (1992). *Ivan Illich in conversation: The testament of Ivan Illich*. House of Anansi.
- Di Martino, P. & Zan, R. (2010). 'Me and maths': towards a definition of attitude grounded on students' narratives. *Journal of Mathematics Teacher Education*, 13(1), 27–48.
- Epstein, D., Mendick, H., & Moreau, M. P. (2010). Imagining the mathematician: Young people talking about popular representations of maths. *Discourse: Studies in the Cultural Politics of Education*, 31(1), 45–60.
- Fellus, O., & Glanfield, F. (2017). Reflections on the FLM preconference. *For the Learning of Mathematics*, 37(1), 15–18.
- Graven, M., Hewana, D., & Stott, D. (2013). The evolution of an instrument for researching young mathematical dispositions. *African Journal of Research in Mathematics, Science and Technology Education*, 17(1-2), 26–37.
- Harré, R., & Gillett, G. (1994). *The discursive mind*. Sage Publications.
- Heath, S. B. (1983). *Ways with words: Language, life, and work in communities and classrooms*. Cambridge: Cambridge University Press.
- Hogan, M. P. (2008). The tale of two Noras: How a Yup'ik middle schooler was differently constructed as a math learner. *Diaspora, Indigenous, and Minority Education*, 2(2), 90–114.
- Holland D, Lachicotte W., Skinner D, & Cain C. (1998). *Identity and agency in cultural worlds*. Cambridge, MA: Harvard University Press.
- Holquist, M. (1983). Answering as authoring: Mikhail Bakhtin's trans-linguistics. *Critical Inquiry*, 10(2), 307–319.
- Ivanič, R. (1998). Writing and Identity: *The discursive construction of identity in academic writing*. Amsterdam: John Benjamins.
- Ivanič, R. (2006). Language, learning and identification. In R. Kiely, P. Rea Dickens, H. Woodfield, and G. Clibbon (Eds.), *Language, culture and identity in applied linguistics*, (7-29). London: British Association of Applied Linguistics and Equinox.
- Jorgensen, R. (2014). Social theories of learning: A need for a new paradigm in mathematics education. In J. Anderson, M. Cavanagh, & A. Prescott. (Eds.), *Curriculum in focus: Research guided practice* (Proceedings of the 37th annual conference of the Mathematics Education Research Group of Australasia, pp. 311–318). Sydney: MERGA.
- Langer-Osuna, J. M. (2017). Authority, identity, and collaborative mathematics. *Journal for Research in Mathematics Education*, 48(3), 237–247.
- Lutovac, S., & Kaasila, R. (2017). Future directions in research on mathematics-related teacher identity. *International Journal of Science and Mathematics Education*, 1–18.
- McAdams, D. P., & Bowman, P. J. (2001). Narrating life's turning points: Redemption and contamination: Narrative studies of lives in transition. In D. P. McAdams, R. Josselson, & A. Lieblich (Eds.), *Turns in the road: Narrative studies of lives in transition* (p. 3–34). Washington, D.C.: American Psychological Association Press.
- McAdams, D. P., & Olson, B. D. (2010). Personality development: Continuity and change over the life course. *Annual review of psychology*, 61, 517–542.

- McLean, K. C., Breen, A. V., & Fournier, M. A. (2010). Constructing the self in early, middle, and late adolescent boys: Narrative identity, individuation, and well-being. *Journal of Research on Adolescence*, 20(1), 166–187.
- Mendick, H. (2004). A mathematician goes to the movies. *Proceedings of the British Society for Research into Learning Mathematics*, 24(1), 43–48.
- Ochs, E., & Capps, L. (1996). Narrating the self. *Annual Review of Anthropology*, 25, 19–43.
- Polkinghorne, D. E. (1988). *Narrative knowing and the human sciences*. Suny Press.
- Ricoeur, P. (1992). *Oneself as another*. University of Chicago Press.
- Sfard, A., & Prusak, A. (2005). Telling identities: In search of an analytic tool for investigating learning as a culturally shaped activity. *Educational Researcher*, 34(4), 14–22.
- Towers, J., Hall, J., Rapke, T., Martin, L. C., & Andrews, H. (2016). Autobiographical accounts of students' experiences learning mathematics: A Review. *Canadian Journal of Science, Mathematics and Technology Education*, 1–13.
DOI:10.1080/14926156.2016.1241453
- Wenger, E. (1998). *Communities of practice: Learning, meaning, and identity*. New York: Cambridge University Press.

THINKING IN MOVEMENT AND MATHEMATICS: A CASE STUDY

Francesca Ferrara and Giulia Ferrari

Università di Torino, Italy

This paper discusses recent theories in mathematics education that, while studying the role of the body in mathematics, reveal growing interest in the dynamic nature, or flow, of mathematical activity, rather than in what the activity allows to achieve and how. Pursuing the lines of flight offered by these studies on the visceral role of movement, we draw on the idea of “thinking in movement” by Sheets-Johnstone (2009, 2011) to elucidate the interconnection between moving and thinking. We use this perspective to analyse an interview in which a grade 4 student is engaged with spatio-temporal relationships, as a way to better study his encounter with graphical configurations.

INTRODUCTION

Since the corporeal turn prompted by theories of embodied mathematics in the 2000s, studies that take into account the role of the body in mathematics education research often discuss bodily movement in the classroom as a crucial resource for teaching and learning (e.g. Radford, Edwards & Arzarello, 2009; Radford, 2013; Edwards et al., 2014). These studies often strive to code multimodal engagement—hand gesture, eye gaze, prosody in speech (high-low pitch), bodily posture, and so on—as a way to infer correspondences with a particular cognitive stage, level of understanding or step in a learning trajectory, within constructivist or acquisitionist perspectives. But the risk of such associations is to see bodily engagement as a placeholder of cognitive schemas already existing in mind and, therefore, to fall into old body/mind splits, instead of thinking of the body and bodily activity as “genuinely constitutive of knowing” (Nemirovsky *et al.*, 2013).

In the last decade, researchers have offered ways to rethink the notion of embodiment through new perspectives that insist in dissolving any conceptual-perceptual cut or dualism. For example, Nemirovsky and colleagues (2013) take a non-dualistic stance on learning in informal settings to describe how perceptuomotor integration partakes in mathematical thinking about graphing motion, pursuing a phenomenology of lived experience. De Freitas and Sinclair (2014) propose the idea of assemblage within a new materialist perspective to address the issue of the body in learning in a wider sense, which also comprises the body of mathematics. They want to extend thinking beyond the single individual effort and to show how it occurs as distributed through material encounters of human and non-human bodies. Enactivist researchers (e.g. Maheux & Proulx, 2015) posit a shift in the way they consider enacted mathematical activity as knowing itself, in a dynamic process that involves the learner acting and immersed in the environment. Others explore the image of a growing-making math-

ematics (Roth, 2016), use theories on material phenomenology (Hwang & Roth, 2011) or extend the idea of sensuous cognition (Radford, 2013), in an attempt to investigate classroom situations arguing for monistic views of cognition.

This brief review on recent theories of embodiment in mathematics education makes the following point clear: there is growing interest in the dynamic nature, movement or flow, of the mathematical activity rather than in what the activity allows learners to achieve and the way it does so. To say it differently, focus is more and more shifting to the proper encounters of learners with mathematical concepts and to the relational entanglement of movement and thinking in these encounters. We pursue here this line of flight, drawing attention to the way in which movement and thinking are contiguous and push each other forward in mathematics. In particular, focus is on an informal conversation about graphical representations of motion in the context of an interview between a student and a researcher.

MOVEMENT AND THINKING

In a very recent work, Roth and Maheux (2015) propose a “dynamic approach to mathematical thinking”, addressing the issue of how we might exhibit mathematical thinking in movement in a way that learning and movement are not reduced to schemas. De Freitas and Ferrara (2015) take a similar, more philosophical, stance as they show that mathematical concepts themselves are mobile, but the most freedom of movement belongs to thought. The dynamic, mobile nature that is to be characterised in these studies belongs not just to the process of knowing or to the body but to thought, and to mathematics itself, in resonance with Châtelet’s (1993/2000) view of the virtual dimension of mathematics. In this paper, we share with these authors the same concerns around the non-representational character of gesture and bodily movement and a visceral interest in the way in which movement might be better characterized and studied in the context of classroom situation, in order to embrace the mathematical activity of students and teachers (and researchers) in its whole complexity and profundity. In particular, we aim to contribute to this area of research by showing how movement and thinking sustain and build up each other in mathematical situations. To do so, we draw on Sheets-Johnstone (2009; 2011), whose work is mainly dedicated to elucidating the nature of movement. Grounding her studies in Husserl’s phenomenology and Merleau-Ponty’s theory of perception, she expands their vision and elucidates the primacy of movement in the life of animate beings. For Sheets-Johnstone, movement is not equivalent to a mere local change in position, but is our primary way of making sense of the world at both human and evolutionary scale (that is to say, in terms of human development and with respect to the evolution of animate forms). In particular, she proposes a ground-breaking interconnection among moving, feeling and thinking. Even though we recognise the immense power of her theory of affect grounded in movement (affective/tactile-kinesthetic body), in light of the purposes of our paper, we are more interested in deepening co-constitutional relationships between moving and thinking. Sheets-Johnstone (2011) examines the experience of “thinking in movement” and its foundational character to the creation of a *kinetic bodily logos*. By

proposing a first-person experience of an improvisational dance, she describes a paradigmatic example of thinking in movement. She shows how in the dance “[q]ualities and presence are enfolded into [her] own ongoing kinetic presence and quality” (*ibid.*, 2011), engaging her directly with the here and now, without any gap between the “global dynamic world” which is perceived and “the kinetic world” in which she is moving. The world that she is *exploring* in movement cannot be separated from the world she is *creating* in movement: “the idea that thinking is separate from its expression—a thought in one’s head, so to speak, existing always prior to its corporeal expression—is a denial of thinking in movement”. By the same token, saying that thinking in movement is a way of being in the world and a natural mode of being a body, the author is also challenging representational visions of the body, that is, of “a body that mediates its way about the world by means of language”. This standpoint has at least two important consequences: Sheets-Johnstone first proposes that we must rethink what it means ‘to have meaning’, and, secondly, argues that movement might be meaningful in itself. Therefore, in our understanding of it, the expression “thinking in movement” can be ‘read’ not only left-to-right but also in the opposite direction. In both ways, it implies not just a temporal overlapping but the mutual constitution and implication of the two processes: movement is thinking and thinking is moving. Moreover, there is a manifold of possibilities “contained” in any movement, which can be disclosed through “certain felt tensional quality, linear quality, amplitudinal quality, and projectional quality” (Sheets-Johnstone, 2011). These four primary qualitative structures of movement relate to force or effort, to space and to time. They are “separable only reflectively, that is, analytically, after the fact; experientially, they are all of a piece in the global qualitatively felt dynamic phenomenon of self-movement.” (*ibid.*, 2011)

We draw here on these theoretical aspects to take a perspective on moving-thinking that helps us examine the dynamic nature of mathematical activity of students, who deal with graphical representations of spatio-temporal relationships.

CASE STUDY AND METHODOLOGY

The data we present in this paper come from a pilot experiment, which was conducted in 2016 for a wider study on the role of movement in mathematics. The experiment was designed as a classroom-based intervention (Stylianides & Stylianides, 2013) and carried out with the twofold aim of (1) improving classroom practices proposing a graphical approach to functional relationships at primary school and (2) deepening the ways in which a specific technology that required bodily movements might be fuelling understanding in this context. The pilot study involved a class of grade 4 students (aged 8-9 years old) in graphing motion activities with the software WiiGraph. WiiGraph leverages two remote controllers of the game console Wii to graphically capture the positions of two students as they move in an interaction space, while pointing the remotes to a sensor bar. When the students move back and forth, farther and closer with respect to the bar, two real-time graphs originate on a Cartesian plane. Each graph captures one user’s distance in time, for a duration of usually 30 seconds. In our setting,

we used a 4-meter masking tape strip to create two parallel corridors (orthogonal to the sensor) so that each user could move freely in one of them. During the three 2-hour meetings, the students were mainly involved in using the software in collective discussions led by one of the researchers (the first author). The graphs were projected on an interactive whiteboard. At first, pairs of students moved in the interaction space and the entire class began to explore the encountered graphical representations in terms of movements. Later, the students were asked to move in order to obtain specific configurations on the screen (e.g. a couple of horizontal straight lines, or of parallel slanted lines), and to imagine and draw the graphs eventually produced by two people moving without the real-time feedback of the software. These activities aimed at creating space for mathematical explorations on spatio-temporal relationships to take place in the classroom, through bodily interactions and narratives around movements and experiments with the technology. The students also worked in groups solving written tasks. We filmed the classroom discussions with two cameras and collected the students' written protocols. After a 6-month period from the intervention, five students participated in 10-minute individual semi-structured interview with one researcher (the second author). The interviews were informal conversations regarding memories of the students about what they experienced and liked during the pilot experiment. The focus of this paper is on one of these interviews, which were all filmed. The data were transcribed and analysed following a microethnographic methodology (Streeck & Mehus, 2005), in the attempt of taking into account micro-movements that emerge in the activity through the body (talk, gesture, posture, diagramming, gaze, rhythm, and so on). This choice is in line with our theoretical commitment on movement, and with the crucial role of movement in the students' activity with the technology.

CROSSING LINES

In the following we present and then analyse a 1-minute segment of the interview in which a student, Luca, discusses with the researcher the event of crossing lines. We chose this episode for various reasons. One is that Luca was involved—months before, during class discussion—in the following experiment. He moved together with a classmate, Giulia (Fig. 1a), to fulfil the request of the researcher, that of producing two parallel (slanted) lines with WiiGraph. At the beginning of the experiment, the children seemed to be coordinating with each other to move at the same pace while going farther from the sensor. But, as Giulia decelerated approaching the far end of the corridor, Luca reached her, provoking the software to display a pair of intersecting lines (Fig. 1b) and creating amazement in all the students.

Making sense of the intersection and, more generally, dealing with two distances plotted in Cartesian coordinates is not trivial at all. Studies in the literature highlight that it is difficult for young children “to work out relation of different positions plotted in this way” (Bryant, 2009). There are important, intertwined aspects to be taken into account. For each user, WiiGraph captures spatio-temporal relationships that involve one variable (distance) which depends on another (time). In an experiment, it captures

two distances at the same time: two variables that change together without depending on one another. Similarly, two movements occur simultaneously but independently. Therefore, relations between movements can be established but need to be directly explored and maintained by the students (e.g. the same pace) to produce a specific pair of graphs that preserve specific relationships.

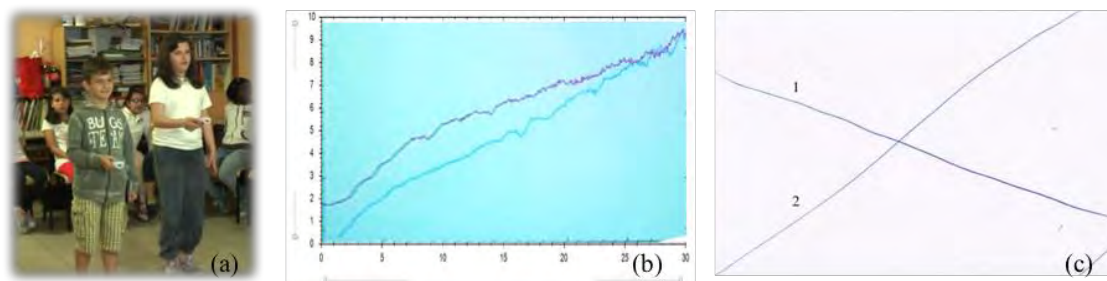


Figure 1. (a) Luca and Giulia moving; and (b) their graphs; (c) two crossing lines.

Going back to Luca and Giulia's experiment, a *choreography* in which two people start close to the sensor, 1 metre away from each other, and, moving at a constant speed, both walk away from the sensor, is one that allows for the creation of parallel straight lines with positive slope. It is not the only one: there is an entire bundle of possible pairs of movements that gives rise to a similar diagram (two parallel lines). There is a manifold of interactions between movers and nuances or little variations in movement virtually contained in that *configuration*. A similar point may be stressed if we speak of crossing lines: Figures 1b and 1c show two of the possible configurations.

What follows specifically refers to Luca's interview, during which the issue of crossing lines comes again to the fore. In the first part of the interview, Luca is asked what he liked and remembers about the classroom intervention (the software is not in use). On the table, two remotes and the sensor bar are at disposal, together with some pens and a sheet of paper. He begins telling that "two children held the remotes, and they had to do lines on that graphical area, pointing the remotes to the sensor". Then he speaks of the case of parallel slanted lines as that in which "two students had to go forward keeping the distance fixed". Holding the remotes, Luca and the researcher simulate this experiment following the indication given by Luca.

After few minutes, the interviewer asks Luca about the crossing lines (in the transcript we use R = researcher; L = Luca; L/RH = left/right hand):

- 1 R: What if I wanted to create two-o lines that cross each other, at some point?
- 2 L: It is needed that a child goes forward (*RH, holding a pen, moves rapidly towards his torso, then comes back to the starting position, in front of him*), th-, the other goes faster (*LH goes shortly back and then with impulse reaches RH*) and then they have to meet (*slowing down speed, LH reaches RH. Looks at R*) (pause) in a point (*still gazes at R*)
- 3 R: Can we try out? What would you do? (*takes one remote in RH and keeps it pointed to the sensor in front of her*)
- 4 L: (*takes the other remote with RH, gazes at R's remote*) I start ahead, then you go faster (*moves LH index finger from the R's remote position towards*

- the sensor*), I go slowly and then they meet (*LH reaches his remote, fingers extended and kept in the same position for few seconds: Fig. 2a*)
- 5 R: And do we both move forward? (*LH rapidly points to the sensor*)
- 6 L: No, then they meet (*LH goes back, then slowly goes forth again and overtakes his RH*), then you go forward and I stay behind (*RH zigzags moving a little closer to his LH*)
- 7 R: Ok
- 8 L: So, you do, (*LH points to the sensor*) you go ahead
- 9 R: Tell me when to go (*keeps the remote still*)
- 10 L: Go (*gazes at R's remote. R and L move the remotes towards the sensor*). You do like this (*moves his remote a little back*), you overtake me and I stay behind (*looks at R, moves again his remote towards the sensor*)
- 11 R: Ok (*interrupts her movement*). So, how are the lines showing up?
- 12 L: (*puts the remote on the table*) Criss-crossed (*cross arms: Fig. 2b*)
- 13 R: How?
- 14 L: Hm (*cross arms again, turned to a different slope, takes a pen*), do I draw them...? (*softly speaking*)
- 15 R: Yes, yes, as you want (*puts her remote on the table*)
- 16 L: One like this (*draws line 1 in Fig. 1c*), the other one like this (*draws line 2 in Fig. 1c*)
- 17 R: (pause) (*gazes at the drawing*) So-o (*points to the drawing*)
- 18 L: Hm, no (*with closed fists, RH ahead and LH back are swapped in position*), I start ahead and then (pause) I start ahead (*points ahead with RH*), you start from behind (*points back with LH*) and then (*suddenly, swaps hands' positions again*) they cross each other
- 19 R: So, is this drawing (*points to it again*) of another movement, for you?
- 20 L: Yes (*takes the remote*)
- 21 R: So (*takes the remote too*)
- 22 L: I start ahead, you [start] from behind, they cross, then (*Fig. 2c captures the experiment performed by Luca and the researcher*)

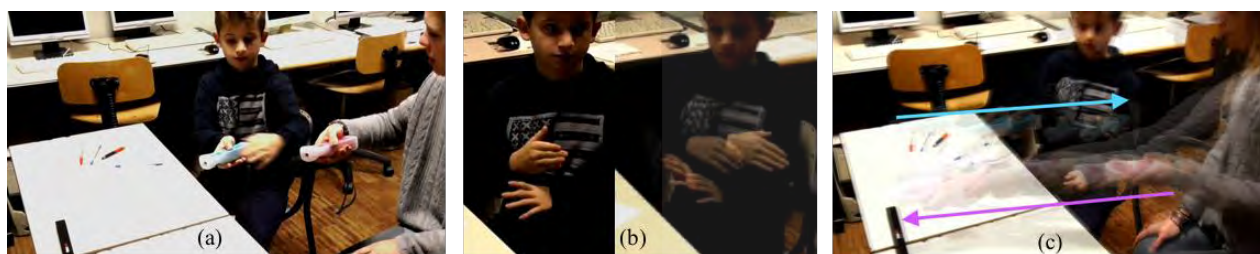


Figure 2. (a) 1st choreography; (b) gesture for intersecting lines; (c) 2nd experiment.

We see how, in the interview, two different mathematical events mainly resonate with the question and the experience of crossing lines (cf. diagrams in Fig. 1b and 1c). These events emerge as intertwined out of movement and fuel Luca's thinking in movement. Bodily movements actualize specific choreographies, perform simulations of experi-

ments, establish shapes for diagrams and arrangements of lines (configurations), or even mix the three aspects together. We can capture a sequence from the episode. A first choreography sees hands, people, remotes going in the same direction, then meeting and eventually one overtaking the other (3 times: [2], [4], [6]; Fig. 2a). Then, a first experiment also engages the researcher in performing such choreography: [10]. The following configuration (2 gestures, 1 diagram: [12], [14], [16]; Fig. 2b and 1c) with the emerging diagram is a turning point as it reconfigures previous movements and engenders a second choreography. The new choreography (2 times: [18]) is still evoking the crossing relational movement, but now involves two hands, people, remotes swapping positions. Finally, a second experiment, rhythmically dictated by Luca's narrative, closes the episode ([22]; Fig. 2c). Each moment fluently evolves into the next in the experience of thinking in movement, which we characterize as follows. On the one side, the diagram reconfigures boundaries between the two choreographies by unfolding a new point of view that is also able to capture a crossing event. Hesitation and suddenness destabilize homogeneous continuity in the temporal overlapping of the two possibilities, as well as of the processes of thinking and moving. On the other side, repetitions of a choreography entail little variations within movements, as it is in the case of the first choreography, where a zigzagging of the remote is added in a way that stresses relative positions between hand and remote, and therefore in the two movements. This sheds light on the complexity within the process of movement in thinking and the potential dimension of both moving and thinking. Ambiguity between the choreographies is generative of new meanings that are still open to mobility within the mathematical event, which is at the core of the episode. Such mobility and openness resonates with the creative power of explosion attributed from Leibniz to points, when thought of as generated by the intersection of two lines or curves, in his account of the virtuality of mathematical concepts (as shown in Châtelet, 1993/2000).

CONCLUSIVE REMARKS

Further research may elucidate how explorations of crossing lines could be considered pivotal in thinking of couples of graphs with WiiGraph. Discussion around this point in the classroom created new meanings for the intersection as “swapping places” or “overtaking the other”, which are the configurations captured by the choreographies in the segment of Luca's interview. As researchers, this point made clear for us the importance for students of experiencing and making sense of the intersection of lines in order to relate not simply each of the graphs to an individual movement, but the graphs themselves, as well as the movements, to each other. Examining movement in thinking in Luca's interview, we offer a way of drawing attention to how the flow of movements implicates dynamic thinking about pairs of graphs and their relations, being generative of mathematical meanings beyond its own meaningfulness. We use superposition of subsequent video frames with increasing transparency filter (see Fig. 2b and 2c) to induce a sense of movement which cannot be otherwise grasped by still images. In fact, there arises a delicate methodological issue that needs to be further examined: to develop ways that allow for better addressing and capturing the complexity of movement

without reducing it. This also points out the richness and hidden beauty that emerge from the challenging matter of movement in/of mathematical concepts, which may be infinite source of delight or, as Châtelet would say, “enchant(e)ment”.

References

- Bryant, P. (2009). Understanding Space and its Representation in Mathematics. In T. Nunes, P. Bryant & A. Watson (Eds.), *Key Understandings in Mathematics Learning*. London: Nuffield Foundation. Online: <http://www.nuffieldfoundation.org/sites/default/files/P5.pdf>
- Châtelet, G. (1993/2000). *Les enjeux du mobile*. Paris: Seuil (English Transl. by R. Shore & M. Zagha, *Figuring Space: Philosophy, Mathematics and Physics*. Dordrecht: Kluwer, 2000).
- de Freitas, E., & Sinclair, N. (2014). *Mathematics and the body: Material entanglements in the classroom*. New York: Cambridge University Press.
- de Freitas, E., & Ferrara, F. (2015). Movement, memory and mathematics: Henry Bergson and the ontology of learning. *Studies in Philosophy and Education*, 34(6), 565–585.
- Edwards, L., Ferrara, F., & Moore-Russo, D. (Eds.) (2014). *Emerging Perspectives on Gesture and Embodiment in Mathematics*. Charlotte: Information Age Publishing.
- Hwang, S., & Roth, W.-M. (2011). *Scientific & Mathematical Bodies. The Interface of Culture and Mind*. Rotterdam: Sense Publishers.
- Maheux, J.-F., & Proulx, J. (2015). *Doing|mathematics*: Analysing data with/in an enactivist-inspired approach. *ZDM Mathematics Education*, 47(2), 211–221.
- Nemirovsky, R., Kelton, M.L., & Rhodehamel, B. (2013). Playing mathematical instruments: Emerging perceptuomotor integration with an interactive mathematics exhibit. *Journal for Research in Mathematics Education*, 44(2), 372–415.
- Radford, L. (2013). Sensuous cognition. In D. Martinovic, V. Freiman & Z. Karadag (Eds.), *Visual Mathematics and Cyberlearning* (pp. 141–162). New York: Springer.
- Radford, L., Edwards, L., & Arzarello, F. (2009). Introduction: Beyond words. *Educational Studies in Mathematics*, 70(2), 91–95.
- Roth, W.-M., & Mahuex, J.-F. (2015). The stakes of movement: A dynamic approach to mathematical thinking. *Curriculum Inquiry*, 45(3), 266–284.
- Sheets-Johnstone, M. (2009). Animation: the fundamental, essential, and properly descriptive concept. *Continental Philosophy Review*, 42(3), 375–400.
- Sheets-Johnstone, M. (2011). *The Primacy of Movement* (2nd Ed.). Amsterdam: Benjamins.
- Streeck, J., & Mehus, S. (2005). Microethnography: The study of practices. In K. L. Fitch & R. E. Sanders (Eds.), *Handbook of Language and Social Interaction* (pp. 381–404). Mahwah: Erlbaum.
- Stylianides, A. J., & Stylianides, G. J. (2013). Seeking research-grounded solutions to problems of practice: classroom-based interventions in mathematics education. *ZDM Mathematics Education*, 43(3), 333–341.

USE OF STUDENT-PRODUCED VIDEOS IN THE TEACHING OF COMBINATORICS

Anders Støle Fidje

University of Agder

This study aims to identify and characterise different orchestrations used by a teacher in a mathematical discussion with regards to student-produced videos. Brown's (2009) degrees of artifact appropriation and the documentational approach (Gueudet & Trouche, 2009) was used to identify key aspects of these orchestrations. The findings show that the teacher used the videos and their contents in distinctly different ways, capitalising on the affordances of the videos.

INTRODUCTION AND CONTEXT

How can technology be used efficiently and innovatively in mathematics teaching? This question is at the forefront of the Digital Interactive Mathematics Project (RFF Agder, 245499). The project aims to develop innovative teaching in a digital interactive learning environment at lower secondary mathematics teaching.

This paper reports from a pilot study within the DIM-project investigating teachers' use of student-produced videos in plenary sessions. While there has been some research on teacher produced instructional video (Bruce & Chiu, 2015; Guo, Kim, & Rubin, 2014) and students' production of their own videos (Kearney & Schuck, 2004), the notion of how the teacher can use student-produced videos to facilitate students' learning of mathematics with the aim to promote conceptual understanding (Hiebert & Lefevre, 1986) is still an unexplored area of research in mathematics education.

For this research report, empirical material from one teacher's use of students' produced videos was analysed. The videos consisted of explanations and thoughts regarding inquiry-based combinatorics group tasks. The focal point is how the teacher uses the videos in various instructional episodes in a full class discussion. An instructional episode is a sequence of the classroom activity directed towards a set instructional goal (Brown, 2009). The next episode starts when the goal, or in some cases the method of instruction, changes. Within these episodes, I define orchestrations as the different the interactions between teacher, videos and/or students.

The research question addressed in this paper is: Which orchestrations can be identified in a full class discussion using student-produced videos, and what are their characteristics?

THEORETICAL FRAMEWORK

In this paper, full class mathematical discussions are analysed using the documentational approach to mathematical didactics (Gueudet & Trouche, 2009), and Browns

(2009) notion of teacher's pedagogical design capacity. I identified and characterised the instructional episodes in the teacher's orchestration. To do this, I needed to investigate the teacher's use of resources in these orchestrations.

When teachers plan their lessons, they use a variety of resources. Gueudet and Trouche (2009) define everything from textbooks, discussions with peers, online forums to feedback from students as resources for teaching. In this paper, the student-produced video was an important resource for planning a full class follow-up discussion on combinatorics. It would, however, be cynical to think that this was the only resource used by the teacher to plan the lesson.

The teacher's work of gathering a set of resources is however not enough for teaching mathematics in a classroom. The teacher also needs to know how and when to utilise the affordances of the various resources. Gueudet and Trouche (2009) explain that the teacher builds schemes of utilisation to guide actions towards the resource in the classroom. Through the process of mutual adaptation called the documentational genesis, the teacher learns the potential of each resource, culmination in what the researchers call a document. As an example, a teacher might introduce the Pythagorean Theorem in a number of ways. He might use a puzzle (as a resource) to show the geometric proof of the relationship. He might rather use a textbook (as a resource) with a traditional text-based argument, or he might want to utilise the dynamic features of GeoGebra (as a resource), showing the students how the relationship maintains while changing the sides and angles of the right-angled triangle.

In search for theory I have come to realise that the teachers' use of the student-produced videos as a resource for instruction mirrors how curriculum materials and artifacts are evaluated and appropriated by the teacher. While all teachers use resources to plan and enact instructional episodes in pursuit of their instructional goals, they use resources in different ways. The teacher's process of making meaning of an artifact is a dual process of influence. The artifact, "with its constraints and affordances, influence the teacher, and the teachers, through their perceptions and decisions, mobilise the curriculum artifacts" (Brown, 2009, p. 23). This process of mutual adaptation mirrors the process of the documentational genesis described above.

Brown (2009) distinguishes three degrees of curriculum resource orchestration; offloading, adapting and improvising. If the teacher offloads, the curriculum material is used as it is. The teacher might instruct the students to read about a particular topic in the book or watch a specific video, choosing not to add to the material's presentation. If the material needs clarification or modification, the teacher might adapt the content. The teacher might disregard the material, and craft instructional episodes without specific guidance from the resources, what Brown labels improvising. The ability to perceive and mobilise the resources is what Brown defines as the teacher's pedagogical design capacity. The idea being that while the resources are important in the setting of the classroom, the teacher's ability to use them suitably is at least as important.

METHODOLOGY

In preparation for this study, I collaborated with three Norwegian lower secondary school ninth grade mathematics teachers, all participants in the DIM-project. The DIM-project is a developmental research project (Gravemeijer, 1994), where researchers and teachers work together to create inquiry-based mathematics tasks with a technology aspect. By inquiry-based tasks, I refer to tasks that engage the students in problem-solving, conjecturing, exploration and generalisation (Diezmann, 2004).

In this paper, I present an analysis of the classroom discussion of one of these teachers. The reason using this one teacher is that the other two suffered from technology mishaps and time constraint, and could not carry out the lesson as planned.

The whole class discussion analysed was the ending of a two-part lesson in combinatorics with tasks designed by the three teachers. The lesson started with warm-up tasks and a whole class discussion about these specific tasks. The following excerpts are warm-up tasks used in the combinatorics lesson.

“How many different passwords can we make using four digits?” and “How many different passwords can we make using four symbols from both digits and letters?”

The students then split into groups of four (or less) and worked on inquiry-based combinatorics tasks about different lock-passwords as above. After an extended work period, the class reconvened and discussed the tasks they had solved. This session was used to shed light on some of the strategies. The groups then solved new similar tasks and used these to explain their strategies in student-produced videos.

A few days later, the topic of combinatorics was revisited in a 45-minute full class discussion. The teacher used the student-produced videos while planning this section. This follow-up discussion is the basis of this paper.

I collected qualitative data on all the classroom activities. In the full class sessions, I positioned myself in the back of the classroom with a video camera and placed an audio recorder on the front side of the classroom. In the group work sessions, I followed one group and documented their process of solving tasks and the production of a video explaining their strategies. The teacher gave me his notes for the final discussion. This included a detailed lesson plan and specific prompts the teacher wrote for himself. Excerpts of this document are used when appropriate.

In this pilot study, I aim to identify key aspects of the teacher’s orchestrations and construct a framework for categorising, describing and recognising the various aspects of the teacher's orchestration of instructional episodes. All the data was imported into nVivo. To get an overview of the data I started with data reduction and used this to frame a rough sketch of the categories presented in the next section. After recognising key aspects of the categories, I refined the framework applied in this paper.

RESULTS AND DISCUSSION

I start this section with a characterisation of the teacher's way to organise and use the instructional episodes in the full class discussion, and use this to differentiate between the categories regarding the orchestrations with the student-produced videos.

The teacher's organisation and use of instructional episodes

All the instructional episodes of the teacher followed a fixed pattern. The teacher organised the episodes in three main parts. Part one was the *presentation* of a given problem. In the following section, the teacher *elaborated* on the issue at hand with the students. In this section, the teacher used inquiry-based prompts to get a grip on the students' ideas. The excerpts beneath are examples of such questions, taken from the teacher's lesson plan.

“Why is it like that?”, “Did you mean..?”, “Let me repeat what you are saying, and tell me if you agree”.

The last part of the instructional episodes apparently had two objectives; to reach the specific instructional goal, and to connect the idea of this episode with the next problem, i.e. *conclude*. In this section, the teacher asked the students to connect their understanding of the problem and the goal of the instructional episode. While the teacher facilitated this part, he relegated the burden of conveying the ideas to the students. The teacher often asked two or more students to explain the same idea. Figure 1 shows the fixed framework for the instructional episodes.

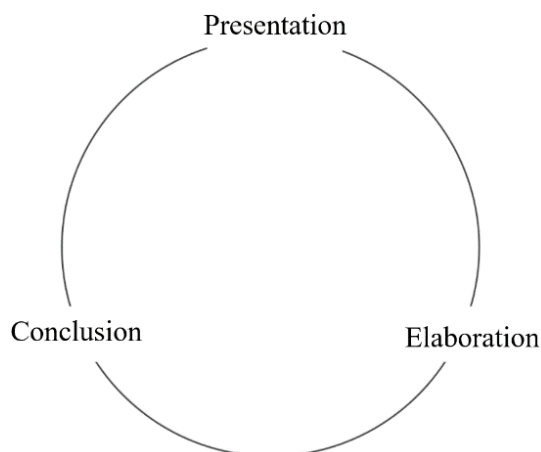


Figure 1: The teacher's fixed pattern for instructional episodes.

While the teacher maintained the fixed pattern of the instructional episodes throughout the lesson, the use of videos varied. In the process of analysis, I looked at the categories from Brown (2009) as a one-dimensional model. This model was used to differentiate the teacher's various orchestrations (see figure 2).

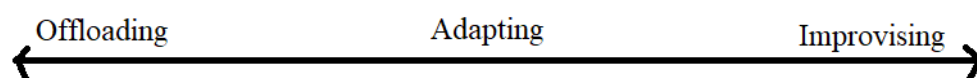


Figure 2: Browns (2009) degrees of orchestrations as a linear model.

Offloading orchestrations

Examples of what I conceive as offloading were mostly present when the teacher used the videos as an introduction to a problem, or as the conclusion to the problem. As the students had worked on the same inquiry-based tasks, there were many different views on the solutions. The teacher then used the videos as a presentation of the problem and possible solutions and follows up with questions to elaborate and lead the discussion into the conclusion. I interpret this as offloading because the teacher used the videos as it is. He did not stop the video to elaborate, and he did not ask the students to evaluate the videos. He used the video as an introduction to expand the problem in the full class discussion.

A student-produced video segment was used to conclude only once during the whole class session. Three students conveying the idea of the multiplication principle in combinatorics live preceded the video, with the video explaining the same idea. The teacher did not add to the video's statements but stated that he thinks it explains the idea thoroughly. The reason for not using more videos as conclusion seems to be that the problems elaborated in the discussion stem from the videos. As the conclusion of each episode is connected to the instructional goals, it would be fair to assume that the videos did not explicitly state a conclusion to the issues at hand.

Adaptation orchestrations

While the teacher did offload some of the instructional aspects of the videos, the most used orchestration was adaptation. Within this category, there were two main orchestrations; video with commentary and student presentation with commentary. In contrast to offloading (use as is), the adaptation of the videos with commentary was an interaction between the teacher and the combination of the video and the producers of said video. The teacher would start the video, pause it, and direct a question to the presenters in the video. Adaptation orchestrations seemed to be used to clarify or to elaborate statements.

The teacher also seemed to use student presentation with commentary when the video suffered from more issues than what the teacher could clarify with questions. The need for a more thorough investigation led the students to redo the presentation with teacher support, mostly in the form of asking questions regarding their statements in the live presentation. While the video was not used to offload the presentation, the teacher still adapted the students' ideas. As the students did not know which questions to expect, this orchestration looked to be strenuous for the students. The teacher's elaboration of the students' thoughts might not be as clear for the students themselves, even though the teacher asked questions directly regarding their presentation. This led to some issues of the presenting students not being able to elaborate on the topic in the direction intended by the teacher. To bring the discussion back on track, towards the instructional goal, the teacher asked the question to another student or the whole class (see hybrid orchestrations).

Hybrid orchestrations

I identified a hybrid form of offloading-adapting as student-led presentations without commentary. The teacher asked a group of students to present something based on their video. While the presenters used their own words to describe the problem at hand, the teacher asked them to clarify and show how they had been thinking. This form of adaptation was mostly used when the video did not show the mathematics the students were presenting. The teacher asked them to “redo” the presentation from the video live with more mediums of instructions. With this orchestration, the teacher adapted the material before the presentation began, and it is therefore not a clear-cut adaptation of the material. This orchestration, as offloading, was mostly used in the presentation part of the instructional episode. Within the model (see figure 3), I have placed this orchestration between offloading and adaptation, as the emphasis is on the students’ presentation without teacher interference.

The other form of hybrid orchestration is adapting-improvising. While the offloading-adapting orchestration was used in the presentation part of the instructional episode, the adapting-improvising orchestration was used to elaborate the issues raised in the presentation part. As the teacher wanted the students to elaborate the issue themselves, and not provide answers, he asked specific questions to specific students. As one student’s ideas might contrast with another, this prompted comments and questions from the other students. In most cases, the teacher’s lesson plan stated the name of the student to be challenged. Whether the teacher selected based on statements in the videos or the teacher’s knowledge of the student’s mathematical competence is hard to determine with the data collected for this paper.

Improvising orchestration

While the first two categories prompted most of the discussion, the teacher did present and prompt some discussions without the student-produced videos as a resource (as offloading or adapting). Even though the videos were not used as a medium for instruction, the issues presented were closely connected to the resource. There is an important distinction to be made between a resource for planning, and a resource for instruction. The improvising orchestrations were all prompted by the presentations in the videos, even though the videos were not used to present or elaborate the questions. If the teacher diagnosed misconceptions in the videos, for example, students use of addition to solve problems calling for the multiplication principle, apparently he used this as a prompt for discussion. This was evident when he was starting an instructional episode by asking “How would we solve this using addition, and how would we solve this using multiplication?”. In this orchestration, the questions were presented to the whole class, and the discussion elaborated the issue of not being able to generalise the use of addition even though it can be used to solve the task. In some cases, this orchestration led to the hybrid of adapting-improvising, as the teacher asked specific students to present their thoughts on the issue.

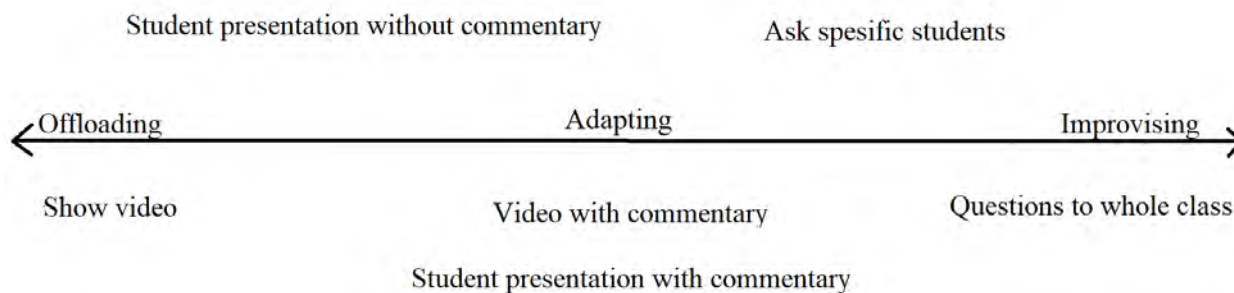


Figure 3: The different orchestrations in the linear model.

FINAL REMARKS AND FURTHER RESEARCH

As this paper reports from a single lesson, only looking at one full class discussion by one single teacher, the orchestration categories presented in this paper will most likely be refined and further elaborated in future research. While videos have been used in mathematics education for some time, the student-produced videos affordances as a resource for following-up teaching needs to be understood to make assertions of their practicality in teaching. By investigating the documentational genesis of teachers, and how the videos fit within a set of resources, I hope to identify aspects of teachers schemes of utilisation while using the resource. The schemes lead to the different orchestrations presented. To fully understand the affordances of the videos, these schemes need to be investigated further. As this paper reports, the teacher chose to adapt the resource in a lot of cases. By investigating teachers' use of videos over a longer period of time, I hope to identify changes in the teacher's documentation work as interactions with the resource become more familiar. The teacher subject in this paper did comment on the strenuous task of planning the discussion while asserting that the oral explanations provided a lot of insight into his students' mathematical competence. This planning period is crucial for evaluating and sectioning the issues elaborated in the discussion.

References

- Brown, M. W. (2009). The teacher-tool relationship. In J. T. Remillard, B. A. Herbel-Eisenmann, & G. M. Lloyd (Eds.), *Mathematics teachers at work: Connecting curriculum materials and classroom instruction* (pp. 17-36). New York: Ruthledge.
- Bruce, D. L., & Chiu, M. M. (2015). Composing with new technology: Teacher reflections on learning digital video. *Journal of Teacher Education*, 66(3), 272-287.
- Diezmann, C. M. (2004). *Assessing learning from mathematics inquiry: Challenges for students, teachers and researchers*. Paper presented at the Mathematical Association of Victoria Conference, Melbourne.
- Gravemeijer, K. (1994). Educational development and developmental research in mathematics education. *Journal for Research in Mathematics Education*, 25(5), 443-471.

- Gueudet, G., & Trouche, L. (2009). Towards new documentation systems for mathematics teachers? *Educational Studies in Mathematics*, 71(3), 199-218.
- Guo, P. J., Kim, J., & Rubin, R. (2014). *How video production affects student engagement: An empirical study of mooc videos*. Paper presented at the first ACM conference on Learning@ scale conference.
- Hiebert, J., & Lefevre, P. (1986). Conceptual and procedural knowledge for teaching on student achievement. In J. Hiebert (Ed.), *Conceptual and procedural knowledge: The case of mathematics* (pp. 1–27). Hillsdale, NJ: Erlbaum.
- Kearney, M., & Schuck, S. (2004). *Authentic learning through the use of digital video*. Paper presented at the Australian Computers in Education Conference.

WHAT VOCABULARY DO TEACHERS USE WHEN ANALYSING THE USE OF REPRESENTATIONS IN CLASSROOM SITUATIONS?

Marita Friesen¹, Carmel Mesiti², and Sebastian Kuntze¹

¹Ludwigsburg University of Education, Germany;

²University of Melbourne, Australia

The International Classroom Lexicon Project has drawn attention to research into the professional vocabulary teachers use when describing classroom phenomena. It is assumed that what teachers identify and interpret when observing classroom situations is not only channelled by their knowledge but also by what they can name. The documentation of professional vocabulary might therefore be of special interest when teachers' analysing of classroom situations is investigated. Building on our prior research, we documented the professional vocabulary used by teachers when analysing classroom situations regarding the use of representations. Although the teachers used specific vocabulary in their analysing results, the documented terms reveal a lack of key aspects regarding theory on dealing with multiple representations.

INTRODUCTION

The professional vocabulary, or lexicon, teachers use when describing classroom phenomena, has currently become a focus of research addressing teachers of mathematics. Research teams in nine countries are documenting the lexicon of their middle school teachers as part of the International Classroom Lexicon Project (e.g., Mesiti & Clarke, 2017). Having a language for describing teaching practice is considered to be not only important in pre-service teacher education, but also for the discussion of in-service teachers' practice and its development within the teaching community. Moreover, Mesiti et al. (2017) argue that teachers' interactions with classroom settings are mediated by their capacity *to name* what they see and experience. The concept of teacher noticing (e.g., Sherin, Jacobs & Philipps, 2011) plays an important role in this context; it is assumed that what teachers identify and interpret in a classroom situation is not only constrained by their knowledge and experience but also by *what they can name* (Mesiti et al., 2017). We therefore conclude that the professional vocabulary teachers use to describe classroom phenomena merits special attention when teachers' *analysing of classroom situations* is investigated. The study presented in this paper brings together two current strands of research on teachers of mathematics with the aim to better access and understand their *analysing of classroom situations*. This includes: (1) research into mathematics teachers' competence of analysing classroom situations regarding the use of multiple representations (e.g., Friesen & Kuntze, 2016) and (2) research into mathematics teachers' lexicon, which is the professional vocabulary they use when describing classroom phenomena (e.g., Mesiti et al., 2017). In this paper, we

describe our approach and the documentation of the vocabulary that $N = 34$ in-service teachers used when analysing the use of representations in mathematics classrooms.

THEORETICAL BACKGROUND AND STATE OF RESEARCH

The use of representations plays a crucial role for students' learning in the mathematics classroom. As mathematical objects are abstract in nature, they can only be accessed by using representations such as: formulae, graphs, diagrams, tables, written and spoken language (e.g., Goldin & Shteingold, 2001). Bruner (1966) coined three stages of representation in which any idea or body of knowledge can be presented to a learner: by action (*enactive* representation), by images or graphics (*iconic* representation) or by symbolic propositions (*symbolic* representation). According to Duval (2006), these stages can be subdivided into so-called representation *registers*, where each register contains some information about the mathematical object it stands for or emphasises certain of the mathematical object's aspects. Since many tasks involve several representation registers and some registers are more efficient for solving problem than others (e.g., Dreyfus, 2002), using *multiple registers* of representations can be regarded as indispensable for the teaching and learning of mathematics: Teachers use different representation registers, for example, when introducing new topics or for explaining whilst students use multiple registers for solving problems and sharing ideas in the classroom (Duval, 2006; Acevedo Nistal et al., 2009).

Numerous studies show, however, that using multiple representations of a mathematical object and changing between them involves high cognitive demands for the learners of mathematics (Ainsworth, 2006; Duval, 2006). Changes between different representation registers, so-called *conversions*, can lead to serious problems in understanding when students fail to see the links between different registers of representations of a mathematical object (Duval, 2006). For this reason, teachers of mathematics have to be able to analyse the use of multiple representations in classroom situations in order to support their students in *connecting* different representation registers when conversions occur (Friesen & Kuntze, 2016). We define such competence of analysing as a teacher's ability "to link relevant observations in a classroom situation to corresponding criterion knowledge so that unconnected changes of representations can be identified and interpreted with respect to their role as potential learning obstacle" (Friesen, 2017, p. 39). In order to elicit teachers' competence of analysing as described above, we used as stimulus material short classroom situations or so-called *vignettes*. These vignettes were presented in three ways, as video material, with comic strips as well as transcribed texts, and teachers were asked open-ended questions to initiate their analysis regarding the use of representations (e.g., Friesen & Kuntze, 2016; Friesen, 2017). The teachers' written responses to this analysing task were coded dichotomously, that means a point was assigned to a teacher's analysing result only if unconnected conversions were identified and interpreted as potential learning obstacles.

In this approach of assessing teachers' competence of analysing, the specific *vocabulary* that teachers made use of in their written analysing results was not taken into

account. Subsequently, in learning of the International Classroom Lexicon Project (e.g., Mesiti & Clarke, 2017), we considered that the documentation of teachers' written lexicon might also provide potential for research into teachers' competence of analysing classroom situations. In the framework of the Lexicon Project, Mesiti et al. (2017) argue that what teachers identify and interpret in a classroom situation is not only constrained by their knowledge and experience but also by *what they can name*. It is assumed that each language user has a personal vocabulary store, or lexicon, from which words for use are selected and to which words he or she encounters are referred to (Field, 2006). Besides automatic associations with closely connected words in a person's lexicon, his or her knowledge in a certain field can create expectations at a more conscious level and thus lead to the activation of relevant vocabulary (Field, 2006). A person's professional vocabulary might consequently be considered as a subset of his or her lexicon which gets activated when corresponding professional contexts are encountered. Following on from this argument, we assumed that the teachers' lexicon for *representations* gets activated when they are asked to analyse classroom situations regarding the use of multiple representations. The teachers' written responses to the analysing tasks could consequently provide insight into the specific vocabulary teachers are familiar with and make use of when analysing classroom situations focusing on representations.

RESEARCH INTEREST AND RESEARCH QUESTIONS

Based on the assumption that teachers' competence of analysing might be related to their use of professional vocabulary, this study aims at documenting the professional vocabulary in-service teachers use when analysing mathematics classroom situations. One feature of the Lexicon documented for Australian middle school teachers is a lack of specificity: all of the 63 lexicon terms refer to general pedagogical practices and none of them identifies a practice unique to the mathematics classroom (Mesiti et al., 2017). We were consequently also interested in the specific nature of vocabulary German teachers use when they are prompted to analyse classroom situations regarding the use of representations, a focus which is particularly specific to the mathematics classroom. Our research questions are the following:

When asked to analyse the use of representations in mathematics classroom situations, what vocabulary do teachers use for articulating their analysing results? In particular: Do they use specific terms related to theory on using multiple representations when they refer, e.g., to stages of representations, representation registers, changes of representations or the connection of registers?

DESIGN, SAMPLE AND METHODS

In order to answer these research questions, we used a vignette-based test instrument comprising of six classroom situations in the domain of teaching fractions in grade six (Friesen & Kuntze, 2016; Friesen, 2017). Three of the classroom situations were designed as texts (teacher-student dialogues) and three in the form of comic strips (see

Friesen, 2017, p. 60 for examples; available online). All classroom situations had a similar plot: A small group of students ask for help and show their teacher how they have started to solve a problem using a certain register of representation (e.g., a calculation). The teacher tries to support the students' understanding by changing to another register (e.g., a pictorial representation, see Figure 4 for examples). The teacher in the classroom situations does, however, not support the students in linking the different representations to each other and leaves the students with unconnected conversions that can be seen as potential further learning obstacles. Each classroom situation was followed by an open-ended question: *How appropriate is the teacher's response in order to help the students? Please evaluate regarding the use of representations and give reasons for your answer* (e.g., Friesen, 2017, p. 63).

The sample of the study consists of $N = 34$ in-service mathematics teachers from seven different secondary schools in the south of Germany (64.7 % of them female, $M_{age} = 38.04$, $SD_{age} = 9.51$). Their experience in teaching mathematics ranges from two to 31 years ($M_{exp} = 8.98$, $SD_{exp} = 5.93$) and most of them (88.2%) have experience in teaching fractions. The majority of the teachers (76.5%) are also teacher educators and supervise young teachers of mathematics in their induction phase (first 18 months of teaching at secondary schools). Therefore, it can be assumed that those teachers are well-experienced in observing classroom situations and discussing phenomena of the mathematics classroom with their colleagues.

As each of the 34 in-service teachers analysed six classroom situations regarding the use of representations, 204 written answers could be examined for this study. In a first step, lexical items (nouns, verbs, adjectives, adverbs) related to our research questions were extracted as they carry the meaning or content of the teachers' analysing results (Field, 2006; Yule, 2016). As the teachers' analysing results also contained terms related to our research questions which were non-lexicalised, i.e. not expressed as a single word (Yule, 2016, p. 275), we included such terms in a second step of analysis (see Figure 1 for examples). The next step was to group the extracted terms in lexical categories which represent increasing levels of specificity according to our research questions: (1) terms related to general pedagogical practices which are not specific to the mathematics classroom, (2) terms related to mathematical content and to practice specific to the mathematics classroom and (3) terms related to representations and their use in the mathematics classroom including stages of representations, representation registers, changes of representations and the connection of registers. If a term could not be clearly allocated to one of the categories, it was discussed with other researchers from our project team until consensus was reached. Based on Field (2006), we finally counted how many times each term occurred within its category in order to find out which terms are used most frequently by the teachers.

<p>→ Darstellung ist stimmig, ebenso verbale Erläuterungen, die Kinder verstehen Bild + Erklärung</p> <p>→ Evtl. hätte es an dieser Stelle genügt zu fragen, wie oft 5 in 13 steckt („2x“), und wie viel Rest dann noch bleibt („3“), dies kann man dann symbolisch aufgreifen</p> $\frac{10}{5} + \frac{3}{5} = 2 + \frac{3}{5} = 2\frac{3}{5}$ <p>hier ist evtl. kein Wechsel der Darstellungsform nötig</p>	<p>→ representation is appropriate, as well as the verbal explanations, the children understand picture and explanation</p> <p>→ it would perhaps have been sufficient to ask how often 5 goes into 13 (“2x”), and how much is left (“3”), this could be taken up symbolically</p> <p>a change of representations is perhaps not necessary in this case</p>	<p>representation verbal explanation (to) understand picture explanation (to) ask (to) take up symbolically change of representations</p>
--	---	---

Figure 1: Sample answer 1 (vignette 5), translation and extracted terms

RESULTS

Data analysis as described above resulted in the extraction of 917 terms which were organised in lexical categories as follows: (1) terms related to general pedagogical practices: 48.7 %; (2) terms related to mathematical content and to practice specific to the mathematics classroom: 17.6 %; and (3) terms related to representations and their use in the mathematics classroom: 33.7 %. Figure 2 shows the terms that were most frequently used to describe general pedagogical practice.

term	frequency	term	frequency
(to) explain, explanation	68	problem	12
(to) understand, understanding	42	example	11
(to) help, help	22	(to) answer, answer	10
(to) ask (a question)	14	task	10

Figure 2: Terms describing general pedagogical practice

The terms that teachers used most frequently to describe content and practices of the mathematics classroom are shown in Figure 3. Five out of the eight terms are content and activities related to the learning of fractions, which reflects the common topic of the analysed classroom situations.

term	frequency	term	frequency
(to) calculate, calculation	32	denominator	9
mathematical language	19	numerator	6
fraction	13	part (of the whole)	6
(to) multiply, multiplication	12	(to) reduce a fraction	6

Figure 3: Terms for mathematical content / practice in the mathematics classroom

The terms that were used most frequently for describing the use of representations in the mathematics classroom were further subdivided based on the theoretical background and the research questions of this study. Accordingly, Figure 4 shows the terms that were most frequently used in the lexical subcategories *stages of representations* and *registers of representations*. The terms used by the teachers for different stages of representations strongly reflect theory as introduced by Bruner (1966, see theoretical background). The teachers used, however, also different terms for describing the same stage (e.g., also *pictorial* and *graphical* for the iconic stage of representation or *formal* and *abstract* for the symbolic stage). Bruner's stages were also complemented using terms such as *verbal* and *real-world*, which can be regarded as registers of representation as introduced by Duval (2006). The subcategory *registers of representations* (Fig. 4) reveals that the rather general terms *representation* and *drawing* were often used to describe pictorial registers shown in the classroom vignettes, whereas more specific terms (e.g., *bar* in vignette 2) occurred less often.

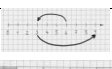
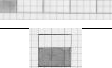
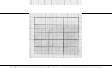



Terms used for describing stages of representations			Terms used for describing registers of representations			
term	frequency	samples	vignette	sample	term	frequency
stage (of representation)	16	work on the formal stage, draw on an iconic stage	1-6		representation	89
enactive	6	enactive representation, enactive example			drawing	18
pictorial	14	pictorial representation of a task, pictorial representation with a rectangle; iconic example, represent iconically, work on the iconic stage; graphical representation, graphical support	1		number line	24
iconic	10		2		diagram / bar	8 / 9
graphical	6		3		rectangle / square	3 / 4
symbolic	5	symbolic stage, symbolic representation; formal stage, formal explanation	4		small boxes	8
formal	5		5		pizza / pie chart	17 / 9
verbal	26	verbal support, verbal explanation for the drawing, verbal help, connect verbal and iconic representations	6		folding	17
real-world	6	real-world situation				
illustrative	21	illustrative example, illustrative drawing; abstract explanation, abstract example				
abstract	5					

Figure 4: Terms for describing stages of representations and different registers

The terms which were used by the teachers most frequently for describing *changes of representations* and *connections* between different registers are shown in Figure 5. It can be seen that only few terms were repeatedly used and that changes of representations were mostly described by using constructions with *explaining*. Teachers referred to changes of representations also with terms such as *illustration* and *visualisation*. Comparably, terms for describing the connection of representations were rarely used by the teachers.

Terms used for describing changes of representations			Terms used for describing the connection of representations		
term	frequency	samples	term	frequency	samples
(to) illustrate, illustration	7 7	illustrate the calculation, illustrate with a drawing, illustration by means of folding	(to) connect to	8	connect to the students' representation/understanding/idea/knowledge
(to) explain (by means of), explanation	10	explain on the number line, explain with a representation, verbal explanation for the drawing, connect verbal and iconic representations	(to) refer to	7	refer to the students' representation/ drawing/problem of understanding/question at hand
(to) visualise, visualisation	3 5	visualise the calculation, visualisation of the problem			
(to) change (representations), change	3 4	change the representation, change to the iconic stage, change from mathematics to a real-world situation			

Figure 5: Terms for changes of representations and the connection of registers

DISCUSSION

The aim of this study was to document the professional vocabulary teachers use when analysing classroom situations regarding the use of representations. The findings reveal that prompting the teachers to pay attention to this specific focus can elicit the use of corresponding specific vocabulary; about one third of the documented terms refer to representations and their use in the mathematics classroom. However, a more careful look at the lexical subcategories indicates that most of these terms were related to stages of representations or registers of representations. The importance of conversions and the connection of registers for students' learning of mathematics was not reflected in the frequency of corresponding specific vocabulary; only about 60 out of 300 terms on the use of representations were related to these essential aspects. This study has taken a small first step towards documenting teachers' lexicon regarding the use of representations, and warrants further research into this topic. Considering the argument that teachers are not likely to identify and interpret classroom phenomena that they cannot name, teachers' lack of capacity to use vocabulary specific to theory on the use of representations might also constrain their competence of analysing. Consequently, the relation between teachers' professional vocabulary and their corresponding competence of analysing should be investigated in more depth, as this might also inform research into competence development.

Acknowledgements

This study is part of the project EKoL and supported by Ludwigsburg University of Education and the Ministry of Science, Research and the Arts in Baden-Wuerttemberg.

References

- Acevedo Nistal, A., van Dooren, W., Clareboot, G., Elen, J., & Verschaffel, L. (2009). Conceptualising, investigating and stimulating representational flexibility in mathematical problem solving and learning: a critical review. *ZDM Mathematics Education*, 41(5), 627–636.

- Ainsworth, S. (2006). A conceptual framework for considering learning with multiple representations. *Learning and Instruction*, 16, 183-198.
- Bruner, J. S. (1966) *Toward a Theory of Instruction*. Cambridge, US: Harvard University Press.
- Dreyfus, T. (2002). Advanced mathematical thinking processes. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 25-41). Kluwer: New York.
- Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics. *Educational Studies in Mathematics*, 61, 103-131.
- Field, J. (2006). *Psycholinguistics*. Routledge: London.
- Friesen, M. & Kuntze, S. (2016). Teacher students analyse texts, comics and video-based classroom vignettes regarding the use of representations – Does format matter? In Csíkos, C., Rausch, A., & Szitányi, J. (Eds.), *Proceedings of the 40th Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 2), (pp. 259–266). Szeged, Hungary: PME.
- Friesen, M. (2017). *Teachers' competence of analysing the use of multiple representations in the mathematics classroom and its assessment in a vignette-based test*. Dissertation study. <https://phbl-opus.phlb.de/frontdoor/index/index/docId/545>
- Goldin, G., & Shteingold, N. (2001). Systems of representation and the development of mathematical concepts. In A. A. Cuoco & F. R. Curcio (Eds.), *The role of representation in school mathematics* (pp. 1-23). Boston, Virginia: NCTM.
- Mesiti, C., & Clarke, D.J. (2017). Structure in the professional vocabulary of middle school mathematics teachers in Australia. In A. Downton, S. Livy, & J. Hall (Eds.) *Proceedings of the 40th Annual Conference of the Mathematics Education Research Group of Australasia* (pp. 373-380). Adelaide, MERGA.
- Mesiti, C., Clarke, D.J., Dobie, T., White, S., & Sherin, M. (2017). “What do you see that you can name?” Documenting the language teachers use to describe the phenomena in middle school mathematics classroom in Australia and the USA. In B. Kaur, W.K. Ho, T.L. Toh, & B.H. Choy (Eds.) *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education (Vol. 3)*, (pp. 241-248). Singapore: PME.
- Sherin, M.G., Jacobs, V.R. & Philipp, R.A. (2011). *Mathematics Teacher Noticing: Seeing Through Teachers' Eyes*. New York: Routledge.
- Yule, G. (2016). *The study of language*. Cambridge: University Press.

SOLVING ARITHMETIC-ALGEBRAIC WORD PROBLEMS BY 10- TO 12-YEAR-OLD STUDENTS

Torsten Fritzlar and Nadja Karpinski-Siebold

Martin Luther University Halle-Wittenberg, Germany

Arithmetic-algebraic problems are mathematically challenging problems which can be algebraically solved using variables and equations, but also with approaches traditionally described as arithmetic. Therefore, they are particularly suitable for investigating and, simultaneously, encouraging the emergence of algebraic thinking. In the presented study we investigate how 10- to 12-year-old students work on such problems. Analyses are focussed on the dealing with unknowns which is considered as a key element of algebraic thinking.

THEORETICAL FRAMEWORK

Importance and Difficulty of Algebra

Algebra is considered as the key path to higher mathematics. Hence there is a strong concern about many students having difficulties with algebraic contents, a fact which has repeatedly been established in case studies and larger comparative analyses. A main cause for these diverse difficulties is seen in the discontinuities between arithmetic and algebra; moreover, seeming continuities frequently also involve shifts in meaning and expansions (e.g. Carpenter et al., 2005). Various Early Algebra- programmes are being developed worldwide to reduce the obvious difficulties that students have with algebra. These programmes aim to unite arithmetic and algebraic perspectives (with different weights) from a very early stage. To further elaborate this approach, however, it appears necessary to conduct detailed studies on the emergence of algebraic thinking in young students.

Early Algebraic Thinking and Unknowns

Algebraic abilities at primary school age become apparent in early algebraic thinking, for which we have postulated the following components (Fritzlar & Karpinski-Siebold, 2011): *Dealing with operations (as objects) and their inverses; establishing relations between numbers, sets and relations (relational thinking); generalizing; dealing with changes; using symbolic representations; dealing with unknowns.* The last component is obviously of particular importance. In this context we can further differentiate between constellations in which a relation between two unknowns has to be established or used, and those in which unknowns must indeed be treated as known mathematical objects. For the latter, we could consider equations such as, for example, $ax + b = cx + d$, which requires equivalence transformations to be solved. Experience has shown that this is extremely challenging for primary school children, and some scholars even speak of a “cognitive gap” (Herscovics & Linchevski, 1994) or “didactical cut” (Fillooy

& Rojano, 1989) in this context. For Filloy, Rojano & Puig (2008) this is the cut-off point between arithmetic and algebra.

A number of studies about first steps into algebra which concentrate on the formal level of equations already exist (e.g. Herscovics & Linchevski, 1994; Filloy et al., 2008). In our research and development project we focus on solving word problems in a narrow neighborhood of the cut. We want to investigate in detail students' solutions, approaches and strategies in handling arithmetic-algebraic problems which involve aspects of algebraic thinking, but can be approached both algebraically and arithmetically. In a second step, our aim is to develop and evaluate teaching sequences providing experiences in dealing with unknowns and facilitating the advancement of corresponding abilities.

Arithmetic-algebraic problems

There are many reasons for considering problem solving an important approach both to developing students' algebraic competences and to making the emergence of algebraic thinking accessible for scientific studies. Apart from its historical role as pacemaker for algebra and teaching algebraic methods as well as being a motivator, appropriate problems allow students to construct new (partly algebraic) approaches while building upon previous (arithmetic) experiences. Additionally, these new approaches can be experienced by students as useful and efficient compared to often very complex and laborious arithmetic procedures (cf. also Bednarz & Janvier, 1996).

How can suitable problems be found or designed to investigate and foster the emergence of algebraic thinking? There are, for instance, some more traditional context-related problems such as scale problems or specific constellations with number pyramids, which can be already used in primary school. Especially for or older students, Bednarz and Janvier (1996) describe frequently used problem types for the transition from arithmetic to algebra: unequal partition problems, problems involving a magnitude transformation and problems involving a relation between non-homogeneous magnitudes and a rate. Additionally, systems of equations used in pertinent research studies (e.g. Herscovics & Linchevski, 1994) can be used as a basis for constructing corresponding word problems. Also in this case, attention should be paid to a familiar context and the lowest possible linguistic complexity of the problem text.

RESEARCH QUESTIONS AND METHODS

In this paper we can only describe a small part of our ongoing investigations. We will focus on so-called unequal partition problems which describe relatively complex situations with two unknowns; nevertheless, they are accessible also to younger students. In these problems, a known sum is divided into two or more unknown parts, for which additive or multiplicative relations are described. According to Bednarz and Janvier (1996), they can also be referred to as disconnected problems because no direct (or gapless) relations between the unknown and known values are provided. This is why

unequal partition problems are also particularly suitable for studying how young students deal with unknowns.

As a first step in this direction, students' approaches to those problems can be differentiated into *arithmetic* (beginning with given data and connecting them step by step), *algebraic* (representing the problem by equations and working on them) and *explorative* (trying particular data, possibly deriving variations of them). But there is a need for a much more detailed analysis. For instance, arithmetic approaches can be very different concerning the constructed relations between givens and unknowns and the introduced new quantities; and depending on the chosen approach, further requirements on the problem solver arise. A more detailed analysis of algebraic approaches can start from the model of proto-algebraic levels of mathematical thinking by Aké et al. (2013).

We here report on the part of our studies which employed the following unequal partition problems and investigated in detail how high-achieving and interested students from grade 4 to 6 worked on it:

Problem 1: Paul buys a textbook and a dictionary. The textbook costs €3 more than the dictionary. He pays with a €50 note and gets €5 back. Calculate the price for the textbook and for the dictionary!

Problem 2: Amelie buys a book with a €20 note. She gets back exactly €5 less than the book costs. Calculate the price of the book!

From our perspective, it is particularly interesting to give students *both* problems to solve in the same test. On the one hand, they have the same underlying mathematical structure in essence (division into two additively compared parts), on the other hand, we think that this analogy is not easy to recognize. On the contrary, the second problem could be significantly more difficult for less experienced students, for several reasons. First, it is certainly linguistically more difficult; further factors increasing its level of difficulty partly depend on the respective solution strategy. Thus, the price of Amelie's book is not an even amount of Euros, which usually makes an explorative approach more complex, but only comes with minimal extra calculation efforts when taking other approaches. If a problem solver proceeds arithmetically, the first problem, with $50 - 5 = 45$, has an easy entry, which requires the calculation of two given values according to the formulation in the text. Furthermore and in particular, the two unknowns (prices) in the first problem are of the same kind and associated with concrete and equal objects (books). As Amelie is returned €5 less than the book costs, the second problem rather describes a process and requires the mental construction of a second quantity (the change) without reference to a concrete object in the described situation. Only through introducing this second quantity the problem becomes an unequal partition problem. In contrast to the first one, the relation between the book price, the change and the overall sum of €20 remains implicit here.

For describing the problems (in their given form) and later on the students' approaches we use a *structure graph* modified from Filloy et al. (2008) which represents the re-

lations by edges, the data by filled circles and required unknown quantities by unfilled squares; unfilled circles are used for new quantities, established by the problem solver, and dotted edges for new relations. On that basis, both problems can be described by the following graphs visualizing similarities and differences:



Fig. 1: Graphs describing problem 1 and 2

39 fourth-graders worked on the problems in a federal state mathematics Olympiad. They had qualified through outstanding performance in the preceding rounds at school, city and district level. Additionally 21 fifth- and 23 sixth-graders attending a special grammar school with a focus on mathematics and science took part in the investigation.

Our research interest was to investigate the students' solution strategies and approaches and their ways of dealing with the unknowns. Therefore, the students were asked to solve the problems individually and their worksheets were analysed in detail. In a first step, both authors analysed and categorised the students' approaches individually i.e. independently of each other. In the second round, the authors met to discuss and agree on cases which had not been allocated to the same category (consensual validation). Categories were derived in a deductive, theory-based approach (see above) and then inductively specified further by drawing on the empirical material. This creates opportunities to validate and extend the current conceptual framework (cf. Hsieh & Shannon, 2005).

To cover and appreciate as much of the observable variety in the presented approaches as possible, we investigated high-achieving and interested students up to grade 6. This age group appears particularly interesting for our purposes as they have not been systematically introduced to algebra yet.

IMPORTANT RESULTS

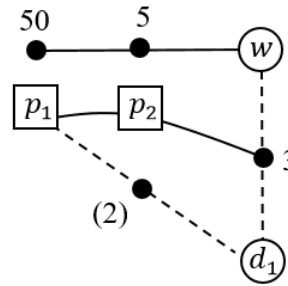
Problem solving approaches of 4th-graders

The majority of the participating 39 mathematically high performing 4th-graders was able to solve both problems. As expected, while 87% of the students successfully solved the first problem, the success rate was considerably lower for the second problem at 62%. With two exceptions (due to calculation errors), all students who solved the second problem also successfully worked on the first problem.

In the following, we describe all identified approaches successfully applied by 4th-graders to solve the two problems:

“sum of parts” (Problem 1)

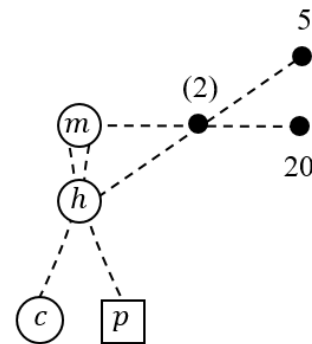
$$\begin{aligned}
 50€ - 5€ &= 45€ \\
 45€ - 3€ &= 42€ \\
 42€ : 2 &= 21€ \\
 21€ + 3€ &= 24€
 \end{aligned}$$



First, the difference between the two parts is subtracted or respectively added to the sum. Dividing this sum by 2 results in the smaller or respectively larger part. On this basis, it is then possible to determine the remaining part.

“share and adjust” (Problem 2)

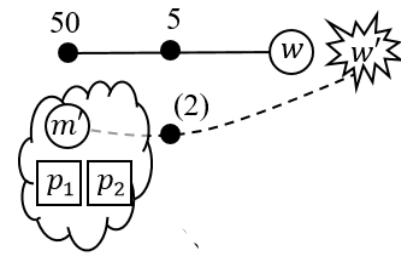
$$\begin{aligned}
 20 : 2 &= 10€ \quad 56,50€ : 2 = 28,25€ \\
 10,00€ - 2,50€ &= 7,50€ \text{ (Rückgeld)} \\
 10,00€ + 2,50€ &= 12,50€ \text{ (Buch)}
 \end{aligned}$$



The overall sum is first divided into two equal parts. These are then corrected by adding or respectively by subtracting half the difference.

“improve approximate solution” (Problem 1)

A variation of this method is shown in the following figure. Since uneven numbers cannot be divided by 2, approximate values are used as a first approach and then corrected (partly probing).



$$\begin{aligned}
 50€ - 5€ &= 45€, 44 : 2 = 22 + \text{Cann probieren mit z.B. } -1 \text{ oder } +2 \text{ und den einen Euro wieder dazu rechnen?} \\
 44 : 2 &= 22
 \end{aligned}$$

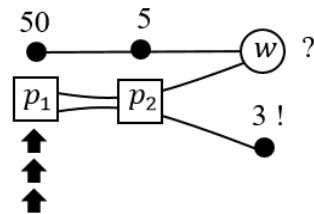
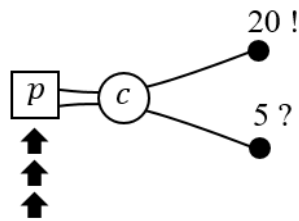
Fig. 2: Some solution strategies by 4th-graders

Fig. 2 shows that students find approaches for both disconnected problems which are traditionally referred to as arithmetic. They establish new relations and new quantities which are calculated stepwise until, *in the end*, the required unknowns can be determined. This takes a minimum of three steps – indicating the mathematical complexity of these approaches – which constantly have to be checked semantically. As the approaches also use given and new relations between unknowns, they can be considered as early algebraic thinking according to Fritzlar & Karpinski-Siebold (2011). However, we could not identify any *operating* with unknowns.

In the following solution strategies, values are assumed for the unknowns and test calculations are carried out.

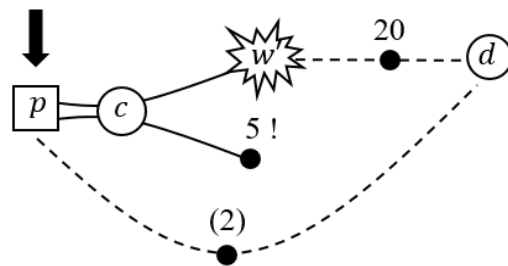
“guess – check – improve” (Problem 1 and 2)

Students try systematically in so far as supposed start values fulfilling one condition are adjusted until also the other condition is fulfilled.

“method of false position (additive)”

(Problem 2)

*A Antwort: Das Buch kostet 12,50 €.
 $12 - 5 = 7$ $12 + 7 = 19$ $(20) 19 + 1 = 20$
 5 € 5 €*



Based on the given relation, concrete values for the two parts are assumed. The difference between the resulting sum and the given total value is then used to derive the necessary correction of the initial assumption.

Fig. 3: Further strategies used by 4th-graders

As many students did not note down their solution strategy, only 25 correct solutions to the first problem could be assigned to one of the described solution process categories. In 12 out of 25 cases, the “sum of parts” approach was used; 11 students used the “share and adjust” approach which seems to be much more difficult referring to the structural graph. Possibly, it is encouraged by the context of a simple purchasing act.

The number of categorizable solutions was even lower for the second problem. What is noteworthy is that considerably more students applied probing to solve this problem. Even though we think that the initial mentioning of the overall sum of €20 and the division into price and change would lead students to using the “share and adjust” approach (at least for mathematically experienced problem solvers), it is only used by three students in our sample. Both can be seen as further indicators of the higher difficulty level of the second problem for 4th-graders.

Only four students used the same solution strategy for both “buying book” problems. This implies that the structural analogy between both problems could not be constructed, or at least not used, by the large majority of fourth-graders.

Problem solving approaches of 5th- and 6th-graders

The mathematically strong 5th- and 6th-graders participating in our study reached very high solution rates in both problems, ranging between 85% and 96%. The results of this group of students no longer point towards a higher difficulty level of the second problem.

The only new approach we could identify was the use of letters. The left part of Fig. 4 shows an interesting solution proposed by 5th-grader Sophia. She first proceeds arithmetically using the “share and adjust”-approach, while the subsequent algebraic solution strategy draws on the “sum of parts”-approach. This change obviously presents no difficulty for her; she also uses both approaches to solve the second problem. Her second solution starts with the construction of a global cognitive model of the situation involving the specified quantities, the unknowns and their relevant relations. The model will be formalized in an algebraic expression. In the further solution process, Sophia replaces one unknown by another, using a relation between the unknowns. Her notation however does not show an operation with unknowns like given numbers. Therefore, Sophia’s solution can be assigned to level 2 of algebraization according to Aké et al. (2013). By contrast, 6th-grader Nina is the only student in the sample who employs both substitution and equivalence transformations (both operations have not been explicitly dealt with in previous mathematics classes). Her solution can be assigned to level 3 of algebraization.

$50 - 5 = 45 \text{ €}$ $45 : 2 = 22,50 \text{ €}$ $3,00 \text{ €} : 2 = 1,50 \text{ €}$ $\begin{array}{r} 22,50 \\ - 1,50 \\ \hline 21,00 \end{array} \quad \begin{array}{r} 22,50 \\ + 1,50 \\ \hline 24,00 \end{array}$	$50 - 5 = 45$ $\text{smaller (€)} + \text{coarser (W)} = 45$ $S = W + 3$ $W + W + 3 = 45 \quad -3$ $W + W = 42$ $42 : 2 = 21 = W$ $21 + 3 = S = 24$	$X + Y = 50 - 5 = 45$ $X + Y = 45$ $X = Y + 3$ $Y + Y + 3 = 45$ $2Y = 45 - 3 = 42 \quad / : 2$ $Y = 21$ $21 + 3 = 24 = X$ $X = 24$
--	---	--

Fig. 4: Solutions of 5th-grader Sophia (left) and 6th-grader Nina (right)

Both in the fifth and the sixth grade, students use variables only in individual exceptional cases. We were not able to detect a significant increase in the usage of variables between 5th and 6th-graders.

Taking into account that approximately a third of the 5th- respectively the 6th-graders used the same approach for both problems, we assume that the students easier recognize the problems’ analogy than the 4th-graders. A connection with the use of algebraic expressions is not demonstrable.

DISCUSSION AND OUTLOOK

A great many of 4th-graders can solve the first problem, distinctly more than half of them also the second problem on buying books. Although these problems are disconnected, students find different sophisticated approaches which are classically considered as arithmetic. Thereby, students have to introduce new quantities and establish

and use relations between unknowns; so, algebraic thinking is emerging. However, there is neither an operating with unknowns as with known mathematical objects, nor will variables used for representing the problem situation.

Despite the very high cognitive effort, formal algebraic approaches (in the classical sense) are hardly identifiable in the whole sample of mathematically high-performing students. Even with 5th- and 6th-graders, an operation with unknowns as with known quantities is not yet apparent (with one exception in our sample). However, some students succeed in formalizing a comprehensive situation model by algebraic expressions.

Solution approaches to arithmetic-algebraic problems can be accurately and clearly described by structural graphs which visualize commonalities and differences as well as their different degrees of complexity.

Due to lack of space we can only give an outlook closely related to the presented case study. From our perspective, further investigations are needed with larger samples and further arithmetic-algebraic problems with different degrees of mathematical complexity and varying characteristics concerning e.g. number and type of knowns, unknowns and relations, relation structure, reification of unknowns, dynamic of the situation or existence of entry points. Additionally, clinical interviews could allow detailed insights in involved processes, e.g. concerning discontinuities as in Sophia's solution.

References

- Aké, L. P., Godino, J. D., Gonzato, M., & Wilhelmi, M. R. (2013). Proto-Algebraic Levels of Mathematical Thinking. In A. M. Lindmeier & A. Heinze (Eds.), *Proceedings of the 37th Conference of the IGPME* (vol. 2, p. 2-8). Kiel: PME.
- Bednarz, N., & Janvier, B. (1996). Emergence and Development of Algebra as a Problem-Solving Tool. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to Algebra: Perspectives for Research and Teaching* (pp. 115–136). Dordrecht: Kluwer Acad. Publ.
- Carpenter, T. P., Levi, L., Franke, M. L., & Zeringue, J. K. (2005). Algebra in Elementary School: Developing Relational Thinking. *ZDM*, 37(1), 53–59.
- Filloy, E., & Rojano, T. (1989). Solving Equations: the Transition from Arithmetic to Algebra. *For the Learning of Mathematics*, 9(2), 19–25.
- Filloy, E., Rojano, T., & Puig, L. (2008). *Educational Algebra*. Boston: Springer.
- Fritzlar, T., & Karpinski-Siebold, N. (2011). Algebraic Thinking of Primary Students. In B. Ubuz (Ed.), *Proceedings of the 35th Conference of the IGPME* (vol. 2, p. 345-352). Ankara: PME.
- Herscovics, N., & Linchevski, L. (1994). A Cognitive Gap between Arithmetic and Algebra. *Educational Studies in Mathematics*, 27(1), 59–78.
- Hsieh, H.-F., & Shannon, S. E. (2005). Three Approaches to Qualitative Content Analysis. *Qualitative Health Research*, 15(9), 1277–1288.

ACADEMIC PROCRASTINATION IN THE TRANSITION FROM SCHOOL TO UNIVERSITY MATHEMATICS

Sebastian Geisler and Katrin Rolka

Ruhr-University Bochum, Germany

The transition from school to university is a challenging process for many students, especially in mathematics. This is illustrated by high dropout rates during the first year. In this contribution we discuss the connection between mathematic-related affective variables (different facets of interest in mathematics, self-concept and beliefs concerning the nature of mathematics) and academic procrastination. Moreover, the effect of procrastination on students' dropout intention and their satisfaction with their studies of mathematics during the first semester is analysed. Procrastinating students show less interest in university mathematics. While the beliefs concerning the nature of mathematics turn out to be a predictor of students' procrastination, procrastination itself predicts students' satisfaction with their studies of mathematics.

INTRODUCTION

High dropout rates during the first year at university reveal students' problems within the transition from school to university mathematics. Dieter and Törner (2012) calculate that up to 80% of German mathematics students quit their studies without graduation (35% during their first year). Chen (2013) reports similar numbers for the United States. Furthermore, only few students manage to succeed in their first mathematics exams at university. At Ruhr-University Bochum, where our research took place, over 70% of the students who attended their first exam failed and 25% did not even attend the exam in 2017. Regarding these numbers, the necessity of understanding students' problems in the transition process becomes obvious. In this contribution, we discuss the role of students' procrastination in this process. In particular, we are interested in the connection between mathematics-related affective variables and procrastination as well as in the effects of procrastination on students' dropout intention and their satisfaction with their studies of mathematics.

PROCRASTINATION

Procrastination is a well-known behaviour, not only but especially, among university students (Steel, 2007). Numerous authors with different research traditions have worked on the conceptualisation of procrastination, leading to many definitions with different focuses. In a meta analytic review Steel (2007) suggests a definition of procrastination which is consistent with recent research. He concludes "that to procrastinate is to voluntarily delay an intended course of action despite expecting to be worse off for the delay" (p. 66). This is in line with Glöckner-Rist et al. (2014) who define procrastination in academic settings as a behaviour which occurs when personally

important actions are delayed in favour of less important actions which are not suitable to reach one's objectives. This includes that students begin study-relevant work too late or do not finish it in time.

Asked for their reasons to procrastinate, university students name a wide range of factors from social problems over laziness and lack of motivation to communication problems with the lecturers (Hussain & Sultan, 2010). In this paper, we focus on the relation between affective variables and (academic) procrastination. Especially the role of self-beliefs is addressed in many studies. Significant negative correlations between students' self-esteem, general and academic self-efficacy, academic self-concept and procrastination are found (Klassen et al., 2008; Kiamarsia & Abolghasemi, 2014; Lohbeck et al., 2017). However, the predictive power of these self-beliefs is rather unambiguous and weak. Klassen et al. (2008) argue that self-beliefs which are more domain-specific can be better predictors of procrastination. In a qualitative study, Schraw et al. (2007) identify students' interest to be the most important self-related cause of procrastination. It is therefore not surprising that procrastinating university students show less interest in the subject they are studying and in tasks related to their studies (Lohbeck et al., 2017; Ackermann & Gross, 2005).

The effects of academic procrastination are as diverse as the causes of procrastination. Besides social and psychological effects, the influence on academic achievements is discussed in several studies - with inconsistent results (Kim & Seo, 2015). A survey among Pakistani university students shows that over 90% of the students see low achievements as a result of procrastination. 75% also mention the risk of dropout as a consequence (Hussain & Sultan, 2010). Furthermore, academic procrastination is negatively connected with students' academic life satisfaction (Balkis, 2013).

Regarding the learning of mathematics, academic procrastination is considered to play an important role. This seems plausible, since procrastination occurs when a task is experienced as difficult and many students think of mathematics as a complicated subject (Akinsola et al., 2007). Academic procrastination correlates negatively with academic achievements in mathematics (in terms of grades) in high-school as well as in university (Bakhshayesh et al., 2016; Akinsola et al., 2007). New studies show a connection between mathematics anxiety and academic procrastination among pupils (e.g. Adimora et al. 2017). However, little is known about the relation between affective variables and procrastination in the special case of university mathematics.

THE TRANSITION FROM SCHOOL TO UNIVERSITY MATHEMATICS

Since we are interested in the connection between affective variables, like interest, and procrastination, it seems plausible that the change in the nature of mathematics – as illustrated below – and its effects on these variables have to be considered. The change in the nature of mathematics within the transition from school to university has been widely discussed before (e.g. Daskalogianni and Simpson, 2001; Ufer et al., 2016). Mathematics at school is often focused on solving real world problems and applying previously learned content. New concepts are illustrated with many examples and yield

on an intuitive understanding. Mathematics at university usually starts with the lectures Calculus I and Linear Algebra I. Both are rather theoretical as well as oriented on proof and therefore differ quite much from mathematics taught in school. New definitions are often presented in a formal way and are less illustrated than in school.

Several authors suggest that this change in the nature of mathematics has to be considered when dealing with affective variables in the context of transition (e.g. Ufer et al., 2016). One variable that should be taken into account in this context are the students' beliefs concerning the nature of mathematics. Some students see mathematics as a rather static summary of rules, facts and algorithms. Others consider mathematics to be a dynamic and creative field of research (Törner & Grigutsch, 1994). The students' established beliefs may influence their experiences during the transition. Some students feel a big gap between mathematics in the two institutions school and university (Geisler, 2017). This feeling can be the result of incongruences between their beliefs, established at school, and the mathematics they get to know at university. Daskalogianni and Simpson (2001) talk about "beliefs overhang" in this case. The change in the nature of mathematics also effects students' interest. Ufer et al. (2016) argue that, when examining students' interest in mathematics during the transition, the term "mathematics" has to be specified because students' interest in school mathematics might differ from their interest in university mathematics.

RESEARCH QUESTIONS

The focus of the paper at hand is twofold: we want to shed light on the connection between mathematics-related affective variables and procrastination, especially in university mathematics, and we are interested in the effects of procrastination on the students' satisfaction with their studies of mathematics and their dropout intention.

Connection between mathematics-related affective variables and procrastination

As mentioned above, the transition from school to university mathematics usually goes hand in hand with a change of the nature of mathematics. Therefore, we are interested in the effect of students' beliefs concerning the nature of mathematics and different interest facets on procrastination. We follow the approach of Ufer et al. (2016) and distinguish between interest in school and university mathematics as well as interest in proof and formal representation (as important mathematical content at university). Besides, we want to investigate the connection between students' mathematical self-concept (as a domain-specific self-belief) and their procrastination.

- 1) *In which way do students who tend to procrastinate differ from those who do not procrastinate in their interest in school mathematics, university mathematics, proof and formal representation, their mathematical self-concept and their beliefs concerning the nature of mathematics?*

We expect that procrastinating students show less interest in university mathematics, proof and formal representation (H1). Since mathematics at university differs much from school, we have no special hypothesis concerning the interest in school mathema-

tics. We expect that procrastinating students have a lower mathematical self-concept (H2) as well as a lower acceptance of dynamical beliefs concerning the nature of mathematics but a higher acceptance of static beliefs (H3).

2) In which way can these affective variables predict the students' procrastination?

Based on recent research (Klassen et al., 2008; Kiamarsia & Abolghasemi, 2014; Lohbeck et al., 2017), we do not expect mathematical self-concept to be a strong predictor (H4). We have no special hypothesis about the predictive power of the other variables.

Effects of procrastination on subjective study success in mathematics

Concerning students' success in their studies of mathematics, we distinguish between rather objective measurements of success (grades and examinations) and subjective ones (satisfaction with their studies of mathematics and dropout intention). In this paper, we focus on the latter two – since the students' achievements (in terms of grades and examinations) have been addressed by many studies before with rather inconsistent results as aforementioned.

3) In which way can the level of procrastination predict the students' satisfaction with their studies of mathematics and their dropout intention?

We anticipate procrastination to predict both, the students' satisfaction with their studies of mathematics (H5) and their dropout intention (H6).

METHODS

We used two questionnaires in the lectures Calculus I and Linear Algebra I which take place in the first semester at university. The students' procrastination, interest, self-concept and beliefs have been measured within the first questionnaire during the first week of the semester (T1). 144 students voluntarily participated. Due to incomplete questionnaires only N=141 data-sets could be analysed (Mean(Age)=20.2, 45% male). The students' satisfaction with their studies of mathematics and their dropout intention was captured by the second questionnaire after eight weeks (T2). N=120 data-sets from the second questionnaire could be used in our analysis.

All items had to be answered on a five-point likert scale (1=totally disagree, 5=totally agree) and had at least satisfying reliabilities (Table 1), except the static beliefs scale which has been excluded from further analysis. To answer research question 1, we split the students in two groups: on the one hand students with a rather low level of procrastination (mean of the scale <3) and on the other hand students with a rather high level ($M \geq 3$). We conducted independent t-tests to analyse the differences between these groups. We used linear regressions (methods: stepwise and inclusion) to answer research questions 2 and 3.

Variable	Source	Number of items	α	Item-Example
Interest Facets*	Ufer et al. 2016	18	0.76-0.87	In school, mathematics was very important for me.
Self-Concept	Kauper et al., 2012	4	0.84	I am very good in mathematics.
Dynamic Beliefs	Laschke & Blömeke, 2013	6	0.73	Mathematics involves creativity and new ideas.
Static Beliefs	Laschke & Blömeke, 2013	6	0.57	Mathematics means learning, remembering and applying.
Satisfaction with Studies of Mathematics	Schiefele & Jacob-Ebbinghaus, 2006	4	0.79	All in all, I'm satisfied with my studies of mathematics.
Dropout Intention	Own formulation	1	-	Recently, I really thought about quitting my studies of mathematics.
Procrastination	Glöckner-Rist et al. 2014	4	0.89	I only start with my work, when I get under pressure.

Table 1: Instruments used in the questionnaires (α = Cronbach's α); *3 subscales School (5 items), University (5 items), proof and formal representation (8 items)

RESULTS

Connection between mathematics-related affective variables and procrastination

Variable	Not Procrastinating N=76		Procrastinating N=65		Cohens d
	<i>M</i>	<i>SD</i>	<i>M</i>	<i>SD</i>	
Interest School	3.63	0.77	3.47	0.68	0.22
Interest University	3.18	0.78	2.83	0.89	0.42*
Interest Proof/Repres.	3.39	0.67	3.05	0.75	0.49**
Dynamic Beliefs	3.81	0.58	3.59	0.65	0.36**
Self-concept	2.99	0.67	2.78	0.73	0.3

Table 2: Results of the t-test: means, standard deviations and effectsizes, * $p < 0.05$
** $p < 0.01$, Answers between 1 (totally disagree) and 5 (totally agree)

The t-tests (Table 2) show that procrastinating students do not differ significantly from those who do not procrastinate concerning their interest in school mathematics. Like expected we find significant differences in the interest in university mathematics, proof and formal representation (H1). Our hypothesis regarding the self-concept cannot be confirmed (H2). However, procrastinating students show lower acceptance of dynamic beliefs. The static beliefs scale was excluded from our analysis, due to its lack of reliability. That's why our third hypothesis can only be confirmed partially.

A stepwise linear regression (Table 3) with all affective variables shows that only the dynamic beliefs can predict students' procrastination significantly in the sense that dynamic beliefs are going ahead with less procrastination. The interest facets and the self-concept do not predict students' procrastination (H4).

Predictor	Regression coefficient B	SE	Beta
Dynamic Beliefs	-0.37	0.13	-0.23**

Table 3: Linear regression (method: stepwise) $R^2=0.06$, $**p<0.01$

Effects of procrastination on subjective study success in mathematics

The linear regressions (Table 4) show that the procrastination is able to predict students' satisfaction with their studies of mathematics in the sense that students with higher levels of procrastination are less satisfied with their studies of mathematics (Beta=-0.22, $p<0.05$) (H5). The students' dropout intention is not predicted significantly by their procrastination, which refutes our hypothesis (H6).

Dependent Variable	Regression coefficient B	SE	Beta	R^2
Study Satisfaction	-0.18	0.08	-0.22*	0.05
Dropout Intention	0.128	0.1	0.1	0.01

Table 4: Results of the linear regressions with predictor *procrastination* (method: inclusion) $*p<0.05$

DISCUSSION AND OUTLOOK

The findings concerning the different facets of interest in mathematics support our hypothesis and are in line with recent research about the connection between procrastination and students' interest in their subject and related tasks (Lohbeck et al., 2017; Ackermann & Gross, 2005). They also confirm the necessity for research concerning the transition from school to university to distinguish between students' interest in school and university mathematics (Ufer et al., 2016). Moreover, these results underline the relevance to arouse and support students' interest in university mathematics, especially in proof and formal representation. We find no significant differences with regard to the self-concept of the two groups of students, which is surprising since recent research found significant correlations between self-beliefs and procrastination (Klassen et al., 2008; Kiamarsia & Abolghasemi, 2014; Lohbeck et al., 2017). It seems

possible that, according to the interest in mathematics, it is necessary to distinguish more clearly between school and university mathematics when dealing with self-concept. Procrastinating students show lower acceptance of dynamic beliefs concerning the nature of mathematics. Moreover, these beliefs turn out to be the only significant predictor of procrastination. This underlines the relevance of students' beliefs in the transition from school to university mathematics.

The procrastination's effect on students' satisfaction with their studies of mathematics is rather weak (5% explained variance). Furthermore, we find no significant effect on their dropout intention. This might be due to the fact that dropout intention was measured by a single item which possibly not covers all facets. In addition, the questionnaires have been filled in during the lectures. Students who do not (regularly) attend the lectures or those who dropped out before may not be captured by our survey.

Our ongoing research will now focus on the influence of procrastination on actually dropout during the first year at university. Further research on the causes of procrastination in the transition process may use a qualitative approach to gain deeper insights, especially with regard to the influence of the changing nature of mathematics and students' beliefs.

References

- Ackerman, D. S., & Gross, B. L. (2005). My Instructor Made Me Do It: Task Characteristics of Procrastination. *Journal of Marketing Education*, 27(1), 5–13.
- Adimora, D. E., Akaneme, I. N. & Nwokenna, E. N. (2017). Academic Procrastination as Predictor of Mathematics Anxiety of Pupils in Enugu State, Nigeria. *New Trends and Issues Proceedings on Humanities and Social Sciences* 03, 338–346.
- Akinsola, M. A. & Tella, A. (2007). Correlates of academic procrastination and mathematics achievement of university undergraduate students. *Eurasia Journal of Mathematics, Science and Technology Education*, 3(4), 363–370.
- Bakhshayesh, A., Radmanesh, H., & Bafrooe, K. B. (2016). Investigating Relation between Academic Procrastination and Math Performance of Students in First Year of High School. *Journal of Educational and Management Studies*, 6(3), 62–67.
- Balkis, M. (2013). Academic procrastination, academic life satisfaction and academic achievement: the mediation role of rational beliefs about studying. *Journal of Cognitive and Behavioral Psychotherapies*, 13(1), 57–74.
- Chen, X. (2013). *STEM Attrition: College Students' Paths Into and Out of STEM Fields*. Washington, DC: National Center for Education Statistics, U.S. Department of Education. Retrieved from <https://nces.ed.gov/pubs2014/2014001rev.pdf>
- Daskalogianni, K., & Simpson, A. (2001). Beliefs Overhang: The Transition from School to University. In Winter, J. (Ed.), *Proceedings of the British Society for Research into Learning Mathematics Vol. 2*, 97–108.

- Dieter, M. & Törner, G. (2012). Vier von fünf geben auf. *Forschung und Lehre*, 19(10) 826–827.
- Geisler, S. (2017). Dropout & Persistence in University Mathematics. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education, Vol. 1*, 197. Singapore: PME.
- Glöckner-Rist, A., Engberding, M. & Rist, F. (2014). *Prokrastinationsfragebogen für Studierende (PFS)*. Retrieved from [http://zis.gesis.org/pdfFiles/Dokumentation/Gloeckner-Rist+%20Prokrastinationsfragebogen%20fuer%20Studierende%20\(PFS\).pdf](http://zis.gesis.org/pdfFiles/Dokumentation/Gloeckner-Rist+%20Prokrastinationsfragebogen%20fuer%20Studierende%20(PFS).pdf)
- Hussain, I., & Sultan, S. (2010). Analysis of procrastination among university students, *Procedia – Social and Behavioral Sciences*, 5, 1897–1904.
- Kauper, T., Retelsdorf, J., Bauer, J., Rösler, L., Möller, J. & Prenzel, M. (2012). *PaLea – Panel zum Lehramtsstudium: Skalendokumentation und Häufigkeitsauszählungen des BMBF-Projektes*. Retrieved from http://www.palea.uni-kiel.de/wp-content/uploads/2012/04/PaLea%20Skalendokumentation%204_%20Welle.pdf
- Kiamarsi, A., & Abolghasemi, A. (2014). The relationship of procrastination and self-efficacy with Psychological vulnerability in students. *Procedia – Social and Behavioral Sciences*, 114, 858–862.
- Kim, K. R., & Seo, E. H. (2015). The relationship between procrastination and academic performance: A meta-analysis. *Personality and Individual Differences*, 82, 26–33.
- Klassen, R. M., Krawchuk, L. L., & Rajani, S. (2008). Academic procrastination of undergraduates: Low self-efficacy to self-regulate predicts higher levels of procrastination. *Contemporary Educational Psychology*, 33, 915–931.
- Laschke, C. & Blömecke, S. (2013). *Teacher Education and Development Study: Learning to Teach Mathematics (TEDS–M). Dokumentation der Erhebungsinstrumente*. Münster: Waxmann
- Lohbeck, A., Hagenauer, G., Mühlig, A., Moschner, B. & Gläser-Zikuda, M. (2017). Prokrastination bei Studierenden des Lehramts und der Erziehungswissenschaften. *Zeitschrift Für Erziehungswissenschaft*, 20, 521–536.
- Schiefele, U., & Jacob-Ebbinghaus, L. (2006). Lernermerkmale und Lehrqualität als Bedingungen der Studienzufriedenheit. *Zeitschrift Für Pädagogische Psychologie*, 20(3), 199–212.
- Schraw, G., Wadkins, T., & Olafson, L. (2007). Doing the things we do: A grounded theory of academic procrastination. *Journal of Educational Psychology*, 99(1), 12–25.
- Steel, P. (2007). The Nature of Procrastination: A Meta-Analytic and Theoretical Review of Quintessential Self-Regulatory Failure. *Psychological Bulletin*, 133(1), 65–94
- Törner, G., & Grigutsch, S. (1994). Mathematische Weltbilder bei Studienanfängern – eine Erhebung. *JMD*, 15(3–4), 211–221.
- Ufer, S., Rach, S., & Kosiol, T. (2016). Interest in mathematics = interest in mathematics? What general measures of interest reflect when the object of interest changes. *ZDM Mathematics Education*, 49(3), 1–13.

GAP AND CONGRUENCY EFFECT IN FRACTION COMPARISON

Juan Manuel González-Forte¹, Ceneida Fernández¹, and Wim Van Dooren²

¹Universidad de Alicante (Spain)

²KU Leuven, Centre for Instructional Psychology and Technology (Belgium)

Natural number bias in fraction comparison has been studied examining the role of congruency effect. However, the congruency effect has been observed in opposite directions, suggesting that an explanation may lie in the strategies used by students. One of the strategies that may be used by students is gap thinking. We have carried out a cross-sectional study from 5th to 10th grade with 438 participants examining the effect of congruency and gap thinking in students' responses and reasoning. Results show that gap thinking has influenced students' responses and support the claim that gap effects could explain differences between congruent and incongruent items, extending this result from primary to secondary school. However, it seems that this effect disappears (in Spanish students) at the end of Secondary Education.

THEORETICAL AND EMPIRICAL BACKGROUND

Research has shown that primary and secondary school students have difficulties with different aspects of rational numbers (Van Dooren, Lehtinen, & Verschaffel, 2015). Although these difficulties have been explained through different causes, since the 1980s, research has found that students struggle with understanding different aspects of rational numbers because the knowledge of whole numbers interferes with their learning (Fischbein, Deri, Nello, & Marino, 1985). This tendency to regress to a property compatible for whole numbers was termed by Behr et al. (1984) as whole number dominance, and recent research has termed it as *whole/natural number bias* (Ni & Zhou, 2005; Van Dooren et al., 2015). The term *bias* refers to the fact that knowledge of natural numbers can facilitate students' reasoning when activities of rational numbers are compatible with this knowledge, but it does not facilitate their reasoning when rational numbers behave differently from natural numbers (for a review, see Van Dooren et al., 2015).

Research on natural number bias has focused on four dimensions on which rational numbers differ from natural numbers: density, representation, operations and size (Obersteiner, Van Hoof, Verschaffel, & Van Dooren, 2016; Van Dooren et al., 2015). Our study is focused on the last aspect that is centred on the way the number size can be determined. Particularly, it is focused on comparing two rational numbers in the fractional representation. A broader review of the studies focused on students' performances when comparing fractions has been done.

Research on fraction comparison items

Studies that consider fraction comparison items have shown variability in strategies used by students depending on specific item characteristics (Gómez, Silva, & Dartnell, 2017). An important characteristic is the *congruency* with natural number ordering. Some fraction comparison items are congruent (or consistent) with knowledge of natural number ordering (for instance, $2/3$ vs. $7/8$) since the fraction which has the largest numerator and denominator is the largest. Other fraction comparison items are incongruent (or inconsistent) with the natural number ordering (for example, $2/3$ vs. $5/8$) since the largest fraction does not have the largest numerator and denominator.

Research has shown that students' and even adults' accuracies and response times to fraction comparison items are influenced by congruency. In fact, students and adults had longer reaction times and were less accurate on incongruent items with natural number reasoning compared to congruent items (Van Hoof, Lijnen, Verschaffel, & Van Dooren, 2013; Wolf, & Vosniadou, 2011) using incorrect reasoning such as “ $5/8$ is larger than $2/3$ since 5 is larger than 2, and 8 is larger than 3”. However, the congruency effect also has been observed in opposite directions. Gómez and Dartnell (2015) showed that when fractions have no common components (numerator or denominator), congruent items are more difficult and take longer to answer than incongruent items. This opposite direction has also been found in university students (DeWolf, & Vosniadou, 2015) and in expert mathematicians (Obersteiner, Van Dooren, Van Hoof, & Verschaffel, 2013). This raises questions about the effect of congruency and highlights the need to consider other explanations involving, for instance, some of the commonly used strategies for comparing fractions (Gómez et al., 2017).

A documented incorrect strategy used by students when comparing fractions was called by Pearn and Stephens (2004) as *gap thinking*. This reasoning is based on the distance (gap) between numerator and denominator. For example, $3/2$ is larger than $5/3$ because “from 2 to 3 there is one gap and from 3 to 5 there are two gaps” (Pearn, & Stephens, 2004). Gómez, Silva, and Dartnell (2017) distinguish fraction comparison items where gap thinking can lead to a correct answer such as the pair of fractions $2/7$ and $5/8$ ($5/8$ is larger than $2/7$ and there are five gaps between 2 and 7 and only three gaps between 5 and 8), items where gap thinking can lead to an incorrect answer such as the pair of fractions $7/9$ and $2/3$ ($7/9$ is larger than $2/3$ and there are two gaps between 7 and 9 and only 1 gap between 2 and 3) and, finally, items where gap thinking is uninformative, because both fractions have the same gap (e.g. $2/3$ and $3/4$). However, literature has shown that in neutral items, some students can consider that the two fractions are equal “since the difference between numerator and denominator is the same” (Clarke, & Roche, 2009). For instance, when comparing the fractions $2/3$ and $3/4$, students can think that are equal because the difference between numerator and denominator is 1. So the neutral ones can also lead students to an incorrect answer. Gómez et al. (2017) found that gap thinking affected participants' response times, explaining why congruent items without common components are associated with a

worse performance than incongruent items in mathematically-trained individuals (undergrad students of Engineering).

Objective

As indicated above, Gómez et al. (2017) have investigated gap thinking by examining responses times in mathematically-trained individuals. The participants of our study are primary and secondary school students, and instead of looking at responses time, we are interested in students' reasoning.

We examine how Spanish primary and secondary school students solve and justify fraction comparison items in order to get information about the phenomenon Natural Number Bias and the effect of gap thinking (cross-sectional study from 5th to 10th grade). Two conditions in fraction comparison items have been taken into account: the congruence with the natural number ordering (congruency) and the distance between the numerator and the denominator (gap thinking). With regard to the gap thinking condition, we consider items with different distance between numerator and denominator where the gap can lead to a correct answer and the “neutral” (items with the same distance between numerator and denominator) where gap thinking can be uninformative or can lead to an incorrect answer.

METHOD

Participants and instrument

Participants were 438 primary (5th and 6th grade) and secondary school students (7th, 8th, 9th and 10th grade) belonged to two different Spanish primary schools and two different Spanish secondary schools. The participated schools belonged to different cities and students were from mixed socio-economic backgrounds. The instrument was a test consisting of four fraction comparison items (Table 2). In each item, students had to circle the largest fraction and explain why they think that the fraction they have chosen was the largest.

	Pair of fractions	Congruent / Incongruent (C/I)	Same / Different distance (S/D)
Item 1	2/3 vs. 7/8	C	S
Item 2	2/7 vs. 5/8	C	D
Item 3	5/3 vs. 9/7	I	S
Item 4	2/3 vs. 5/8	I	D

Table 2: Characteristics of the items

Analysis

We analysed students' success levels in each item and grade and the type of reasoning used. Students' answers were classified as correct and incorrect:

- If the student correctly encircled the largest fraction, the answer is classified as correct.
- If the student incorrectly encircled the largest fraction, or does not encircle any fraction (blank answer), the answer is classified as incorrect.

A repeated measures logistic regression analysis, using the generalized estimating of equations (GEE), was conducted on the occurrence of correct answers.

Furthermore, we examined the type of reasoning used by students in each of the items. We carried out an inductive analysis to generate categories. Firstly, a subset of students' answers was independently analysed by three researchers. We then compared our results and discussed our discrepancies until we reached an agreement on identified categories. Subsequently, new data samples were added in order to revise our categories. Finally, four categories of correct reasoning and three categories of incorrect reasoning emerged. Since we are interested in the use of incorrect reasoning (based on knowledge of natural numbers or on gap thinking), we only described the three categories emerged of incorrect reasoning:

- Reasoning based on the order of natural numbers. In this reasoning, the largest fraction is the fraction whose numerator and denominator are bigger (Fig. 1).

3. Rodea la fracción mayor

- $5/3$
- $9/7$

¿Por qué crees que la fracción que has elegido es la mayor?

Porque esos números son mayores.

Figure 1: “Because these numbers are bigger” (5th grade student)

- Reasoning based on gap thinking. In this reasoning, the largest fraction is the fraction whose distance between numerator and denominator is the smallest in items with different distance (Fig. 2) or both fractions are equal since the gap (distance between numerator and denominator) is the same, in items with the same distance.

4. Rodea la fracción mayor

- $2/3$
- $5/8$

¿Por qué crees que la fracción que has elegido es la mayor?

porque la diferencia que hay del numerador al denominador es menor.

Figure 2: “Because the difference between the numerator and the denominator is smaller” (10th grade student)

- “Others” that included nonsense reasoning or blank responses.

RESULTS

Figure 3 shows students' success level in each item from 5th to 10th grade. Globally, students were more successful in congruent items (82.56%) than in incongruent ones (50.16%). A repeated measures logistic regression analysis showed that this difference

was significant, $\chi^2(1, N=438)=182.513, p<0.001$. This effect was significant in items with the same distance (76.41% - congruent vs. 46.91% - incongruent) and in items with different distance (88.70% - congruent vs. 53.41% - incongruent). Furthermore, students were more successful in comparisons with different distance than with the same distance (71.06% vs. 61.67%). The repeated measures logistic regression analysis showed that this difference was significant $\chi^2(1, N=438)=30.926, p<0.001$. This effect was significant in congruent items (88.70% - different distance vs. 76.41% - same distance) and in incongruent ones (53.41% vs. 46.91%).

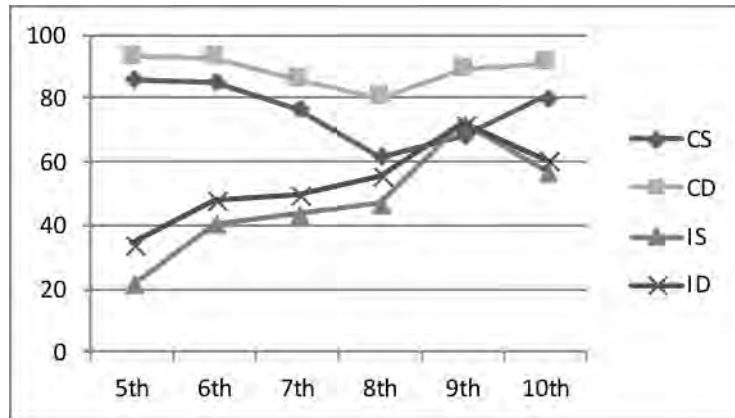


Figure 3: Students' success level in congruent (C) and incongruent (I) comparison items with same (S) and different (D) distance from 5th to 10th grade

Table 3 shows percentages of correct answers in congruent and incongruent items. There was a significant 'grade' \times 'congruence' interaction effect, $\chi^2(5, N=438)=53.002, p<0.001$, revealing significant differences in 5th grade, 6th grade, 7th grade, 8th grade and 10th grade, but not in 9th grade (78.95% vs. 71.93%). Differences in 8th and 9th grade between congruent and incongruent items are smaller, being the last difference not significant.

Grade	Congruent	Incongruent
5 th	89.41	27.65
6 th	88.89	44.44
7 th	81.41	46.79
8 th	70.99	51.23
9 th	78.95	71.93
10 th	85.71	58.93

Table 3: Percentages of students' correct answers in congruent and incongruent items

Furthermore, there was a decrease in performance on the congruent items from 5th to 8th grade and an increase in performance on the incongruent items. These differences were not significant except the difference between 5th and 6th grade in the incongruent items (27.65% vs. 44.44%). From 8th to 10th grade, there was an increase in performance on the congruent items and in the performance on the incongruent items (except from 9th to 10th grade) being only significant the difference between 8th and 9th grade.

Table 4 shows the percentage of use of a reasoning based on the order of natural numbers (ON), reasoning based on the gap thinking (GT) and others (OT). Students used more the reasoning based on the order of the natural numbers (ON) than the reasoning based on gap thinking (GP) in all items. Reasoning based on gap thinking almost disappeared in 10th grade (1.79% in CS, 1.79% in CD, 1.79% in IS and 8.93% in ID). However, reasoning based on the order of natural numbers persisted at the end of secondary education (around a 20% of students used this reasoning in 10th grade)

	CS			CD			IS			ID		
G	ON	GT	OT	ON	GT	OT	ON	GT	OT	ON	GT	OT
5 th	63.53	5.88	11.77	54.18	10.59	14.05	58.82	5.88	21.18	51.76	15.29	14.13
6 th	46.91	4.94	0.73	46.91	11.11	8.64	45.68	3.70	18.52	40.74	11.11	9.88
7 th	39.74	6.41	28.21	35.90	6.41	30.77	32.05	7.69	32.05	33.33	6.41	29.49
8 th	22.22	17.28	29.64	19.75	18.52	29.63	14.81	18.52	34.57	16.05	17.28	40.74
9 th	19.30	3.51	26.31	10.53	19.30	19.29	17.54	3.51	26.32	14.04	15.79	22.80
10 th	21.43	1.79	28.57	21.43	1.79	17.85	19.64	1.79	30.36	17.86	8.93	23.21
Tot	35.52	6.4	20.87	31.45	11.29	20.04	31.42	6.85	27.17	28.96	12.47	23.38

Table 4: Percentages of use of an incorrect reasoning

Comparing Table 3 (success levels in congruent and incongruent items) and Table 4 (percentages of use of incorrect reasoning), we can observe that 5th and 6th graders have the highest percentage of correct answers in congruent items, but a large percentage of these students used an incorrect reasoning based on the order of natural numbers. Therefore, although primary school students' success levels in congruent items are higher, these students use their knowledge of natural numbers or strategies such as gap thinking to solve these items. In fact, the percentage of correct answers of 5th and 6th graders are the lowest in incongruent items (IS and ID) where the knowledge of natural numbers cannot be used.

As we mentioned, differences in 8th and 9th grade between congruent and incongruent items are smaller. Table 4 gives us a possible explanation. Although there is a decrease of the reasoning based on the order of natural numbers in these grades, there was an increase in the use of the gap thinking in 8th grade in all items, persisting in 9th grade in items with different distance.

DISCUSSION AND CONCLUSIONS

Generally, our results show that both primary and secondary school students have difficulties with fraction comparison items, the natural number bias (Ni & Zhou, 2005; Van Dooren et al., 2015) being the main reason for students' failure in fraction comparison.

Congruent comparison items have obtained better results than incongruent comparison items (DeWolf, & Vosniadou, 2011; Gómez et al., 2015). The qualitative analysis of students' reasoning confirms that students have considered that a fraction is larger if the numerator and denominator are bigger (Behr et al., 1984; DeWolf, & Vosniadou, 2011). This reasoning is based on the knowledge of natural numbers since students consider the numerator and denominator as two independent parts (Behr et al., 1984), and apply the natural number ordering to compare fractions. Our results are in line with previous ones although conditions to collect data were different since we use only four fraction comparison items and ask for students' reasoning, instead of controlling time.

Furthermore, fraction comparisons with different distance have obtained better results than fraction comparisons with the same distance. Therefore, gap thinking (Pearn & Stephens, 2004) has influenced the students' responses since in comparisons with different distance, gap thinking leads to a correct answer. This result is confirmed also in the qualitative analysis of students' reasoning where students reason that a fraction is larger where the distance between numerator and denominator is smaller. These students do not consider that the numerator and denominator maintain a multiplicative relationship, comparing numerator and denominator with a subtraction-based rule. As Gómez et al. (2017) found that gap-related conditions affected significantly participants' responses times (undergrad students of Engineering). In our study that condition (same/different distant between numerator and denominator) affected primary and secondary school students' responses.

As in Gómez et al. (2017) study with mathematically-trained individuals, our data supports the claim that gap effects could explain differences between congruent and incongruent items, extending this result from primary to secondary school. However, it seems that this effect disappears (in Spanish students) at the end of Secondary Education. This result invites further research in this line.

Acknowledgments

This research was carried out with the support of Conselleria d'Educació, Investigació, Cultura i Esport (Generalitat Valenciana, Spain) (PROMETEO/2017/135).

References

- Behr, M. J., Wachsmuth, I., Post, T. R., & Lesh, R. (1984). Order and equivalence: A clinical teaching experiment. *Journal of Research in Mathematics Education*, 15(5), 323–341.
- Clarke, D. M., & Roche, A. (2009). Students' fraction comparison strategies as a window into robust understanding and possible pointers for instruction. *Educational Studies in Mathematics*, 72(1), 127–138.
- DeWolf, M., & Vosniadou, S. (2011). The whole number bias in fraction magnitude comparisons with adults. In Carlson, L., Hoelscher, C., & Shipley, T. F. (Eds.), *Proceedings of the 33rd Annual Conference of the Cognitive Science Society* (pp. 1751–1756). Austin, TX: Cognitive Science Society.

- DeWolf, M., & Vosniadou, S. (2015). The representation of fraction magnitudes and the whole number bias reconsidered. *Learning and Instruction*, 37, 39–49.
- Fischbein, E., Deri, M., Nello, M. S., & Marino, M. S. (1985). The role of implicit models in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education*, 16, 3–17.
- Gómez, D. M., & Dartnell, P. (2015). Is there a natural number bias when comparing fractions without common components? A meta-analysis. In Beswick, K., Muir, T., & Fielding-Wells, J. (Eds.). *Proceedings of 39th Psychology of Mathematics Education conference*, Vol. 3, pp. 1–8. Hobart, Australia: PME.
- Gómez, D. M., Silva, E., & Dartnell, P. (2017). Mind the gap: congruency and gap effects in engineering students' fraction comparison. In Kaur, B., Ho, W.K., Toh, T.L., & Choy, B.H. (Eds.). *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education*, Vol. 2, pp. 353–360. Singapore: PME.
- Ni, Y., & Zhou, Y. D. (2005). Teaching and learning fraction and rational numbers: The origins and implications of whole number bias. *Educational Psychologist*, 40(1), 27–52.
- Obersteiner, A., Van Dooren, W., Van Hoof, J., & Verschaffel, L. (2013). The natural number bias and magnitude representation in fraction comparison by expert mathematicians. *Learning and Instruction*, 28, 64–72.
- Obersteiner, A., Van Hoof, J., Verschaffel, L., & Van Dooren, W. (2016). Who can escape the natural number bias in rational number tasks? A study involving students and experts. *British Journal of Psychology*, 107, 537–555.
- Pearn, C., & Stephens, M. (2004). Why you have to probe to discover what year 8 students really think about fractions. In Putt, I., Faragher, R., & McLean, M. (Eds.), *Proceedings of the 27th Annual conference of the Mathematics Education Research Group of Australasia* (pp. 430–437). Sydney, Australia: MERGA.
- Van Dooren, W., Lehtinen, E., & Verschaffel, L. (2015). Unraveling the gap between natural and rational numbers. *Learning and Instruction*, 37, 1–4.
- Van Hoof, J., Lijnen, T., Verschaffel, L., & Van Dooren, W. (2013). Are secondary school students still hampered by the natural number bias? A reaction time study on fraction comparison tasks. *Research in Mathematics Education*, 12(2), 154–164.

GENDER SPECIFICITIES IN A SUPPORT PROJECT FOR ENGINEERING STUDENTS

Birgit Gries

Paderborn University

Supporting students in mathematics requires concepts that match the target group, which can be classified, among others, according to gender. Our study explores data from almost 1,500 engineering first-years in order to identify gender-specific learning behaviour and its impact on academic success, thus investigating if classification by gender does provide insights into criteria for project re-design. Results show that in spite of stereotyped learning strategies, non-gender-specific factors are better suited to explain different examination outcomes.

INTRODUCTION

Mathematics is traditionally regarded as a male domain (e.g. Barkatsas, Forgasz, & Leder, 2001; Fennema & Leder, 1990), although there have been numerous attempts to challenge this stereotype, many of them successful (e.g. Alcock, Attridge, Kenny, & Inglis, 2014; Leder, Forgasz, & Solar, 1996). Still, males often outperform females in mathematics and other subjects from the STEM range (OECD, 2007). As STEM subjects, and particularly engineering, offer promising career options, good salaries and high social status, it is interesting from a perspective of social participation to explore the gender specificities of future engineers with the data from a support project for engineering first-years (Dehling, Glasmachers, & Härterich, 2014).

The project focus on learning strategies allows to assess students' learning behaviour, which, in combination with their educational background, presents a promising source of information for their academic progress. A growing awareness of the strong connection between performance, gender, and affective aspects (Freislich & Bowen-James, 2001; Frost, Hyde, & Fennema, 1994; Forgasz, 1995) has drawn the attention to affect in recent years, which, in our project, is reflected in the choice of learning strategy categories, including items referring to motivation or effort.

THEORETICAL BACKGROUND

Our project for supporting engineering first-years in mathematics is located at a big university in Germany. The basic idea in the conceptualization of the project was to address students' learning strategies with respect to mathematics, as mathematics is a subject where inadequate learning behaviour would take its toll early on. With the aim to avoid unnecessary dropout, students were coached in adequate learning behaviour, always in connection with topics from their mathematics lecture. Considering the basic psychological needs phrased by Deci and Ryan (1990), competence (experience con-

trol), autonomy (perceive oneself as making decisions), and relatedness (feel connected to others), appeared to be a crucial measure for the ongoing re-design of the project interventions. The project was aimed at male and female students alike, with no initial intention to appeal to any gender in particular.

“Learning strategies” is used here as a generic term for all kinds of learning behavior, including affective (like motivation to learn) and metacognitive aspects (e.g. reflecting about your learning behaviour and consequently regulating it). Learning strategies are the factor by which individuals can influence their learning outcome, as they are comparatively modifiable, where talent or apprehension are rather predefined. There are different categories of learning strategies; we opted for a conceptualization (LIST, Wild & Schiefele, 1994) that identifies three main groups (cognitive, metacognitive, and resource-oriented learning strategies). The cognitive strategies in our project cover e.g. elaborating and repeating techniques; planning, monitoring, and regulating are classed as metacognitive; and the resource-oriented learning strategies refer to external (e.g. use of sources of information, studying with peers) as well as internal (time management, effort, attention) learning resources. The affective aspects can be found among the internal resource-oriented learning strategies.

Our research questions are to be viewed as relating to a university mathematics course for engineering first-years and the support project described above, focusing on learning strategies.

- RQ1 Are there gender-specific distinctions concerning project participation and examination outcome?
- RQ2 Are there gender-specific differences in the educational backgrounds or in the learning strategies employed?
- RQ3 Is there empirical evidence for a gender-specific connection between different examination outcomes and learning strategies?

METHODOLOGY

The learning strategies were self-reported in a pre-post design using the LIST questionnaire (Wild & Schiefele, 1994), which originally consists of thirteen scales with three to eight items in each scale. The data collected in four project years from 2011 to 2014 was analysed in detail. Data from LIST was available, as well as demographic information and the examination statistics, including results from mini tests during the semester. Variables belonging to the educational background (basic or advanced mathematics course attended at school, type of school attended, school mark in mathematics, result of first mini test at university, attendance of preparatory course offered by the university) were analysed as well. Depending on the sources of the different data (from official records or from different paper-and-pencil surveys), the number of data sets varied. As much data as possible was used for the various analyses.

As a first step, descriptive statistics were employed on the official records to identify gender-specific distinctions concerning project participation and examination out-

come. The calculation of expected numbers, based on the overall percentages, in contrast to the actual numbers, indicated where to take a closer look.

Before entering the LIST data into further analysis, it was ascertained that it met the usual quality standards of internal reliability by computing Cronbach's α and executing an explorative factor analysis (maximum likelihood extraction of factors, pairwise exclusion, orthogonal rotation), allowing for appropriate adaptations. Scale scores were computed on these grounds. One-way independent ANOVA, including Levene's test for homogeneity, was used to compare the scores of male and female students.

As a next step, multiple linear regression was utilized, separately for males and females, choosing those LIST scale scores with a significant correlation to examination success as predictors, and the examination outcome as dependent variable. Correlations between a variable on taking enough time to study concluded the computations.

RESULTS

Unsurprisingly, there are more males (77%) than females (23%) among our sample of 1,495 engineering first-years, though the percentage varies over the years and particularly between the different engineering courses.

	Pass	Fail	Sum		Part.	Non-part	Sum
Male	282 (282)	198 (199)	480	Male	41 (52)	439 (429)	480
Female	100 (100)	70 (70)	170	Female	29 (18)	141 (152)	170
Sum	382	268	650	Sum	70	580	650

	Pass	Fail	Sum
Part.	51 (41)	19 (29)	70
Non-part.	331 (341)	249 (240)	580
Sum	382	268	650

Table 1: Contingency tables for 2012/2013, n=650 complete data sets, remarkable numbers in bold face.

The real and expected (in brackets) numbers of students in the categories *examination success*, *project participation* and *gender* were explored, with the numbers for 2012/2013 given in Table 1. There, the top left contingency table for male / female and examination pass / fail conveys the information that both males and females passed or failed the examination in the expected numbers, so the cohort does not suggest gender-specific distinctions concerning examination outcome (RQ1). The top right table, though, uncovers the fact that proportionally more females (29, 18 expected) than males (41, 52 expected) participated in the project (RQ1). Finally, the bottom table shows that project participants passed the examination more often (51) than expected from their proportion (41). This result is unique for the project cycle in 2012/2013, see

Griese (2016, section 5.5.1); no other cycle reveals similarly distinct gender-specific differences in these respects. Therefore, we will only explore the data from this year more deeply, as in the following years, some effort was undertaken to make project participation more attractive to males, e.g. by introducing elements of gamification (see Griese, 2016, section 4.6, for details).

Concerning the educational background, the variables describing basic or advanced mathematics course attended at school, the type of school attended, the school mark in mathematics, the result of the first mini test at university, and attendance of preparatory course offered by the university were scrutinized. No significant differences between males and females were found, with two exceptions: First, females were underrepresented (42, 61 expected, $n=421$ with some missing data) in the group from the traditional *Gymnasium*, and thus overrepresented (47, 28 expected) in the group from less-prestigious school types. Second, female students were found to having disproportionately seldom (25, 39 expected, same sample) attended the preparatory course.

The exploration of LIST data made some adaptations necessary (deleting one *Attention* item and the metacognitive scales), but finally delivered acceptable (i.e. > 0.7) Cronbach's α values.

	Male			Female		
	M	SD	n	M	SD	n
Organizing	44.82	19.28	202	58.84	20.38	106
Elaborating	59.00	15.77	206	50.87	19.20	115
Repeating	44.78	16.21	211	51.53	17.14	118
Effort	58.14	16.72	199	61.47	17.87	109
Attention	55.08	15.87	211	56.05	15.93	113
Time Man.	33.69	22.65	234	31.62	22.20	117
Environment	62.33	18.85	215	62.98	18.72	116
Peer Learn.	56.19	21.77	219	62.04	18.22	109
Reference	72.29	21.46	231	74.72	23.57	121

Table 2: Scale scores describing frequency of use, pre survey 2012/2013, $n=421$: [0; 16.67] very seldom,]16.67, 50] seldom,]50, 83.33] often,]83.33, 100] very often.

The scale scores of LIST, describing how often a learning strategy was used, were compared between the males and females, see Table 2. It emerged that female students reported more frequent learning behavior on the whole, with significantly higher ($M_f=58.84$, $M_m=44.82$, $F(1, 306)=35.34$, $p<0.001$) scores in *Organizing* techniques (meaning highlighting important formulae, making lists of relevant concepts, summarizing algorithms etc.), *Repeating* ($M_f=51.53$, $M_m=44.78$, $F(1, 327)=12.63$, $p>0.001$)

and *Peer Learning* ($M_f=62.04$, $M_m=56.19$, $F(1, 326)=4.29$, $p<0.05$). Only one learning strategy was employed significantly more often by males than by females ($M_f=50.87$, $M_m=59.00$, $F(1, 319)=8.96$, $p<0.001$): *Elaborating*, which describes e.g. finding examples, visualizing procedures, or connecting new facts to previous knowledge. Levene's tests revealed that the variances were equal for all scales.

Male	Pearson's r	Sig.	n	Female	Pearson's r	Sig.	n
Effort	-0.248	0.015*	96	Elaborating	0.374	0.019*	39
Attention	0.302	0.002**	99				

Table 3: Significant Pearson's r values in relation to *Examination Success*.

Two (for males) respectively one (for females) LIST scale(s) were found to correlate significantly with the examination outcome, see Table 3. This limited the choice of predictors for the linear models considerably, but allowing for more predictors made no difference, as then no other predictors were entered into the final models.

Male	b	SE for b	β	Sig.	$R^2=0.091$
(Constant)	2.264	0.317		0.000	Durbin-
Attention	0.018	0.006	0.302	0.003**	Watson=1.785
Female	b	SE for b	β	Sig.	$R^2=0.140$
(Constant)	1.924	0.489		0.000	Durbin-
Elaborating	0.028	0.012	0.374	0.019*	Watson=1.854

Table 4: Regression models for males ($n=95$) and females ($n=38$), outcome variable *Examination Success*, * <0.05 , ** <0.01 , *** <0.001 .

Both for the forward and the stepwise method of including or excluding predictors resulted in identical final models (Table 4), which are different for the two genders: Whereas for male students, the biggest influence on *Examination Success* in relation to learning strategies can be found in the *Attention* scale ($\beta=0.302$, $p<0.01$), *Elaborating* strategies count ($\beta=0.374$, $p<0.05$) for female students. The values of the standardized β s are to be understood in the following way: As the variable *Examination Success* is coded in German school marks (from 1 to 5), where a lower number expresses a better performance, and the LIST scale scores to attain values between 0 and 100, the absolute β values are relatively low. Their positive algebraic signs mean different things for the two predictor variables in the two models: The *Attention* items in LIST are reverse-coded, so a lower value denominates less attention or more concentration. Therefore, $\beta=0.302$ for *Attention* in the model for male students means that more concentration while learning for mathematics results in better examination scores. In

contrast, a more frequent use of *Elaborating* strategies results in a higher score. So $\beta=0.374$ means that more *Elaborating* is connected with a worse examination result.

The item “I took enough time for studying” correlates significantly with *Examination Success* for both males ($r=0.251$, $p<0.01$, $n=119$, $M=2.478$, $SD=0.7825$) and females ($r=0.368$, $p<0.01$, $n=50$, $M=2.610$, $SD=0.6952$), though the mean differences are not significant. The positive algebraic signs of the correlation coefficients mean that the more students agreed to the above statement, the better were their marks.

DISCUSSION

The exploration of the contingencies between gender, project participation, and examination success revealed that in one project year, female students disproportionally often participated in the project (Table 1, top right). As project participation was voluntary and involved an application, this indicates a stronger urge among the females to find support. Together with the understanding that project participants passed the examination more often than expected (Table 1, bottom), the numbers show that in this cohort, female project participants displayed a considerable advantage concerning examination success (RQ1).

In order to identify or exclude reasons for the phenomenon just stated, that females profited from project participation more than males, the gender-specific differences in the educational background and the learning strategies are of interest (RQ2), as both are sensible reasons why some students do better than others. The finding concerning the type of school attended speak for a slightly weaker educational background of female students, which rather contraindicates than supports their greater academic success. The result concerning partaking in the preparatory course hints at females taking preparation less seriously, in contrast to their later increased project participation numbers. Other variables on educational background, e.g. school marks in mathematics, are unremarkable. In sum, none of these results explains why female project participants did better in the examinations.

Investigating learning strategies separately for males and females pointed to more *Organizing*, *Repeating*, and *Peer Learning* strategies for female students, and more *Elaborating* techniques for males (Table 2), which is in keeping with other research (Khanal, 2017). Thus, females seem to fulfil the cliché of being more diligent (more highlighting, more making lists, more summarizing, more memorizing) and more socially-minded, whereas males more often report to attempt to get to the bottom of things. Which of these learning strategies might be responsible for passing an examination, however, is hard to decide – the tasks in mathematics for first-year engineering students often consist of routine calculations like computing derivatives, integrals, determinants, or solving systems of linear equations or differential equations, where diligence might pay off at least as much as understanding. Which learning strategies, from an empirical point of view, impact on the examination outcome (RQ3), can be deduced from the correlations and the linear models with the outcome variable *Examination Success*, see Tables 3 and 4. As expected, more concentration while studying

is connected to better marks for the male students. Contrary to common expectations, though, more *Elaborating* strategies (i.e. more finding examples, visualizing, connecting information) is associated with less success for the females. This is hard to interpret, particularly as *Elaborating* is the one learning strategy preferred by males. The variance explained by learning strategies alone is low, though, as, in keeping with other research (e.g. Rach & Heinze, 2011), other factors (like previously acquired skills or knowledge) have a massive impact. So this one disconcerting result must not be overrated.

The meaningfulness of allowing yourself enough time to engage in mathematics in order to meet the standards is reflected in the medium effect ($r=0.368$, meaning 13.5% of variance explained) this variable has on *Examination Success* for females, compared to the small effect ($r=0.251$, 6.3% of variance explained) for males. The variance explained thus is markedly higher than for the learning behaviour covered by the LIST questionnaire.

Other work has identified *Effort* and *Attention* as influential on examination success (Griese, 2016, section 5.7; Schiefele, Streblow, Ermgassen, & Moschner, 2003), thus stressing the importance of affective-motivational aspects. Our new results support the relevance of *Attention* strategies (for male students), i.e. of working efficiently without getting distracted. The fact that this seems gender-specific indicates that distraction might be a weak point for male engineering first-years, thus pointing to appropriate counter-measures recommended for students or teachers.

References

- Alcock, L., Attridge, N., Kenny, S., & Inglis, M. (2014). Achievement and behaviour in undergraduate mathematics: Personality is a better predictor than gender. *Research in Mathematics Education*, 16(1), 1–17.
- Barakatsas, A. N., Forgasz, H., & Leder, G. (2001). The gender stereotyping of mathematics: Cultural dimensions. In J. Bobis, B. Perry, & M. Mitchelmore (Eds.), *Numeracy and beyond (Proceedings of the 24th annual conference of the Mathematics Education Research Group of Australasia)* (pp. 79–86). Sydney: MERGA.
- Deci, E. L., & Ryan, R. M. (1990). *Intrinsic motivation and self-determination in human behavior* (3rd ed.). New York: Plenum Press.
- Dehling, H., Glasmachers, E., & Härterich, J. (2014). MP²-Mathe/Plus/Praxis. *Mitteilungen der Deutschen Mathematiker-Vereinigung*, 22, 112–114.
- Freislich, M.-R., & Bowen-James, A. (2001). Gender, attribution and success in tertiary mathematics. In J. Bobis, B. Perry, & M. Mitchelmore (Eds.), *Numeracy and beyond (Proceedings of the 24th annual conference of the Mathematics Education Research Group of Australasia)* (pp. 231–237). Sydney: MERGA.
- Frost, L. A., Hyde, J. S., & Fennema, E. (1994). Gender, mathematics performance, and mathematics-related attitudes and affect: A meta-analytic synthesis. *International Journal of Educational Research*, 21(4), 373–385.

- Fennema, E., & Leder, G. (1990). *Mathematics and gender*. New York: Teachers College, Columbia University.
- Forgasz, H. (1995). Gender and the relationship between affective beliefs and perceptions of grade 7 mathematics classroom learning environments. *Educational Studies in Mathematics*, 28, 219–239.
- Griese, B. (2016). *Learning strategies in engineering mathematics - conceptualisation, development, and evaluation of MP²-Mathe/Plus* (PhD Thesis). Ruhr-Universität, Bochum.
- Guo, J., Marsh, H. W., Parker, P. D., Morin, A. J. S., & Yeoung, A. S. (2015). Expectancy-value in mathematics, gender and socioeconomic background as predictors of achievement and aspirations: A multi-cohort study. *Learning and Individual Differences*, 37, 161–168.
- Khanal, B. (2017). Gender differences in the preference of learning strategies in mathematics. In B. Kaur, W. K. Ho, T. L. Toh, & B. H. Choy (Eds.), *Proceedings of the 41st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, p. 221–221). Singapore: PME.
- Leder, G., Forgasz, H., & Solar, C. (1996). In A. Bishop, K. Clements, C. Keitel, J. Kilpatrick, & C. Laborde (Eds.), *International handbook of mathematics education, part 2* (pp. 945–985). Dordrecht: Kluwer Academic Publishers.
- OECD. (2007). *PISA 2006 Science competencies for tomorrow's world. Vol. 1: Analysis*. Retrieved Jan. 15, 2018 from http://www.oecd.org/edu/school/programme-for-international-student-assessment-pisa/pisa2006results.htm#Vol_1_and_2
- Rach, S., & Heinze, A. (2011). Studying mathematics at the university: The influence of learning strategies. In B. Ubuz (Ed.), *Proceedings of the 35th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, pp. 9–16). Ankara, Turkey: PME.
- Schiefele, U., Streblow, L., Ermgassen, U., & Moschner, B. (2003). Lernmotivation und Lernstrategien als Bedingungen der Studienleistung: Ergebnisse einer Längsschnittstudie / The influence of learning motivation and learning strategies on college achievement: Results of a longitudinal analysis. *Zeitschrift für Pädagogische Psychologie / German Journal of Educational Psychology*, 17(3/4), 185–198.
- Wild, K.-P., & Schiefele, U. (1994). Lernstrategien im Studium. Ergebnisse zur Faktorenstruktur und Reliabilität eines neuen Fragebogens. *Zeitschrift für Differentielle und Diagnostische Psychologie*, 15, 185–200.

TEACHER'S GAZE BEHAVIOR WHEN SCAFFOLDING PEER INTERACTION AND MATHEMATICAL THINKING DURING COLLABORATIVE PROBLEM-SOLVING ACTIVITY

Eeva Haataja, Enrique Garcia Moreno-Esteva,
Miika Toivanen, and Markku S. Hannula
University of Helsinki

Scaffolding is an important teaching action that can target either at student cognitive or socio-emotional processes. The details of these scaffolding events can be examined through teacher visual attention, which interacts with his pedagogical decisions. In this study we use mobile gaze tracking device to investigate the different types of teacher's gaze behavior while he is scaffolding collaborative problem solving on mathematics lesson. Teacher's attentional focus during scaffolding events was found to relate to the purpose of the scaffolding event and thus to reflect teacher's objectives for the interaction with students. This situational nature of teacher gaze implicates the need of contextual use of gaze tracking analysis in the field of education.

INTRODUCTION

Teacher's experience, values, and pedagogical views affect their attention towards various students and targets in the classroom (McIntyre, Mainhard, & Klassen, 2017). Teacher's ability to notice situations that are crucial for mathematical or interactional learning is an essential skill that develops through practice (Stockero, 2014). The nature of social interaction affects teacher's gaze behavior (Prieto, Sharma, Kidzinski, & Dillenbourg, 2017). Teacher makes decisions on the direction of his attention. By offering adequate attention to students according to their needs, the teacher can foster their engagement to learning (Nizielski, Hallum, Lopes, & Schütz, 2012) and nurture the teacher-student relationship (Dessus, Cosnefroy, & Luengo, 2016). The learning and the social interaction during problem solving are influenced by the content of teacher feedback and guiding as well. Many teachers offer mostly guidance on mathematical contents and procedures, sometimes even neglecting the importance of peer interaction (Ding, Li, Piccolo, & Kulm, 2007). Teacher has the possibility to support interaction between students, which is often the least appearing form of interaction in mathematics lessons (Akkus & Hand, 2011).

Human eyes work as receiver and provider of information (Csibra, 2010). Recent research has shown the unevenness of teacher's gaze attention among students (Dessus et al., 2016), and the connection between teacher expertise and student-orientation of gaze behavior (McIntyre et al., 2017). Teacher's actions and visual perceptions interact and cooperate continuously as she targets her gaze towards relevant areas (Tatler, Kirtley, Macdonald, Mitchell, & Savage, 2014). As the eyes do not participate in ac-

tion, investigating human gaze with mobile gaze tracking cameras offers valuable information on cognitive processes during activities (Shayan, Bakker, Abrahamson, Duijzer & van der Schaaf, 2017). Gaze consist of fixations on targets and brief eye movements called saccades between fixations. A sequence of fixations on a defined area of interest constitutes a "dwell", whose duration is referred as "dwell time" (Holmqvist et al., 2011). A person's characteristics emerge in the proportion, duration, and frequency of the dwell times on separate targets (Yamamoto & Imai-Matsumura, 2013) and thus form the gaze behavior of the subject (Lappi, 2016). In this research, concept gaze behavior refers to the pattern of teacher's dwell times.

This study examines teacher's gaze behavior during scaffolding events with collaboration groups of students. Our research question is:

How does the content of scaffolding event (either mathematical or interaction- oriented) affect a teacher's visual attention?

METHODS

The researchers collected the data during a mathematics lesson of a 9th grade class in Southern Finland. The participating class included nineteen 15– and 16–year-old students (11 boys, 8 girls) and a teacher. Students were sitting in five collaboration groups of two to four students. In this article, we examine three groups which we here call A, B, and C. The teacher was a 30-year-old male who had three years of experience teaching mathematics. We chose this class for our study because of the teacher's wish to participate, and all students were asked to express their willingness to participate in a consent form. The participants received small donations as an acknowledgement of their contribution.

We recorded actions and whole-class conversations during the problem-solving session using three stationary video cameras in the classroom. In addition, eye movements of four students and the teacher were recorded using our self-made gaze tracking devices (Toivanen, Lukander & Puolamäki, 2017). The devices consist of two eye cameras and a scene camera and simple electronics, attached to 3D-printed eyeglasses-like frames. Software computes the gaze target coordinates on the scene video.

The collaborative problem-solving session examined in this study is an 18-minute episode in a 45-minute lesson. The objective of the collaboration in groups of two or four students was to find out the optimal solution to a geometry problem. The teacher encouraged the students to share their ideas both verbally and visually, and to select the solution they preferred. The lesson task was to find the shortest possible way to connect four imaginary cities with electrical cable, located at the vertices of a square. The teacher's role was to support students in the problem-solving process through encouragement and questions without giving hints for solving the task. During the session, the teacher roamed in the classroom and stopped to help one group at a time. These short scaffolding sessions are called 'events' in this research.

In this study, our coding unit is a dwell time. We coded gazes with ELAN software using distinctive codes, and annotated each dwell that was at least 80 milliseconds (two frames) long. The basic principle in the coding system was to code those targets that contained information on teacher-student interaction and teacher's pedagogy precisely. Less informative targets, such as classroom furniture, were coded with less attention to accuracy.

After the coding, we exported all the dwell times to a spreadsheet for further analysis. We compare distributions of targets of teacher's visual attention in several guiding events quantitatively. To cover the explorative and descriptive research questions, the analysis includes both a statistical representation and a qualitative description of the interaction during the session. The qualitative analysis is based on the verbal teacher-student interaction on video data.

RESULTS

This research compares teacher's gaze behavior during scaffolding events that either included mathematical conversation and advice, or focused on interaction and collaboration. This chapter presents teacher's gaze targets during six events with three student groups. The following graph (Figure 1) shows the overall distribution of gaze targets in all these events.

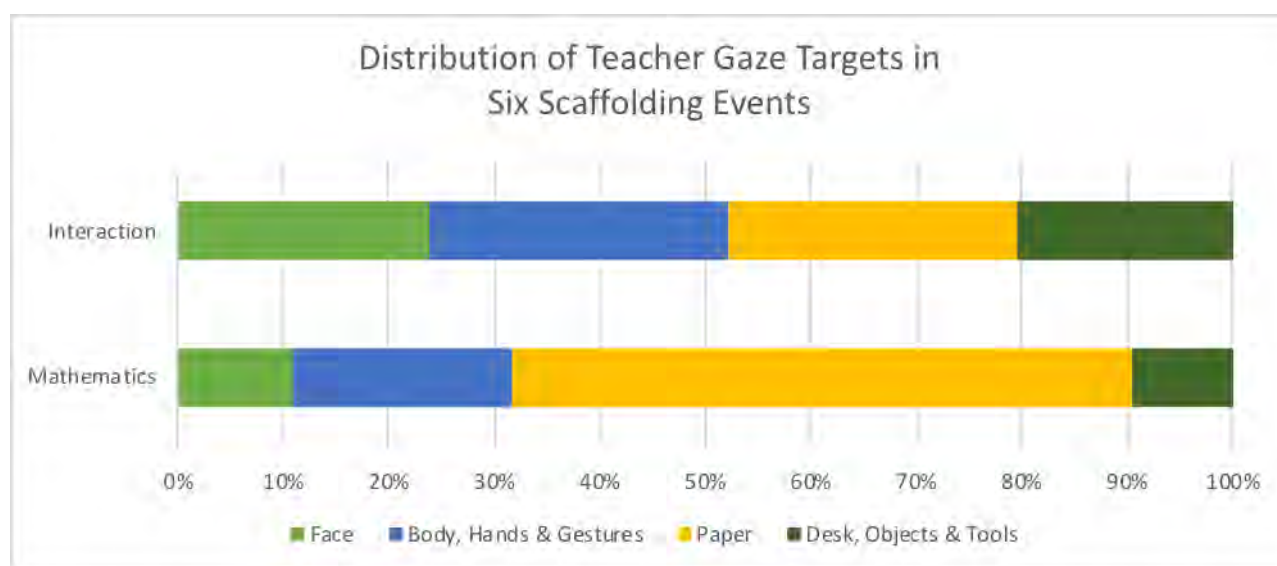


Figure 1: The average of the student-related targets of teacher gaze in groups A, B and C during Interactional and mathematical event.

Teacher gazed at students more as persons during interactional events and at their papers in mathematical events. During mathematical scaffolding, 59 % of teacher's dwell times were directed to students' solutions, while during interactional events the percentage was 27 %. Accordingly, the percentages of gaze at student's face, hands and bodies were 25 % in mathematical and 50 % in interactional events.

In interactional events, the teacher focused more on multiple objects on students' desks. On one hand, he looked at the surface of the desks and school accessories while

he was considering the interaction or listening to students. On the other hand, he scanned students' irrelevant objects such as phones and headphones, especially in those groups where students expressed lack of motivation.

Despite these general tendencies, interesting differences emerged between separate collaboration groups (Figure 2).

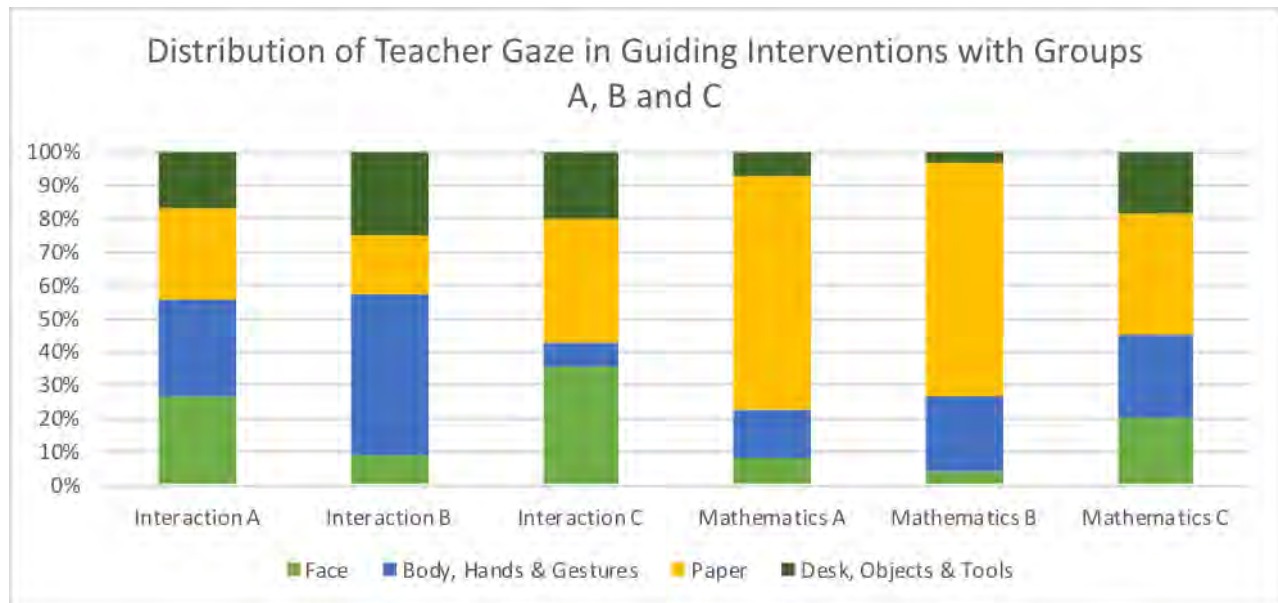


Figure 2: Student-related targets of teacher gaze in collaboration groups A, B and C during Interactional and Mathematical scaffolding events.

Group A consisted of four boys, Konsta, Daniel, Unto, and Henrik. Their interaction-oriented event (50 seconds) started when the teacher noticed them sitting quietly and intervened to encourage them to share the ideas. The group was divided into two pairs, as Konsta and Daniel in the front row focused on solving the problem together while Unto and Henrik behind them were drawing comic solutions. The teacher requested Konsta and Daniel to turn to Unto and Henrik and present the solutions to them. Konsta wanted to continue his efficient collaboration with Daniel, and disagreed with the teacher. During this event, the teacher's gaze distributed quite evenly between different targets. When students focused on their solutions, the teacher also looked at the papers or students' hands. Face-targeted gazes occurred with those students who responded to teacher initiatives of eye-contact.

The mathematical event with group A took place at the end of the collaboration session. The students had drawn several solutions on their papers and wanted teacher feedback. The teacher gazed at students' papers with long dwell times to observe the solutions and while he listened to students' description of the solutions. These gazes lasted 2.4 seconds on average, while the mean of all gazes in collaboration session was 0.5 seconds. As Konsta presented a specific question on the comparison of measurement and calculation of the solution they assumed optimal, the teacher took the paper and re-measured it. During this measuring, he naturally looked strictly on the paper. Also, while he discussed the calculation with Konsta and Daniel, his gaze was mostly

directed to boys' papers. In this event, the students wished for confirmation to their solution, and the teacher wanted to help them finishing the problem solving process as the first group in lesson. Reflecting teacher's instructional objectives to this event, these paper-targeted gazes formed 70 % of teacher's dwell times.

Group B consisted of one girl, Ingrid, and three boys, Tauno, Aarne, and Jani. In the interactional event (30 seconds), the teacher instructed them to turn towards each other and share their solution ideas by discussing. At the end of the event, Jani turned to others and started to describe his solution to them. In this event, the teacher focused a large amount of his gaze at student's bodies, hands and desks. On her desk, Ingrid had her pencil case, headphones and a phone. Bodies, however, were in the teacher's focus as he tried to capture the students' attention. Despite this, Tauno and Aarne ignored the teacher's initiatives and continued drawing, and the teacher directed his gaze at their hands. Only Jani made eye contact with the teacher by turning to him for a short while. Other students had their gaze targeted downwards during the whole event.

In the end of the collaborative phase, Tauno asked the teacher to intervene. This mathematical event (110 seconds) included mainly conversation on the task, as Tauno had managed to find out the optimal solution, and wanted teacher's confirmation of it. The teacher spent time focusing on Tauno's solution, and asked clarifying questions to help Tauno to develop it further. Teacher's gaze was 84 % of the time directed to Tauno during this event, and 94 % of the Tauno-oriented gazes targeted at his paper, the ruler pointing the solution, or hands starting to calculate. The teacher did not try to have eye contact with him, and looked at his face only when Tauno already had started re-measuring the solution.

In **group C** (two girls), the distribution of teacher gaze was more complex than in the two other example groups. The essence of the interaction event (38 seconds) was the teacher's attempt to motivate Aino and Annikki to work on the problem task. The students responded to the teacher's initiatives of eye contact, and the teacher directed his gaze at their faces while talking to them. When Annikki described her opinion about the task and its solutions, the teacher took turns looking at her paper and her face. After that, the teacher's focus turned back to the girls' faces. This event included a strong emotional component, as the students expressed their frustration and refused obeying the teacher's instructions. The teacher seemed to focus on the students' faces to better interpret their emotions and attitude. The teacher also paid attention to the phones on the table, which were not used for calculating.

The teacher returned to group C 2.5 minutes later when he noticed that Aino and Annikki were still sitting passively. This event (37 seconds) was mathematical, and teacher opened the conversation by telling the students that their solution was not the optimal. The solution papers received same proportion of teacher's gaze in this event (36 %) as in the earlier (37 %), probably because the solutions did not represent the girls' expected achievement level. However, the teacher focused less on girls' faces and more on their hands because Aino showed her solutions and concentrated in handling her headphones. The students also had several objects on their desk that captured

the teacher's attention occasionally. Despite the mathematical content of the conversation, its actual purpose was motivating students. This explains the similarities in teacher's gaze behavior in both events with group C.

DISCUSSION

This research indicates that teacher's gaze behavior is situational nature. As in Yamamoto and Imai-Matsumura's (2013) results, our informant teacher focused his gaze on those areas of interest that were relevant for his objectives. Generally, he gazed at students' solution papers during mathematical scaffolding, and at student faces, bodies, and belongings during events that included motivating and instruction on collaboration skills. During this lesson, the teacher offered guidance on two important aspects of mathematical problem solving: procedural and interactional objectives (Ding et al., 2007). In the beginning of the collaboration phase, he stressed peer interaction and shared reflection of the solution ideas. Towards the end of the phase, the emphasis of instruction moved towards mathematical goals. Thus, also his gaze behavior varied during different scaffolding events, and comparison of six individual events revealed the relation between the interaction and visual attention. To discuss the results, we will first examine face targets and secondly paper and hand targeted teacher gazes.

Gazes towards students' faces were probably the most difficult component of teacher's gaze behavior to analyze, as it seemed to be affected by the social characteristics of the scaffolding event. As Prieto et al. (2017) found out, looking at students' desks and back decreased teacher's cognitive load, while looking at their faces increased it. Our research indicates similar result. Face-targeted gazes include a great amount of information on student's emotional state, and the teacher used these gazes to either receive or give relevant information on the collaboration. Another aspect of face-directed gazes is eye contact. In the analysis of the six events, the teacher either initiated eye contact without students' response, where he did not seek eye contact at all, or where he established successful eye contact with students. The amount of eye contact carries with it high dwell times of face-targeted gazes. Nevertheless, this study does not explain the reasons for the small amount of eye contact. However, we can suggest that it relates to an already established teacher-student relationship and/or the novelty brought about by the gaze tracking glasses on the teacher's face.

The goal of teacher gazes at students' hands and papers seems to be receiving information on the problem-solving process and students' engagement. The teacher looked at students' hands to follow their actions. During these events, hand-targeted gazes occurred mostly while the teacher directed his attention to students drawing or presenting their solutions. Similarly, the gazes at papers offered a survey of a group's progress on the problem-solving task. Elegant and relevant solutions captured the teacher's attention resulting in longer dwell times than during the explanation of intuitive solution drafts. Groups A and B had proceeded with their problem solving between the scaffolding events, and teacher was interested in focusing on the solutions in latter ones.

Group C, however, presented same undeveloped solutions in both events, and also the amount of teacher's attention to those solutions remained similar.

Mathematical problem solving involves both interactional and mathematical objectives. To reach these goals, the teacher is supposed to pay attention to relevant events in the classroom (Stockero, 2014). This research indicates that the objectives alternate during a collaborative problem-solving lesson, and this variation directs the teacher's visual attention. Thus, he targets his gaze to interpret the phase and plane of student collaboration but also to affect it by focusing on those targets, which he sees relevant for the goals given a certain situation. The teacher can orchestrate the class not only by noticing some events but also by ignoring others.

As a limitation to the study, the method of mobile gaze tracking as well as the teacher's inclination to help create high quality research data may have influenced his attention. Some students may have felt it difficult to look at the teacher's face, and he might have been less task-oriented without this research setting. Gaze tracking is a largely used method for researching cognitive processes (Shayan et al., 2017). Using this method, we were able to examine a teacher's attention in detail. According to our case study, the teacher's gaze behavior is situational by its nature and reflects on his objectives on ever-changing situations. Whether these gaze patterns appear in other teachers and circumstances is a question for future research.

According to this analysis, different stages of interaction would be an interesting aim for future gaze tracking research. As interaction affects the interpretation of objects, this implies a need of flexible areas of interest. First, it is important to classify students' belongings according to their effect on learning process. Phones, for instance, represent completely different goals depending on whether they are used for amusement or calculating. Secondly, students' faces can express e.g. emotional rejection or shared interest regarding to the expression and eye contact. In this research, some students avoided eye contact with the teacher. This raises a further research question on the importance of eye contact to both collaborative and mathematical learning goals in the lesson.

Acknowledgement

The research is part of the MathTrack-project, which has been funded by Academy of Finland (grant 297856).

References

- Akkus, R., & Hand, B. (2011). Examining teachers' struggles as they attempt to implement dialogical interaction as part of promoting mathematical reasoning within their classrooms. *International Journal of Science and Mathematics Education*, 9(4), 975-998.
- Csibra, G. (2010). Recognizing communicative intentions in infancy. *Mind & Language*, 25(2), 141-168.

- Dessus, P., Cosnefroy, O., & Luengo, V. (2016). “Keep your eyes on ’em all!”: A mobile eye-tracking analysis of teachers’ sensitivity to students. In Verbert, K. Sharples, M., & Klobucar, T. (Eds.), *Proceedings of the 11th European Conference on Technology Enhanced Learning* (pp. 72-84). Lyon, France: Springer, Adaptive and Adaptable Learning.
- Ding, M., Li, X., Piccolo, D., & Kulm, G. (2007). Teacher interventions in cooperative-learning mathematics classes. *The Journal of Educational Research*, 100(3), 162–75.
- Holmqvist, K., Nyström, M., Andersson, R., Dewhurst, R., Jarodzka, H., & van der Weijer, J. (2011). *Eye tracking: A comprehensive guide to methods and measures*. Oxford, United Kingdom: Oxford University Press.
- Lappi, O. (2016). Eye movements in the wild: Oculomotor control, gaze behavior & frames of reference. *Neuroscience & Biobehavioral Reviews*, 69, 49-68.
- McIntyre, N., Mainhard, T., & Klassen, R. (2017). Are you looking to teach? Cultural, temporal and dynamic insights into expert teacher gaze. *Learning and Instruction*, 49, 41-53.
- Nizielski, S., Hallum, S., Lopes, P., & Schütz, A. (2012). Attention to student needs mediates the relationship between teacher emotional intelligence and student misconduct in the classroom. *Journal of Psychoeducational Assessment*, 30(4), 320-329.
- Prieto, L.P., Sharma, K., Kidzinski, Ł., & Dillenbourg, P. (2017). Orchestration load indicators and patterns: In-the-wild studies using mobile eye-tracking. *IEEE Transactions on Learning Technologies*.
- Shayan, S., Bakker, A., Abrahamson, D., Duijzer, C., & van der Schaaf, M. (2017). Eye-tracking the emergence of attentional anchors in a mathematics learning tablet activity. In Was, C., Sansosti, F., & Morris, B. (Eds.) *Eye-tracking technology applications in educational research* (pp. 166-194). Hershey, PA: IGI Global.
- Stockero S. (2014) Transitions in prospective mathematics teacher noticing. In Lo, JJ., Leatham, K., Van Zoest, L. (Eds.) *Research trends in mathematics teacher education* (pp. 239-259). Cham, Switzerland: Springer.
- Tatler, B., Kirtley, C., Macdonald R., Mitchell, K., & Savage, S. (2014). The active eye: Perspectives on eye movement research. In Horsley, M., Eliot, M., Knight, B., & Reilly, R. (Eds.) *Current trends in eye tracking research* (pp. 3-16). Cham, Switzerland: Springer.
- Toivanen, M., Lukander, K., Puolamäki, K. (2017). Probabilistic approach to robust wearable gaze tracking. *Journal of Eye Movement Research*, 10(4).
- Yamamoto, T., & Imai-Matsumura, K. (2013). Teachers’ gaze and awareness of students’ behavior: using an eye tracker. *Comprehensive Psychology*, 2(6).



INDEX

INDEX OF AUTHORS AND CO-AUTHORS (VOLUME 2)

A

Abo-Mokh, Dania Jamal	139
Ahl, Linda Marie	3
Ahmadpour Mobarakeh, Fatemeh.....	67
Alderton, Julie	11
Amador, Julie	19
Anabousy, Ahlam Adnan	27, 139
Anderson, Judy Anne	35
Andresen, Mette	43
Andrews, Nick.....	51
Appova, Aina.....	59
Argun, Ziya.....	403
Arteaga, Pedro	131
Asghari, Amir Hossein	67
Askew, Mike.....	75
Awoniyi, Florence Christianah.....	83
Ayalon, Michal	91

B

Baccaglini-Frank, Anna.....	99
Bakos, Sandy	107
Bampili, Amalia-Christina	115
Barmby, Patrick.....	243
Barwell, Richard.....	123
Batanero, Carmen	131
Baya'a, Nimer Fayez.....	139
Bempeni, Maria	147
Bernabeu, Melania.....	155
Böckmann, Matthias.....	163
Bresock, Krista Kay.....	171
Brown, Julian Tom	179
Buform, Àngela	187
Büscher, Christian	195

C

Canavarro, Ana Paula.....	203
Carney, Michele	211
Carotenuto, Gemma.....	219

Carroll, David.....	395
Cawley, Anne	227
Chimoni, Maria	235
Chirinda, Brantina	243
Chorney, Sean	251
Christiansen, Iben Maj	259
Christou, Konstantinos.....	267
Coles, Alf	275
Conner, AnnaMarie.....	283
Connolly, Mary	291
Cooper, Jason	299, 307
Coppola, Cristina.....	219
Crisan, Cosette	315
Cusi, Annalisa	323

D

Daher, Wajeeh Mahmood	139
Dahl, Bettina	43
Datsogianni, Anastasia.....	331
Di Martino, Pietro	99, 219, 339
Dilberoglu, Merve	347
Ding, Liping	355
Dreher, Anika	363
Dreyøe, Jonas	371
Duranczyk, Irene.....	227
Dyrvold, Anneli.....	379

E

Eames, Cheryl Lynn.....	387
Earnest, Darrell	19
Edwards, Antony.....	395
Emre-Akdogan, Elcin.....	403

F

Fellus, Osnat.....	411
Fernández, Ceneida	187, 459
Ferrara, Francesca	419
Ferrari, Giulia	419
Fidje, Anders Støle.....	427

Friesen, Marita..... 435
 Fritzlar, Torsten 443

G

Garcia Moreno-Esteva, Enrique 475
 Gea, María Magdalena 131
 Geisler, Sebastian 451
 González-Forte, Juan Manuel..... 459
 Gregorio, Francesca..... 339
 Griesse, Birgit 467
 Gucler, Beste 403

H

Haataja, Eeva 475
 Hannula, Markku 475
 Haser, Çiğdem 347
 Heinze, Aiso 363
 Helenius, Ola 3

I

Ingram, Jenni 51

J

Jones, Keith 355

K

Karpinski-Siebold, Nadja 443
 Kimai, Patrick..... 227
 Kriesler, Alice..... 35
 Kuntze, Sebastian 435

L

Larsen, Dorte Moeskær 371
 Lim, Dexter..... 227
 Llinares, Salvador..... 155, 187
 López-Martín, María del Mar..... 131

M

Mali, Angeliki..... 227
 Mesa, Vilma 227
 Mesiti, Carmel 435
 Misfeldt, Morten..... 371

Moreno, Mar 155
 Morselli, Francesca 323

N

Nicol, Cynthia 291

O

Olsher, Shai 299
 Österling, Lisa 259

P

Paulding, Katie 211
 Pinto, Alon 307
 Pitt, Andrea 51
 Pitta-Pantazi, Demetra 235
 Pouloupoulou, Stavroula..... 147
 Pratt, Nick 11

R

Rodd, Melissa..... 315
 Rolka, Katrin 451

S

Sakonidis, Charalampos..... 115
 Santos, Leonor..... 203
 Schukajlow, Stanislaw 163
 Sealey, Vicki Lynn..... 171
 Sinclair, Nathalie..... 99, 275
 Singletary, Laura Marie 283
 Sodian, Beate..... 331
 Strom, April..... 227

T

Tabach, Michal..... 27
 Taghi Dastjerdi, Sharareh 67
 Thibault, Mathieu..... 107
 Toivanen, Miika 475
 Tsiplaki, Ioanna..... 147

U

Ufer, Stefan 331

V

Vamvakoussi, Xenia.....	147
Van Dooren, Wim	459
Venkat, Hamsa	75

W

Watkins, Laura	227
----------------------	-----

Wilkie, Karina	91
----------------------	----

Z

Zachariades, Theodosios.....	115
------------------------------	-----



UMEÅ UNIVERSITY